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DYNAMIC
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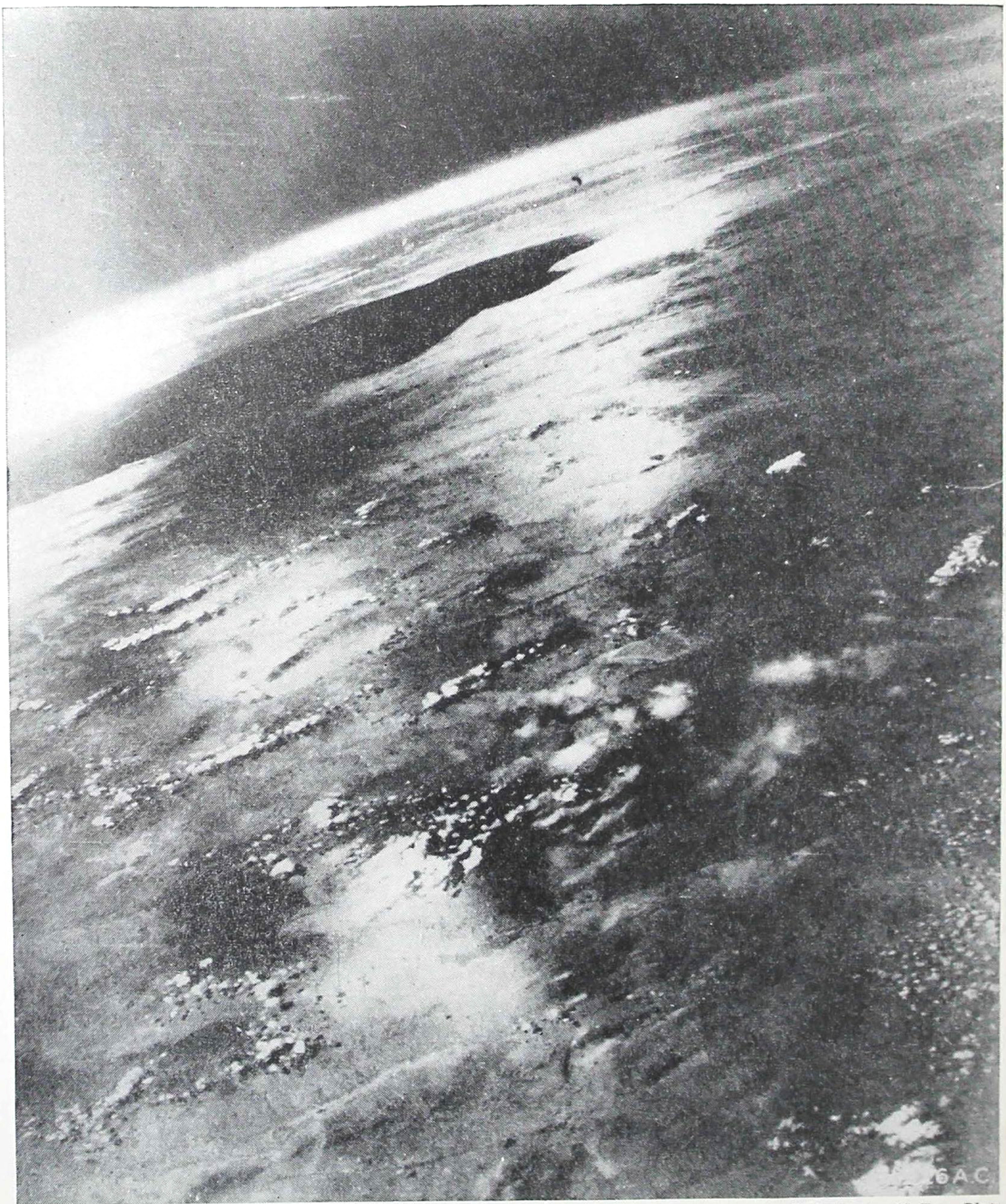

United States Air Force Photo

Rocket's-eye view of 200,000 square miles of the United States and Mexico. This photograph was taken from a V-2 rocket one hundred miles above New Mexico. The line on the horizon, showing the curve of the earth, is formed by clouds over the Pacific, 900 miles from the bottom of the picture. (See Problem **S5** on page 253.)

DYNAMIC PLANE GEOMETRY

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Preface

For many years articles have been appearing in professional periodicals calling for the reformation of the subject matter and methods of presentation of plane geometry to make it more meaningful to students and more functional in general and scientific thinking.

We offer this text, which is the outcome of long experience and experimentation, as an answer to this need. The course upon which it is based has been developed over a considerable number of years. During this time many different methods of presentation and much new subject matter have been evaluated. Those that showed promise were further refined and incorporated in the text. In *Dynamic Plane Geometry*, therefore, will be found the best of the conventional course together with well-tested elements of the new geometry about which so much has been written and so little done. Among its features are:

Stress on Methods. The emphasis on the methods used to develop geometrical concepts is at once evident in the organization of the text. Each chapter contains subject matter appropriate to the development of a given aspect or type of thinking about space relations. The chapter headings indicate the kind of thinking employed as well as the spatial content. Where the traditional arrangement is somewhat altered by this organization, it is believed that the new sequence is itself an improvement.

Generalized Concepts. Teachers have long recognized that only generalized concepts are capable of transfer. In this course, study is not confined to individual definitions and theorems but is extended to *the* definition, *the* theorem, *the* assumption, and so on. In other words, the nature of such concepts and their role in sound thinking have been underscored throughout.

Dynamic Approach. We have tried to broaden the methods of demonstrative geometry through the Gestalt approach. We have supplemented the deductive and static pattern of traditional geometry with inductive and dynamic exploration. By a *dynamic* geometry we simply mean a study of the parts of space and their relations to one another while they are in motion and changing. Beginning early with the rotation of a ray to form an angle, geometry in motion is increasingly stressed until the peak is reached in the study of the circle. Later the dynamic method is applied algebraically to variation and functionality. The inductive method and its relation to deductive thinking is given wide recognition in the study of congruent triangles and in many other appropriate places.

Big Issues. We have attempted to put first things first and make them big issues. Certainly, the nature of postulational thinking is dominant in demonstrative geometry. It is treated in Part One. Part Two is concerned with a number of types of general thinking. Adequate attention is given the converse, indirect proof, necessary and sufficient conditions, and the relationships among these concepts.

Transfer Materials. Not only is the transfer of the mode of thinking taught in the form of applications but related topics are also used as vehicles for developing certain concepts in geometry. Algebra is one field with which the methods of geometry have been tied up. In this way skill in algebra is maintained not by isolated review but by a correlation of the notions of algebra with those of geometry so that they integrate with each other. Our next correlation is with science and shop problems. An adequate carry-over into solid geometry is provided. A large number of aviation problems, inserted where they utilize the geometric principles being studied, are intended to exploit the students' natural interest in this field. Finally, we have correlated in many places with social problems and citizenship, and with general sound thinking.

Aids to Teaching. Throughout the period during which this text was developed constant attention was given to the teachability of the subject matter, and many practical aids were incorporated to assist both teacher and student. For example:

1. In almost all the propositions the given data are marked on the diagram, and the parts of the diagram that enter the conclusion are denoted (as in algebra) by the last letters of the alphabet. This enables the learner to "take in" the whole situation at a glance.

2. The proofs of the important theorems are led up to gradually through the inductive approach. The kind of proof that has some transfer value or some use in the further study of mathematics is given preference. Long and mechanical proofs are avoided. The temptation to memorize theorems and their proofs is minimized by the stress on insight.

3. In addition to the regular material, which affords a complete and simplified minimum course, superior and honor levels (denoted by S and H) have been included for enrichment and to enable the teacher to choose that material suitable to the capabilities of his students and the time available.

4. The emphasis throughout is to enable the student to learn by seeing and doing. To that end introductory and explanatory paragraphs are short and illustrative examples (with figures) are used wherever possible to show just what is to be done.

5. More than 1200 exercises afford a wide variety of problems from which the teacher may choose, and almost 1000 figures enhance the emphasis on the graphic presentation of subject matter.

6. Dependence charts and tables are outstanding helps to understanding. These charts and tables summarize many of the more important theorems, show in graphic form how they depend on certain basic assumptions and each other, and classify them according to their conclusions.

7. The end-of-chapter material has been developed with much thought. A summary of important concepts serves to fix these concepts in the students' minds. Review exercises and tests help them to assess their understanding of the subject matter and enable the teacher to determine the progress of each student. At the end of the book there is a summary of fundamental theorems, classified according to types of figures.

8. Chapters are subdivided into a number of lessons, each one of which covers a single phase of the chapter topic and constitutes an easy steppingstone to the next. Insofar as is possible each lesson is a self-contained unit and includes both text material and a series of carefully graded exercises. These lessons are not to be confused with assignments, as in many cases a single lesson will require more than one day to complete. There are eighty-two such lessons in the text. The division between them is indicated by a space and a small rule. For the convenience of the teacher an outline will be found on page 284 showing the section number with which each new lesson begins.

This text contains most of the general recommendations that have been made for geometry in recent years. It is the author's hope that it will make the study of geometry more understandable and meaningful to the student, and will foster the art of clear thinking so vital for effective living.

The author wishes to express his appreciation to many teachers, students, and friends for their generous help during the period of this text's development. He is particularly grateful to Michael R. McGreal, Assistant Superintendent of Schools, Newark, New Jersey, for his active support and encouragement; to Stanton A. Ralston, Principal of Central High School, and Alan Johnson, formerly Assistant Superintendent of Schools, for originally authorizing the experimental groundwork for the course; to Dr. Leon Mones, Principal of Cleveland Junior High School, Arthur M. Seybold, Director of the College High School of the State Teachers College, and Dr. Heber H. Ryan, Assistant Commissioner of Education in New Jersey, for their evaluation and criticism of the socialized aspects of the book; to Arnold Strassberg and Harold A. Gouss for their valuable suggestions and criticisms while trying out the material in their classrooms; and to Ernest R. Ranucci and N. Howard Ayers, chairmen of mathematics departments in Newark, for their many important contributions.

David Skolnik

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Part One

The Meaning of Proof

SYMBOLS AND ABBREVIATIONS USED IN THIS BOOK

Symbols

∠, ∠s	angle, angles
⌒	arc
⊙, Ⓢ	circle, circles
≅	congruent, is congruent to
°	degree
=	equals, is (are) equal to
≠	unequal, is (are) not equal to
>	is (are) greater than
<	is (are) less than
≗	is measured by

′	minute *or* prime *or* foot
″	second *or* inch
∥	parallel, is parallel to
∥s	parallels
∦	not parallel
▱, ▱s	parallelogram, parallelograms
▭, ▭s	rectangle, rectangles
⊥, ⊥s	perpendicular(s), is perpendicular to
∼	similar, is (are) similar to
△, △s	triangle, triangles

Abbreviations

adj.	adjacent
alt.	altitude
alt. int.	alternate interior
ax.	axiom
comp	complementary
cons.	construct *or* construction
corr.	corresponding
def.	definition
diag.	diagonal
diam.	diameter
ext.	exterior
int.	interior
isos.	isosceles
opp.	opposite
post.	postulate

prop.	proposition
pt.	point
quad.	quadrilateral
rect.	rectangle
req.	required
rt.	right
sq.	square
st.	straight
subs.	substitution
subt.	subtraction
sup.	supplementary
tan.	tangent
trans.	transversal
trap.	trapezoid
vert.	vertical

Introduction

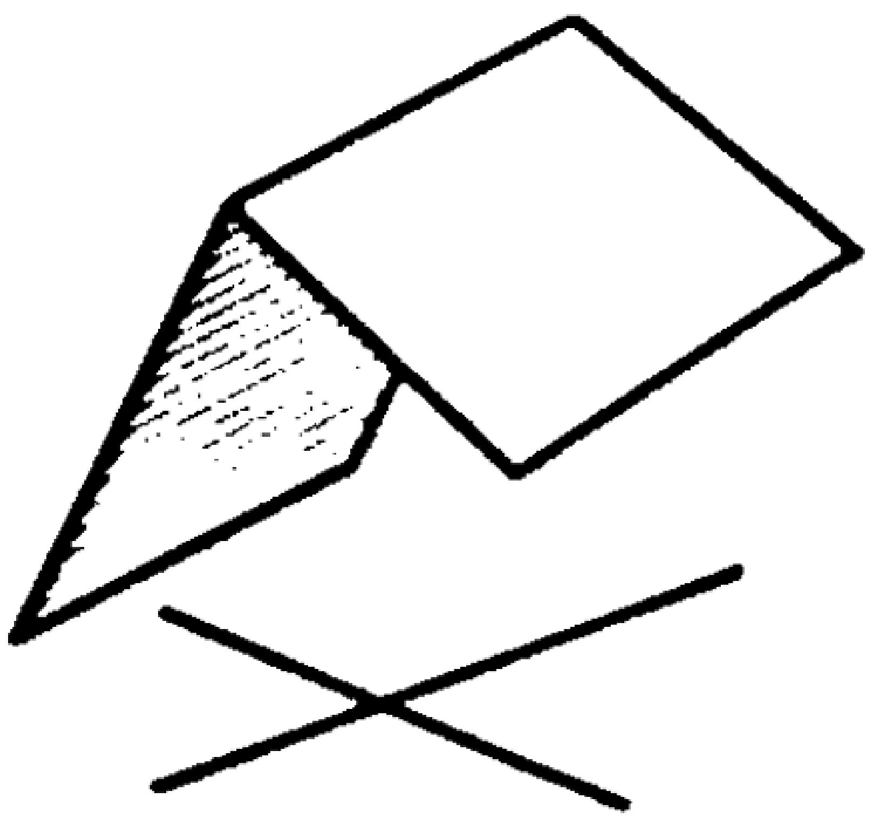
GEOMETRY, A STUDY OF SPACE

Parts of Space

1. Originally geometry meant a study of earth measurement. The word *geometry* comes from two Greek words meaning "earth measure." Ancient peoples needed to measure or to survey land, and it is therefore understandable that geometry was developed in early times.

Through the years geometry has come to mean a study of space and of those figures that move about in space. Think about the flat surface on the wall or on the blackboard. Try to visualize this flat surface as having no thickness and no boundary lines or limits and you have a good idea of a part of space called a *plane*. Most of the figures we shall study in this course in plane geometry can be fitted into a plane and are therefore called *plane figures*. You are already familiar with the shapes of many of these figures, such as the triangle, the circle, the square, and the rectangle.

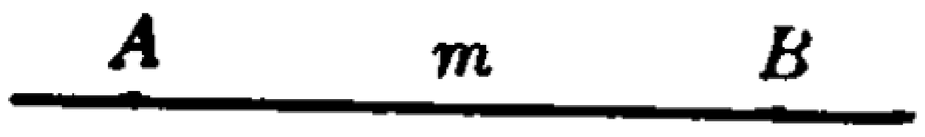
Now imagine a plane formed by a sheet of paper. Fold the paper once. The crease is a *straight line*. The folded paper represents two planes that cut each other in a straight line. In mathematics a straight line has neither width nor thickness. It has only length and much of it. In fact, a straight line has no beginning and no end.



Now think of two straight lines cutting or *intersecting* each other. The part of space that is on both straight lines is called a *point*. In mathematics a point has no size whatsoever, only position.

Naming Points and Lines

2. Since a point is position, it is customary to label a point by a capital letter just as it is customary to capitalize geographical names. Thus, the two points shown below are called point A and point B. A straight line is labeled by a small letter placed on or near the line, or by two capital letters to show two points on the line. Accordingly, the line shown below is called m or AB.



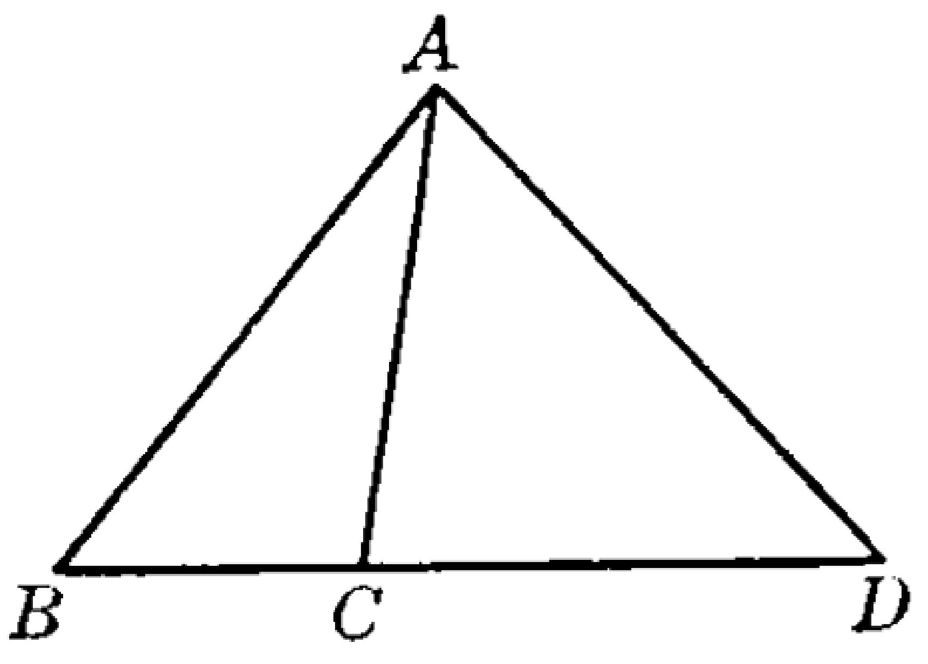
Parts of a Straight Line

3. The part of a straight line that is on one side of a point is called a half-line or a *ray*. A ray has a beginning but no end. This reminds us of a ray of light which starts from a point and goes on indefinitely.

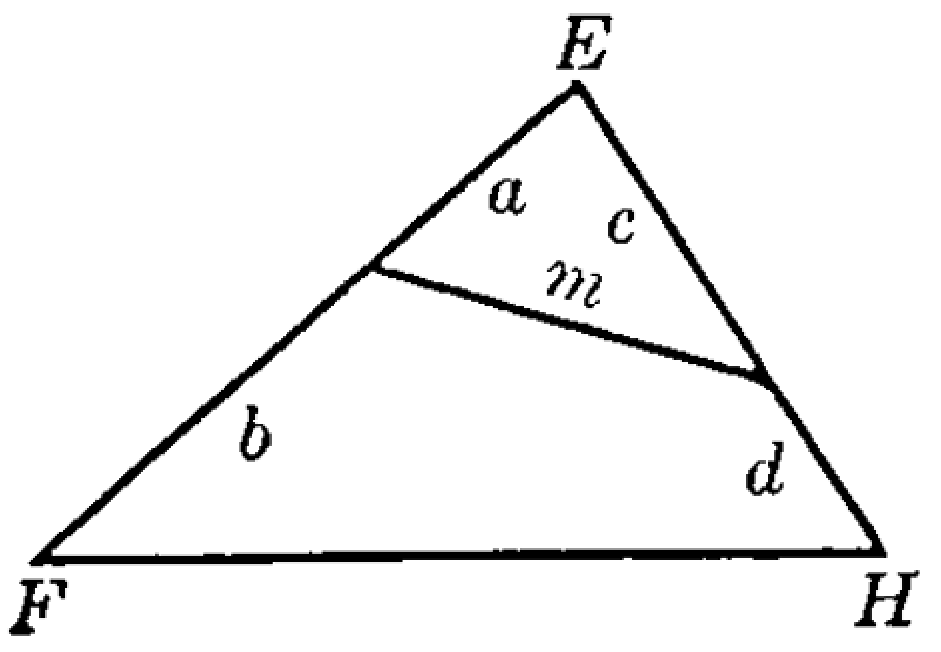
The part of a straight line between two points is called a *line segment*. A line segment has a beginning and an end. When no confusion can result, the expression *straight line segment* AB is often shortened to *straight line* AB, *line segment* AB, *line* AB, or simply AB.

EXERCISES

1. In the figure name the points and the line segments.



2. In the figure name all the line segments.



3. Complete the equations for the figure for Ex. 2:

$$a + b = \text{—?—}$$

$$EH - \text{—?—} = c$$

4. Write an equation stating that a certain line segment in the figure for Ex. 1 equals the sum of two other line segments.

SUPERIOR WORK

S1. If three straight lines cross one another, how many line segments may be formed? Is it always that many? Draw figures to illustrate your answers.

S2. If two points are taken on a line, how many line segments and rays are formed? (Do not forget the overlapping rays.)

S3. Is it possible to draw a true line or a true point? What is meant by saying that lines and points are only ideas of the mind?

GEOMETRY, A STUDY OF MOTION

Motion and Space

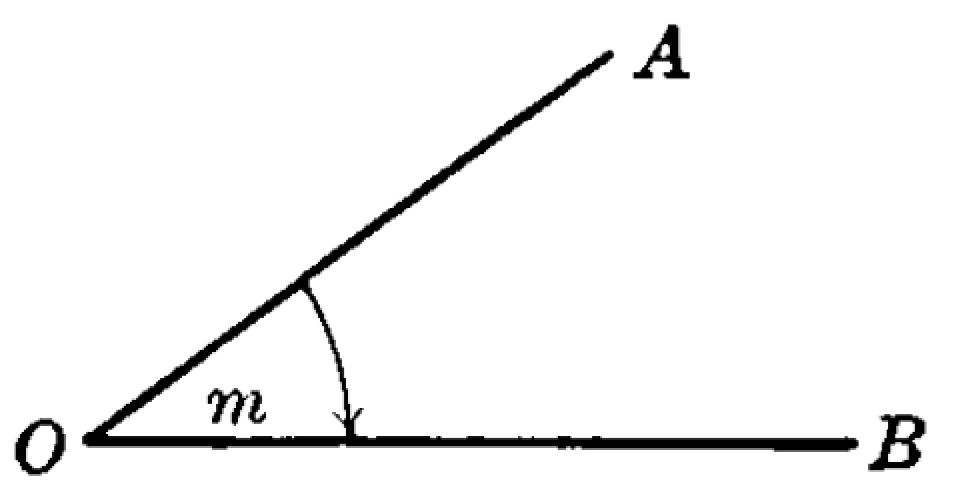
4. To move an object means to change its position in space. Since geometry is a study of space and figures in space, we shall often think of motion when we are working with the parts of space.

The idea of a point *moving* so that it generates or forms a line segment is an example of geometry in the study of motion. The completed line segment is an example of geometry in which the figures are at rest or *stationary*. Drawing a line segment is a study of motion. The completed line segment is a stationary figure.

In using geometry to study motion, we think of the parts of space as *moving*, *changing*, or *being formed*. In applying geometry to figures at rest, we think of the parts of space as *stationary*, *fixed*, or *completed*.

What an Angle Is and How It Is Labeled

5. If a ray turns about its initial point from one position to another like a hand of a clock, it forms a figure called an *angle*. We may think of the angle shown as having been formed by the rotation of a ray about the point O from the position OA to the position OB. In studying figures at rest, we think of the angle as having been formed by the two rays OA and OB from the common point O.

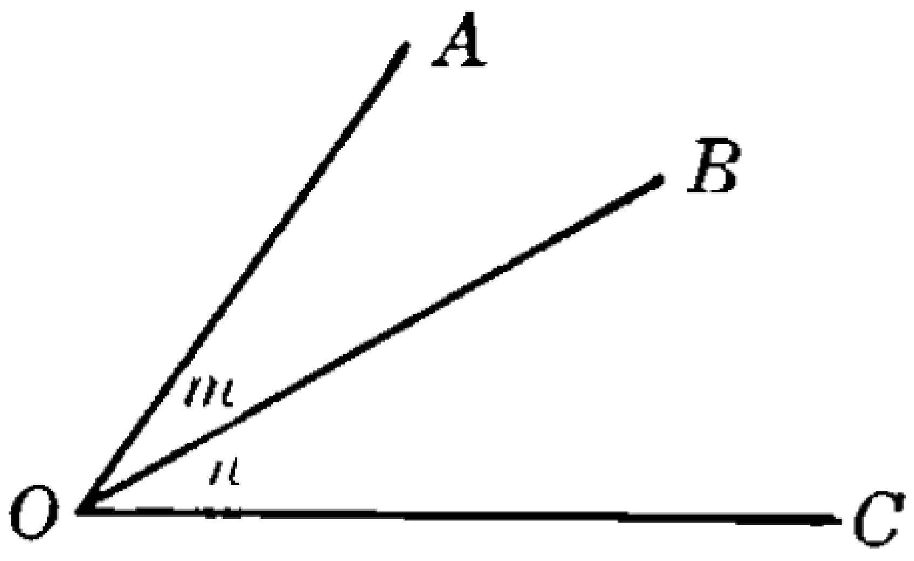


The rays OA and OB are called the *sides* of the angle and the point O is called the *vertex* (plural: *vertices*) of the angle.

The symbol for angle is ∠; for angles,

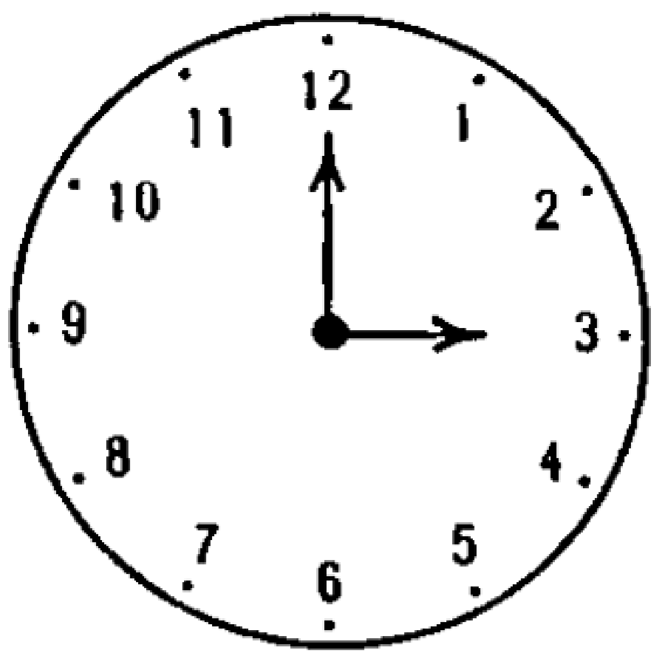
$\angle$s. (*Note:* A complete list of symbols appears on p. xii.) The angle above may be called one of the following: $\angle AOB$, $\angle BOA$, $\angle O$, or $\angle m$. If three letters are used, the vertex letter must always be in the middle.

Do you see three angles in the following figure? They are $\angle COB$ or $\angle n$, $\angle BOA$ or $\angle m$, and $\angle COA$. To save time, we can use $\angle O$ to mean the entire angle COA. This can be done only when there is no possibility of confusion. Thus, we should not label either $\angle COB$ or $\angle BOA$ simply $\angle O$.



Measuring and Drawing Angles

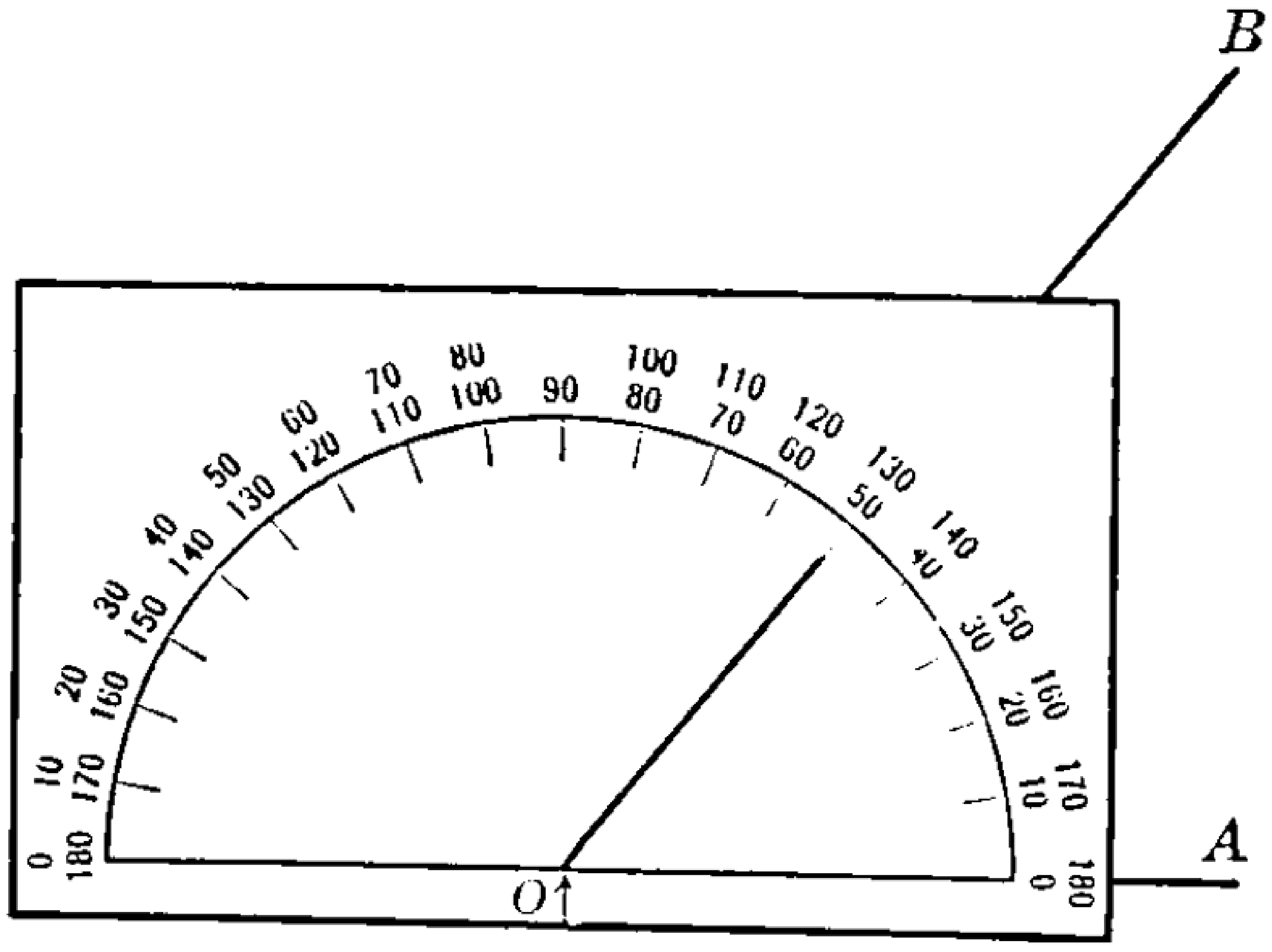
6. The size of an angle depends only on the amount of rotation of the ray that forms the angle, and not on the length of that ray. Remember that a ray has no length. Thus, the angle formed by the hands of a wrist watch at 3 o'clock is as large as the angle formed by the hands of the famous tower clock of the English Houses of Parliament at the same time.



The most common unit of measure of an angle is the *degree*. A complete revolution is a rotation of 360 degrees. For example, the long hand of a clock turns through a rotation of 360 degrees when it turns from 12 all the way around to 12 again. When it turns from 12 to 3, it turns through one quarter of a complete revolution, or 90 degrees. One degree is $\frac{1}{360}$ of a complete revolution.

The instrument used to measure angles is a *protractor*. To measure an angle, such as $\angle AOB$, place the center of the protractor on the vertex of the angle and the straight edge along one side of the angle. The zero mark on the scale will fall on one side of the angle. The size of the angle is shown on the scale where the *other* side of the angle crosses the protractor. If the sides of the angle are not long enough to extend beyond the scale, extend them before making the measurement. If, however, you are measuring an angle in your textbook, use a ruler or the edge of a piece of paper to show the extension. Do not write or draw in the book.

There are usually two scales on a protractor, one reading from left to right, the other from right to left. Read the scale whose zero point is on the side of the angle that lies along the straight edge of the protractor. The angle in the figure is 50 degrees (50°).

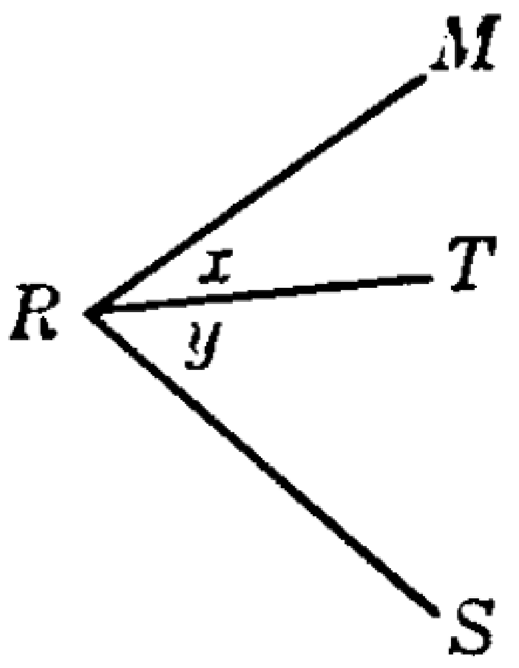


The protractor is used not only for measuring angles, but also for drawing angles. To draw an angle, take the following steps: Draw a straight line and choose a point on it to use as the vertex.

Place the center of the protractor on this vertex and the straight edge of the protractor along the line. With a pencil put a dot where the number of degrees needed for your angle is shown on the scale of the protractor. Remember to use the scale that has its zero on the *initial* side of your angle. Remove the protractor and draw a line from the vertex through the point showing the measurement. The angle formed is the desired angle.

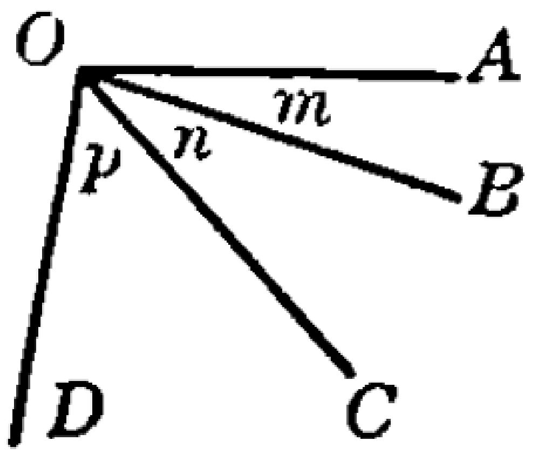
EXERCISES

1. Complete the equations for the figure below:



(a) $\angle x + \angle y = ?$
(b) $\angle R - \angle x = ?$
(c) $\angle SRT + \angle ? = \angle MRS$

2. Complete the equations for the figure below:



(a) $\angle n + \angle p = \angle ?$
(b) $\angle O - \angle m = \angle ?$
(c) $\angle n + \angle ? = \angle AOC$
(d) $\angle O - \angle ? = \angle COD$
(e) $\angle O - \angle n = \angle ? + \angle ?$

3. Through how many degrees does the long hand of a clock rotate in 10 minutes? in 30 minutes? in $1\frac{1}{2}$ minutes?

4. Using a protractor, draw angles of 40°, 72°, and 130°.

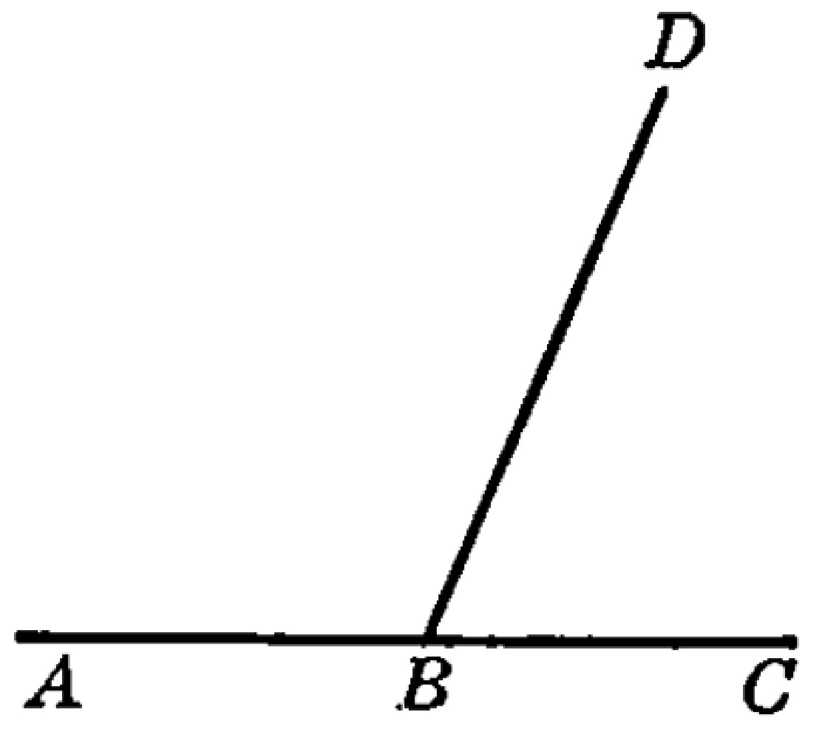
5. By estimating draw angles of the following approximate sizes: 90°, 45°, 30°, 60°, 20°, 120°, and 135°. Then measure these angles with your protractor and write the actual measurement inside each angle.

6. Draw a triangle, making two sides 3 inches each. With a protractor measure the angles *opposite* these sides. What do you discover?

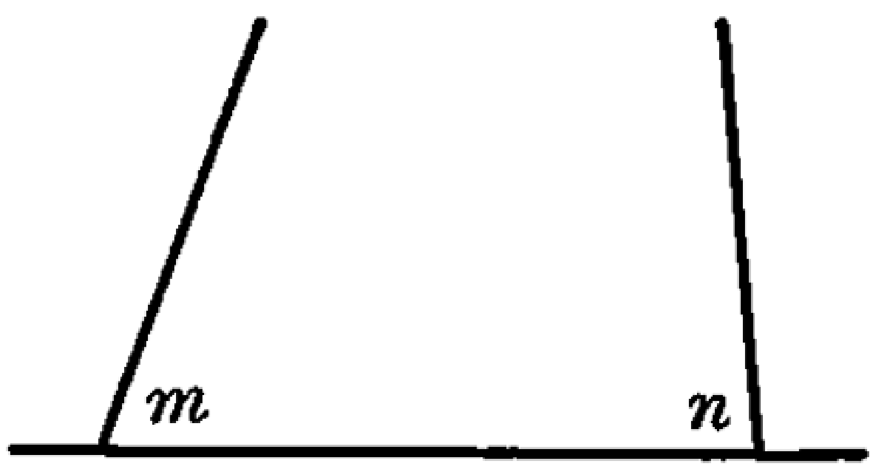
SUPERIOR WORK

S1. With a protractor construct a diagram like the one for Ex. 2 on this page, making $\angle m = 80°$, $\angle n = 60°$, and $\angle p = 40°$. How large is $\angle AOD$?

S2. Write an equation stating that a certain angle in the figure below equals the sum of two other angles.



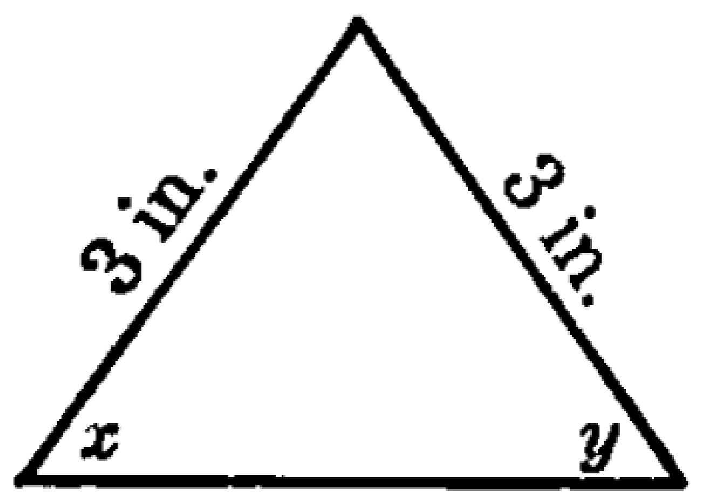
S3. With a protractor construct a diagram like the figure below, making $\angle m = 70°$ and $\angle n = 110°$.



GEOMETRY, A STUDY OF REASONING

Demonstrative Geometry

7. Certain facts about space can be discovered by observation. If we draw a triangle in which two sides are each 3 inches long, we can easily discover that the angles opposite the equal sides are equal. We can do this by measuring the angles with a protractor or by cutting out the angles and matching them with each other. By repeating this experiment a number of times, we are led to the



conclusion that no matter how long the two measured sides are and no matter what angle they form between them, as long as these two sides are equal to each other, then the opposite angles are equal.

If someone asked us how we know this fact, we would answer that we have "tried it out." We have experimental proof. In this course we shall have to give other kinds of reasons for our statements. We shall say that certain angles are equal, not because we measured them with a protractor, but because it follows by reasoning from some other fact. In mathematics we are interested in finding out what conclusions follow from certain facts. If these conclusions can be supported by reasons, we call them *necessary conclusions.*

Mathematics is the science of necessary conclusions.

Many relationships in space cannot be easily discovered by observation alone. To discover and to prove relationships that cannot be discovered by the eye alone, as well as relationships that can only be suspected by the eye, mathematics calls in the power of the mind. This is the power to think and to reason. A geometry that uses this power is called *demonstrative geometry,* because the conclusions are *demonstrated* or *proved.*

Demonstrative geometry is the geometry in which the necessary conclusions are proved by reasoning.

The Proposition

8. In everyday reasoning we make many statements that are conditional, or that lead to other statements or conclusions. For example: *If* a person is born in the United States, *then* he has a right to American citizenship.

In geometry an "if then" statement of this type is called a *proposition.* An example of a geometric proposition is the following: If two sides of a triangle are equal, then the angles opposite the equal sides are equal. The conditional part of the proposition, the statement of what is assumed to be true, is called the *hypothesis.* The part of the proposition that follows from the hypothesis is called the *conclusion.* The hypothesis in the proposition just given is that *two sides of a triangle are equal.* The conclusion is that the *angles opposite the equal sides are equal.*

In a conditional statement the condition is called the *hypothesis;* what follows from the condition is called the *conclusion.*

If the conclusion follows from the hypothesis through the force of reasoning, the conclusion and the whole proposition are said to be *valid.* A conclusion can be valid without being true if the hypothesis on which it is based is false. A true conclusion must have a hypothesis that is true. For our purposes we are primarily concerned with validity or correctness of the reasoning rather than the truth of the proposition.

Consider this proposition in algebra: If we accept $3x - 10 = 35$ as our hypothesis, we are led through the force of reasoning to the conclusion that $x = 15$.

Demonstrative geometry requires us to use our reasoning power. It may also help us to think more clearly in nonmathematical situations if we use this same reasoning power in everyday situations whenever possible.

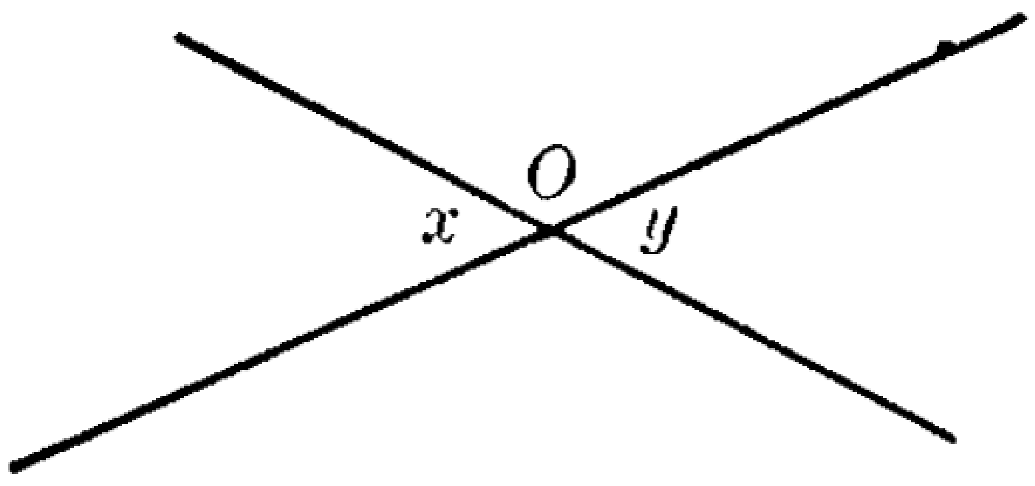
EXERCISES

1. If $5x - 7 = 3x + 1$, what is the necessary conclusion about the value of x?
2. If $x = 5$, then $x^2 = 25$. Is the conclusion valid?

3. If $x^2 = 25$, then $x = 5$. Is the proposition valid?

4. If a person is redheaded, then he is ill-tempered. Is the proposition valid?

5. Two straight lines intersect at O. Can you discover a necessary conclusion about the angles x and y?



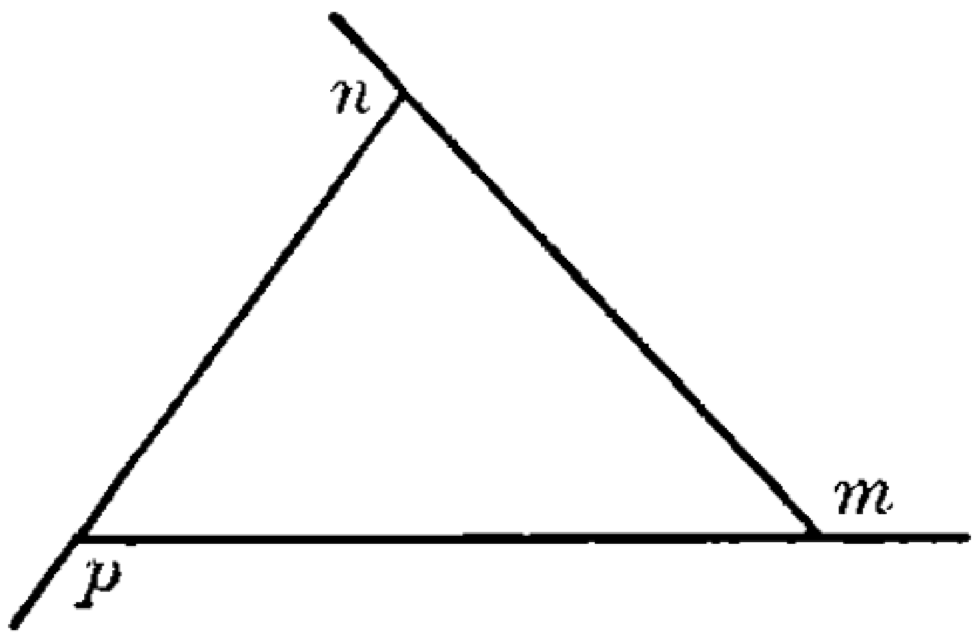
Think of the angles x and y as being formed at the same time by the rotation of a line about the vertex O. Does this suggest a reason for your conclusion?

SUPERIOR WORK

S1. Write two mathematical and two non-mathematical propositions that are valid.

S2. If a certain hypothesis leads to a certain conclusion, does the conclusion always lead to the hypothesis? Write two propositions that illustrate your answer.

S3. Draw a figure as nearly as you can like the one shown; by measuring the angles m, n, and p with a protractor, find their sum.



Draw several other figures like it, but change the sizes of the three angles. Measure the angles. What do you discover about the sum of the angles in each figure?

S4. Is the following proposition valid? If $xy \neq 0$, then $x \neq 0$. (The symbol $\neq$ means "not equal to.")

S5. In each group, what two ideas are most closely related?

(a) Geometry, hypothesis, proposition, proof, and conclusion.

(b) Experimenting, guessing, valid proposition, hypothesis, and necessary conclusion.

SUMMARY

9. The geometry that we are studying has to do with space, and motion in space. In studying motion, we are thinking about figures that grow larger or smaller or move about in space. To understand figures in space better, we should think of them as they change as well as when they are stationary.

Some simple parts of space are points, straight lines, and planes. Line segments and rays are parts of straight lines. An angle is a figure made by two rays from the same point.

A point is labeled with a capital letter. A straight line, a line segment, and a ray are labeled either with two capital letters or with one small letter. An angle is labeled with three capital letters, one capital letter, or one small letter.

In demonstrative geometry propositions are proved by reasoning. A proposition states that if a certain statement, called the *hypothesis*, is accepted, then a certain *conclusion* must also be accepted. In a valid proposition the conclusion is necessary because it follows by the law of reason.

RECOGNIZING NEW TERMS

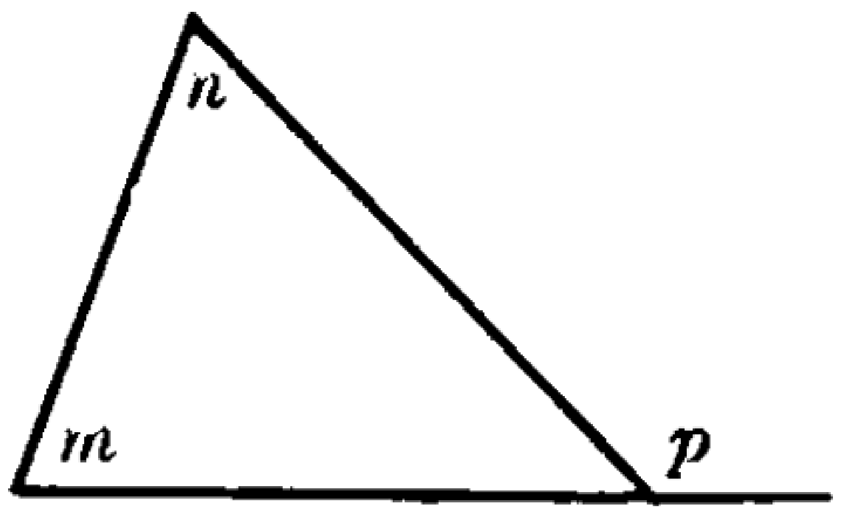
Can you describe each word or term below? If not, go back to the section indicated.

geometry § 1
plane § 1
space § 1
point (pt.) § 1
straight line (st. line) § 1
line segment § 3
ray § 3
angle (∠) § 5
  sides
  vertex
protractor § 6
demonstrative geometry § 7
proposition § 8
  hypothesis
  conclusion
valid conclusion § 8
valid proposition § 8

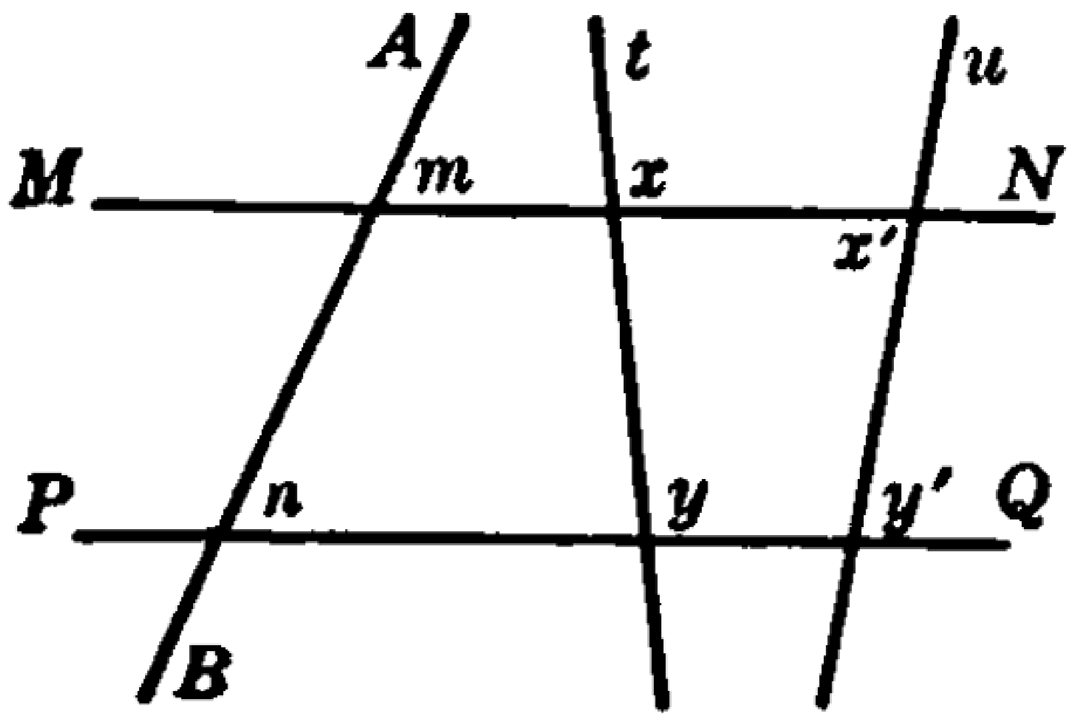
Note: A complete list of abbreviations appears on p. xii. Learn each abbreviation as you proceed through the text.

REVIEW EXERCISES

1. In the figure estimate the number of degrees in $\angle m$, $\angle n$, and $\angle p$.



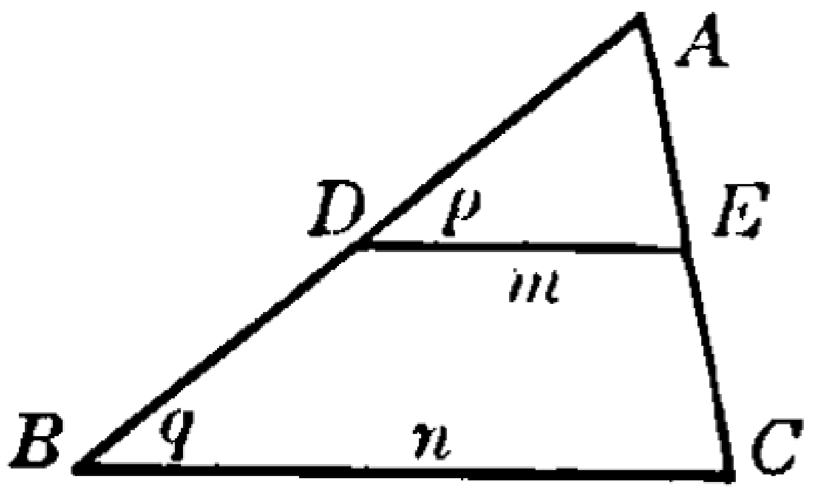
2. Using a protractor, draw a triangle with two angles of 40° each. Measure the sides opposite the equal angles. What do you discover about the length of those sides?
3. With the aid of the protractor, draw a closed four-sided figure in which three angles are each 90°. Estimate the size of the fourth angle. Check your estimate by measuring with the protractor. What do you discover?
4. (*a*) Using the figure below as a guide, draw a line AB. Then, using your protractor, draw two lines MN and PQ so that $\angle m$ and $\angle n$ are each 65°.
   (*b*) In the same figure, draw two other lines such as t and u. What is the measurement of $\angle x$, $\angle y$, $\angle x'$, and $\angle y'$?



5. (*a*) At 9:30, the hands of a clock make an angle of —?—°.
   (*b*) A rotation of a ray from a vertical to a horizontal position covers an angle of —?—°.
   (*c*) As the small hand of a clock rotates 10 degrees, the large hand rotates —?— degrees.
6. (*a*) In a proposition it makes no difference which part we call the hypothesis and which the conclusion. True or false?
   (*b*) If a clock strikes the hour as the hands form an angle of 90°, it is three o'clock. Is this proposition valid?

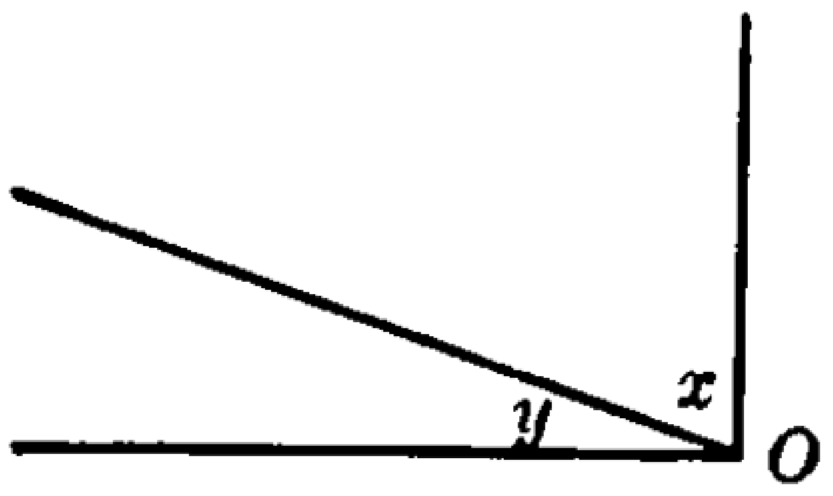
SUPERIOR WORK

S1. Using the figure below as a guide, draw a figure making $AD = DB = 1\frac{1}{2}$ inches and $AE = EC = 1$ inch.

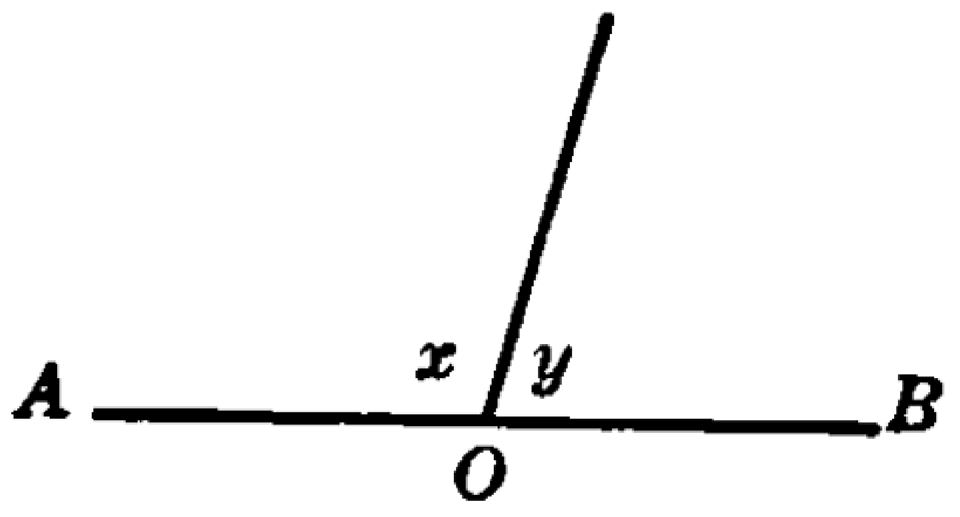


Now measure, and write on the figure you have drawn, the size of $\angle p$ and $\angle q$, and the length of line segments m and n.

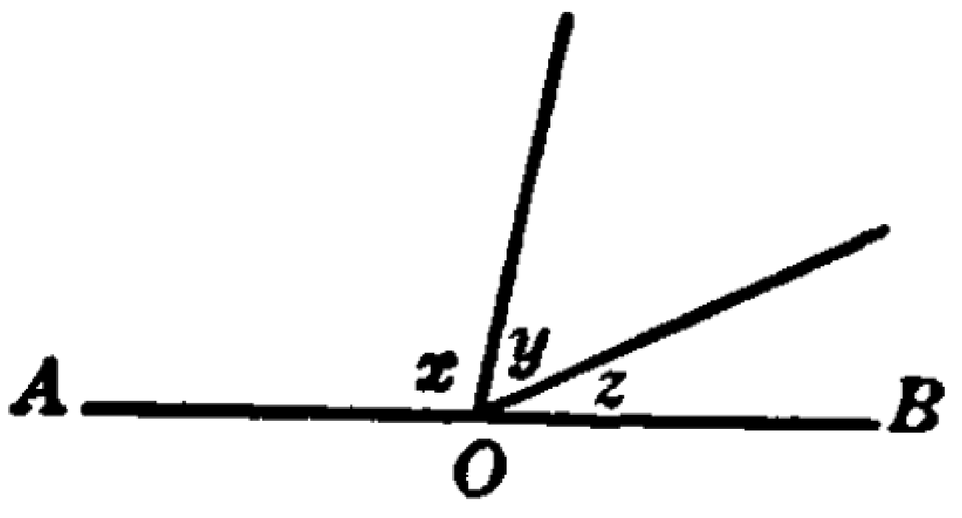
S2. In the figure, $\angle O = 90°$ and $\angle x = 4 \cdot \angle y$. Find the number of degrees in $\angle x$ and $\angle y$. (The raised dot between the "4" and "$\angle y$" indicates multiplication.)



S3. In the figure, AB is a straight line and $\angle x - \angle y = 30°$. Find the number of degrees in $\angle x$ and $\angle y$.



S4. In the figure, AB is a straight line, $\angle x = 2 \cdot \angle y$, and $\angle y = 2 \cdot \angle z$. Find the number of degrees in $\angle x$, $\angle y$, and $\angle z$.



S5. Examine each proposition below. Then decide whether each one is *valid* or *not valid*.

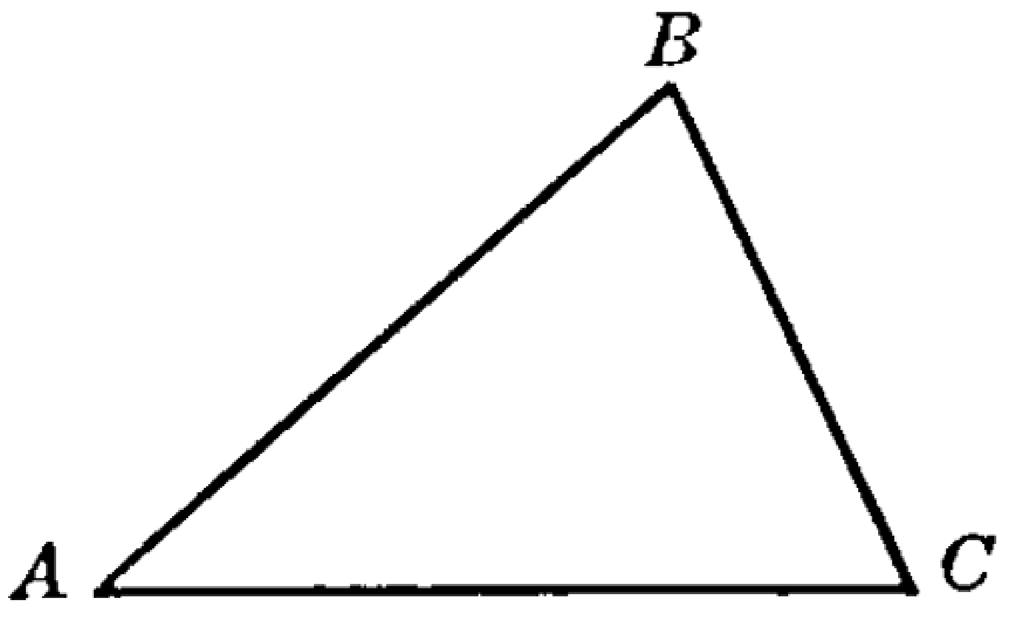
(*a*) If $x + 2x = 3x$, then $x = 10$.

(*b*) If $(x - 2)(x - 3) = 0$, then $x = 2$.

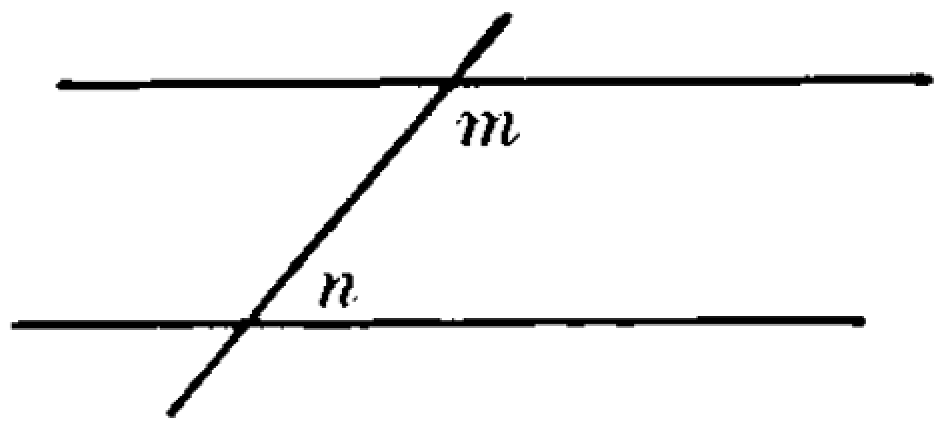
(*c*) If $x = 3$, then $(x - 2)(x - 3) = 0$.

TEST

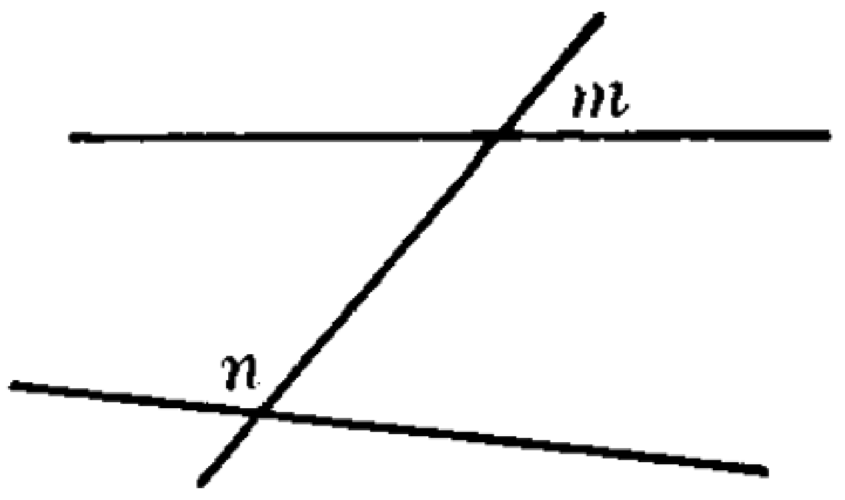
1. Estimate the size of each angle in the figure, then complete the following equation: $\angle A + \angle B + \angle C =$ —?—°.



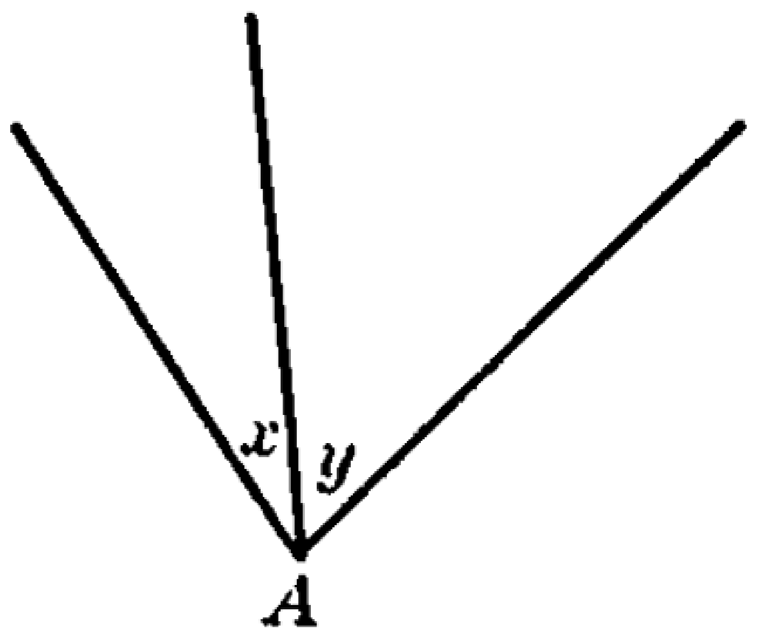
2. Complete the following equation by estimating the size of the angles in the figure below: $\angle m + \angle n =$ —?—°.



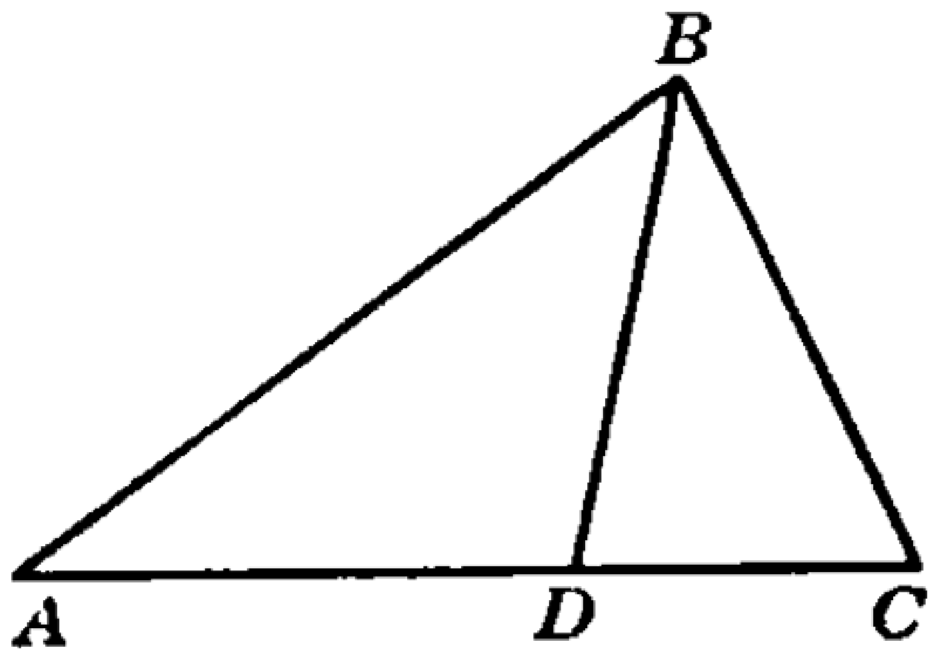
3. Using a protractor, draw a diagram like the one shown in the figure below making $\angle m = 30°$ and $\angle n = 150°$.



4. In the accompanying figure, complete the equation $\angle x + \angle y = \angle$ —?—.



5. Name all the line segments in the figure below.



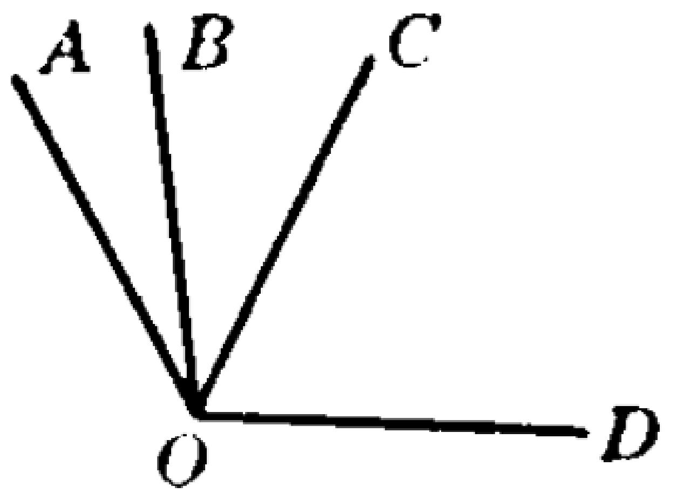
6. Write an equation for the above figure stating that the sum of two line segments equals a third line segment.

7. Write an equation stating that the sum of two angles in the figure for Ex. 5 equals a third angle.

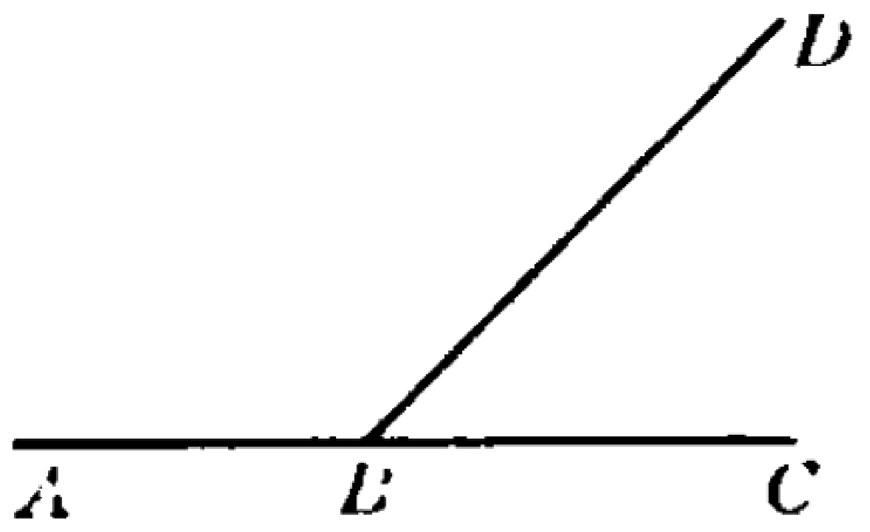
8. Name the largest angle in the preceding figure that is less than 180°.

9. Through what angle does the minute hand of a clock turn in 30 minutes?

10. Name all the angles in the figure below.



11. In the figure below, complete the equation: $\angle ABD + \angle DBC = \angle$ —?—.



Complete the following statements by supplying one word for each blank.

12. In the figure above, there are —?— rays from the point B.

13. A point has only —?— but no —?—.

14. A statement H leads to a statement C. Statement H is called the —?—; statement C is called the —?—.

15. If $\sqrt{x} = 7$, then the necessary conclusion is that $x =$ —?—.

16. If $x^2 = 16$, the necessary conclusion is that $x =$ —?— or —?—.

17. The score at the half of a football game was Princeton 6, Yale 0. The final score of 7–6 implies that —?— won the game.

18. If the game in Ex. 17 were a baseball game, the conclusion would not be —?—.

19. The geometry in which propositions are proved by reasoning is called —?— geometry.

20. A rotation of the compass needle from north to east is an angle of —?—°.

1

The Assumption

Axioms and Definitions

THE AXIOM

Proved and Unproved Propositions

10. In our geometry we shall prove propositions by reasoning. We shall argue that certain statements are true because other statements that came before them are true.

You may now ask, "How shall we prove the first proposition?" We cannot argue from an earlier proposition because there is none. The answer is, therefore, that we must accept the first proposition without proof. In practice we accept a number of propositions without proof.

If we accept a statement without proof, we are merely assuming it in order to get our thinking started. We call such an assumed statement an *axiom* or a *postulate*. In everyday thinking there is no clear distinction between an axiom and a postulate. In mathematics, postulates usually have to do with space, whereas axioms are more general. Sometimes we merely call such statements *assumptions*. If we mean to accept a statement only after proving it, we call it a *theorem*.

A proposition that is accepted only after it has been proved is called a *theorem*.

A proposition that is accepted without proof is an *axiom* or *postulate*.

The idea of having to assume certain things in order to prove others may seem strange to you. All it really means is that before you can prove anything (or even discuss it intelligently) you must agree to accept certain conditions. You must try to understand this so that you may learn how to build a logical argument.

An example of a proposition accepted without proof is the following: If two points are given, one and only one straight line can be drawn through them. This postulate may be expressed in another way as follows: Through two given points, one straight line and only one can be drawn. This postulate will be referred to as the two-point postulate.

11. Two-Point Postulate. Two points determine a straight line.

Determining a Geometric Figure

12. A geometric figure is said to be determined when one and only one figure can result from a set of conditions.

Three points do not determine a straight line because a straight line cannot always be drawn through three points. One point does not determine a straight line, because more than one straight line can be drawn through that point.

EXERCISES

1. Does the equation $7x - 5 = 23$ determine the value of x? Why?
2. Does the equation $5 + x = x + 7$ determine the value of x? Why?

3. Does the equation $7x + 3x = 10x$ determine the value of x? Why?

4. (*a*) You need $2 made up of quarters and half-dollars. Is the number of coins determined? Why?
   (*b*) You need $2 made up of the same number of quarters as half-dollars. Is the number of coins determined? Why?
   (*c*) You need $1 made up of quarters and half-dollars. Is the number of coins determined? Why?

5. (*a*) If you are given the values of two numbers x and y, can you always find the sum $(x + y)$?
   (*b*) Can someone find a sum different from yours for the same two numbers?
   (*c*) What is the missing word in the following statement? Two numbers —?— a sum.

6. (*a*) If you are told that two lines pass through the same two points, what is your conclusion about the lines?
   (*b*) Complete the following statement: If two straight lines have more than one point in common, then . . . .

SUPERIOR WORK

S1. Draw two circles in a position to show that two points do not determine a circle.

S2. You are given a quarter and asked to buy some 3-cent and some 5-cent stamps. Is the number of stamps that you are to purchase determined?

S3. (*a*) Does the value of x determine the value of x^2?

(*b*) Does the value of x^2 determine the value of x?

S4. (*a*) If you are given any two numbers x and y, can you always find their difference $(x - y)$? Why?

(*b*) If you are given any two numbers x and y, can you always find the quotient $\frac{x}{y}$? Could you do it if you knew nothing about fractions? Could you do it if $x = 0$? or if $y = 0$? or if both x and $y = 0$?

S5. (*a*) What is meant by saying that two lines determine a point of intersection?

(*b*) Does one line determine a plane?

(*c*) Do two lines determine a plane?

(*d*) How many points determine a plane?

(*e*) Read and explain the following quotation: "I do not agree with your conclusion because it is based on a false premise." If you do not know the meaning of the word *premise*, look it up in a dictionary.

HONOR WORK

H1. Is it sometimes necessary at the outset of a debate for debaters to accept without proof certain propositions? Suppose that you are about to debate the proposition that eighteen-year-old citizens should be allowed to vote. What propositions might you and your opponents accept as axioms?

H2. In his Gettysburg Address, Lincoln said that our nation is "dedicated to the proposition that all men are created equal." Find something in the United States Constitution that is based on this proposition.

H3. Point out how some things that you learn in school now will influence or determine the kind of government that we shall have in the future.

The Addition Axiom

Just as two points in space determine a straight line, so do two numbers determine a sum. If we know that two lines are both determined by the same two points, we can be sure that the two lines are the same. In like manner, if two numbers determine a sum, then two numbers that are equal to these numbers will determine the same sum. This is called the addition axiom and is expressed as follows.

13. Addition Axiom. If equals are added to equals, the sums are equal.

In geometry the addition axiom is used in much the same way as in arithmetic and in algebra, but it is applied to lines and angles as well as to numbers and letters.

Do you remember how the addition axiom is used in algebra in solving an equation like $2x - 5 = 7$? The reasoning is as follows:

If equals $2x - 5 = 7$
are added to equals, $5 = 5$
then the sums are equal. $2x = 12$

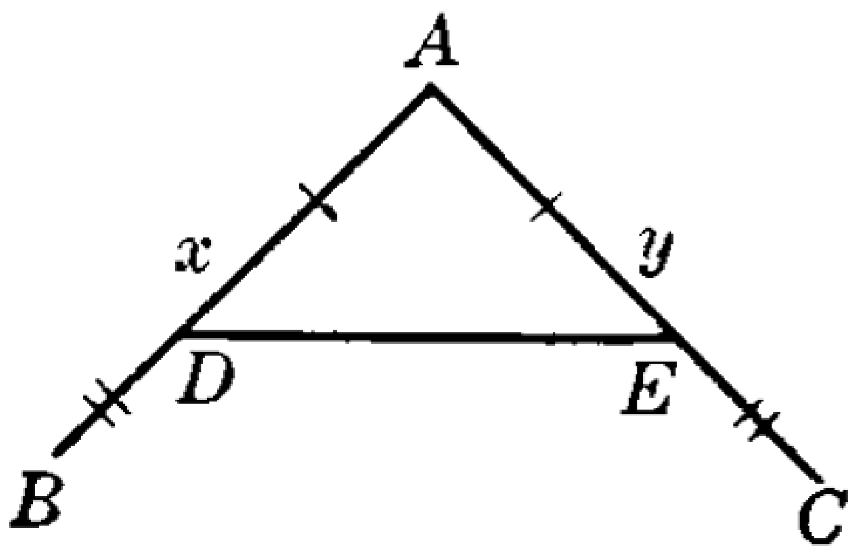
Again, in solving the simultaneous equations $x + y = 7$ and $x - y = 3$, the addition axiom is used. The reasoning is as follows:

If equals $x + y = 7$
are added to equals, $x - y = 3$
then the sums are equal. $2x = 10$

The examples that follow show how the addition axiom is applied in geometry. Notice that the axiom is referred to in shorthand form as Add. Ax.

The conditions that are *given* make up the hypothesis. What must be *proved* is the conclusion.

Illustrative Example



Given:
$AD = AE$
$DB = EC$
Prove:
$AB = AC$

PROOF

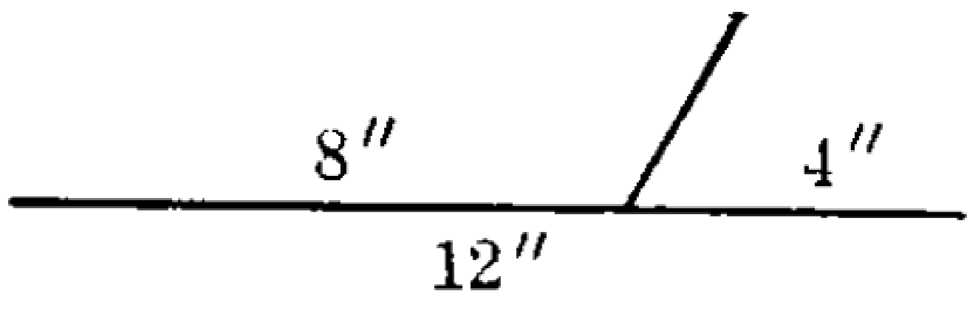
Statements	*Reasons*
1. $AD = AE$ $DB = EC$	1. Given.
2. $AB = AC$	2. Add. Ax.

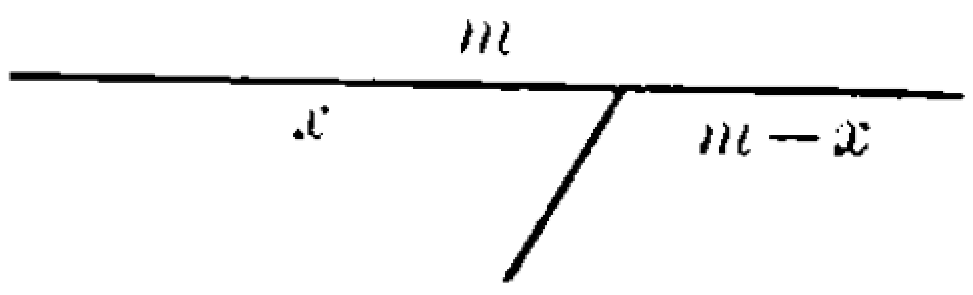
The figure has been marked to emphasize the given facts. Single marks placed on AD and AE show that they are given equal, and double marks on DB and EC show that they are given equal.

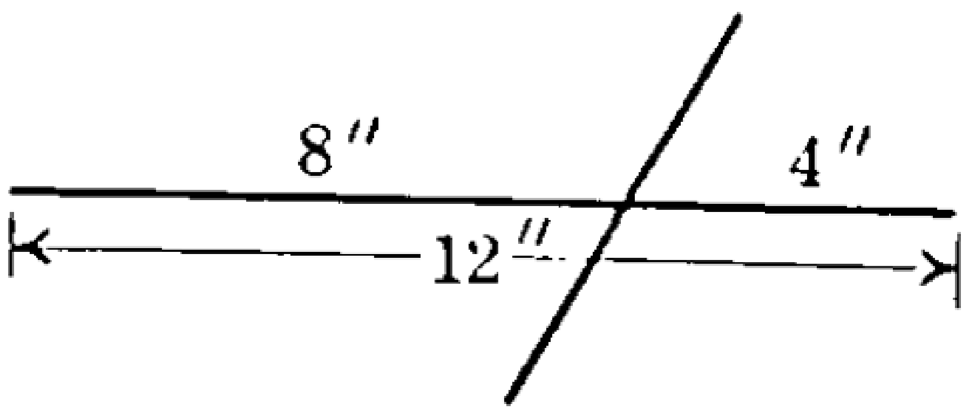
When the left-hand members AD and DB of the two given equations are added, we see *from the diagram* that their sum forms the line segment AB or x. In the same way, when the right-hand members AE and EC of the two given equations are added, their sum is AC or y. From the addition axiom we may conclude that these two sums AB and AC are equal. We have lettered the two line segments in the conclusion x and y. This follows the practice in algebra of using the last letters of the alphabet for unknown quantities.

To avoid mistakes and to show that you understand the use of the addition axiom, explain the last step of the proof by reciting the axiom orally and at the same time pointing out the parts in the figure to which the axiom refers, as follows: "If equals" (pause while you point out the equal line segments AD and AE) "are added to equals" (point out the equal line segments DB and EC), "then their sums" (point out the line segments AB and AC) "are equal."

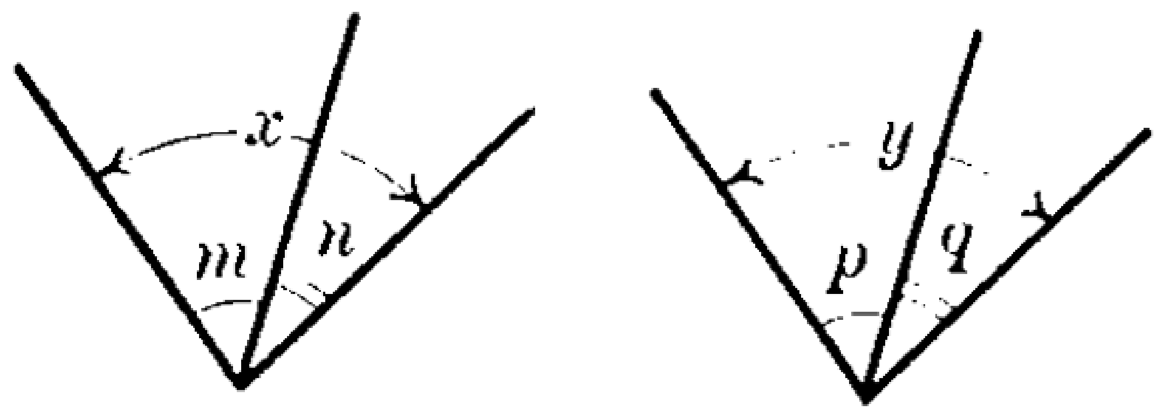
When a line segment is marked by only one letter, its end points should be noted carefully. The method we shall use is illustrated in the diagrams below. In case of doubt, a dimension line is used, as in the third figure.







Illustrative Example



Given:
$\angle m = \angle p$
$\angle n = \angle q$
Prove:
$\angle x = \angle y$

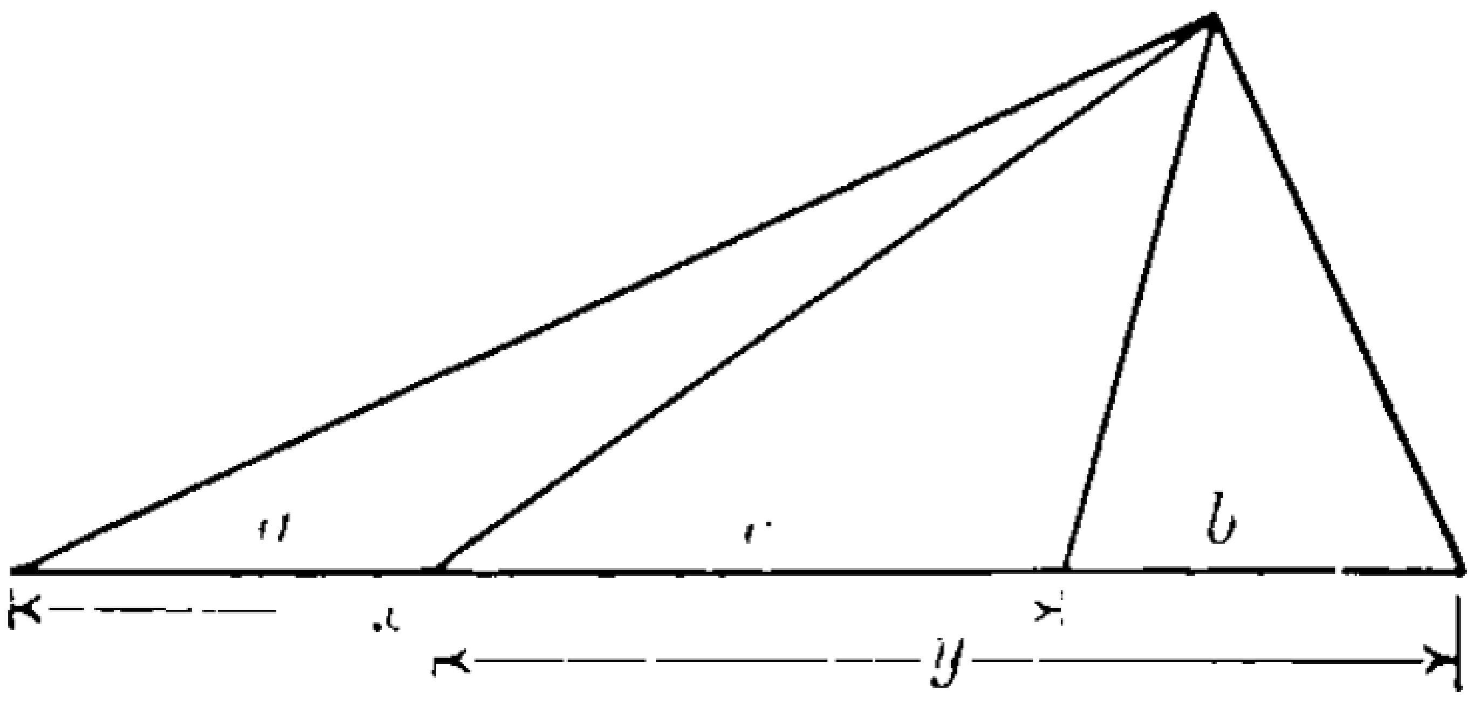
PROOF

Statements	*Reasons*
1. $\angle m = \angle p$ $\angle n = \angle q$	1. Given.
2. $\angle x = \angle y$	2. Add. Ax.

As you go over the proof, repeat the addition axiom and point out the angles to which the axiom refers.

Illustrative Example

Given: $a = b$
Prove: $x = y$



PROOF

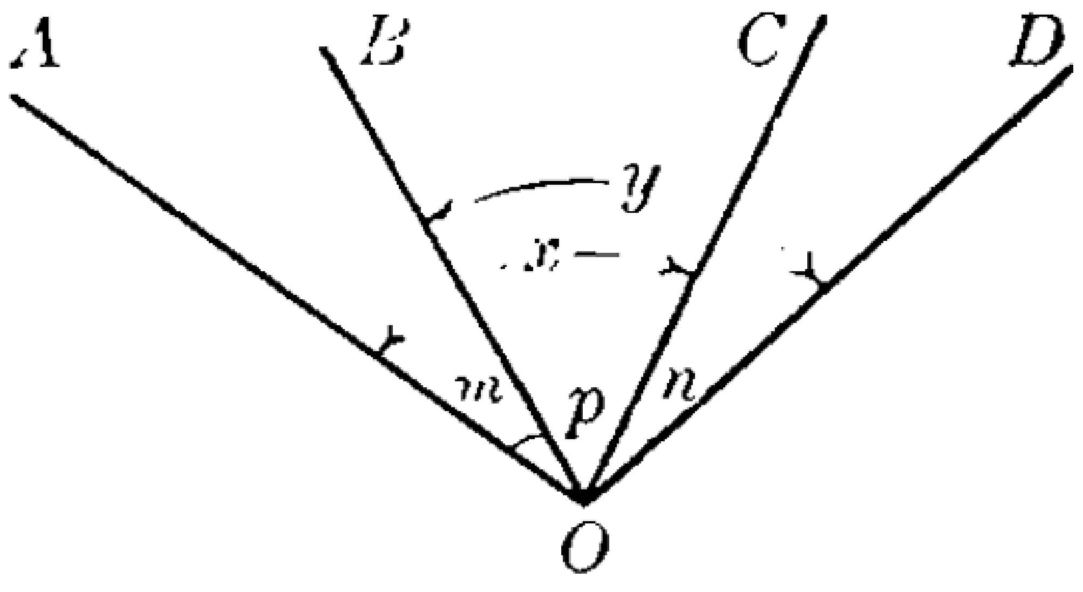
Statements	*Reasons*
1. $a = b$	1. Given.
2. $c = c$	2. Identity.
3. $x = y$	3. Add. Ax.

In the second statement use is made of the fact that a quantity is equal to itself. This special kind of equation is called an *identity.* The same quantity c is added to each side of the given equation.

As you go over the proof, repeat the addition axiom and point out the line segments to which the axiom refers.

Illustrative Example

Given: $\angle m = \angle n$
Prove: $\angle AOC = \angle BOD$



PROOF

Statements	*Reasons*
1. $\angle m = \angle n$	1. Given.
2. $\angle p = \angle p$	2. Identity.
3. $\angle AOC = \angle BOD$	3. Add. Ax.

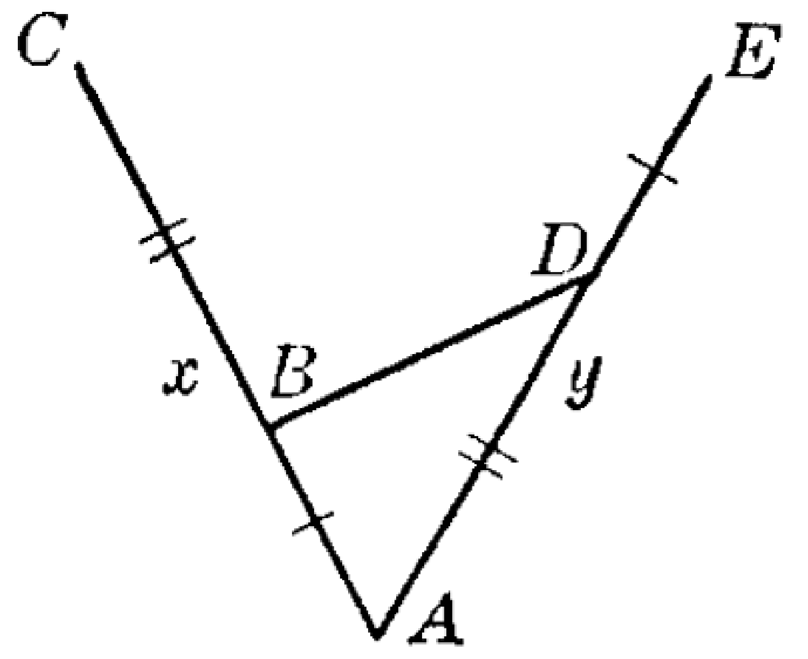
As you go over the proof, repeat the addition axiom and point out the angles.

Before deciding that a conclusion can be supported by the addition axiom, we must make sure that there are equations that can be added and their sums are the quantities that we must prove equal.

EXERCISES

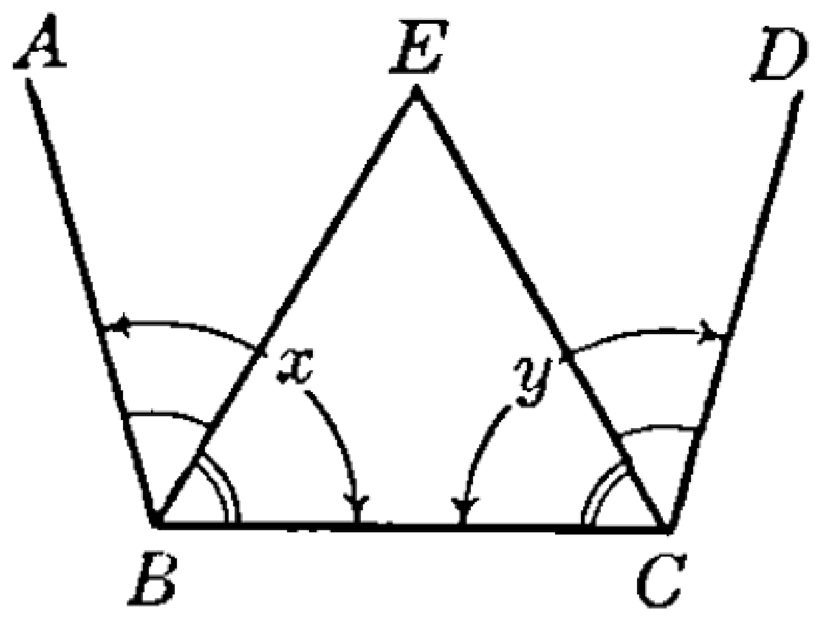
Write out the proof for each of the following exercises. Use the form given in the illustrative examples of the use of the addition axiom. Write down what is given and what is to be proved; then under "Proof" list statements and reasons in separate columns.

1.



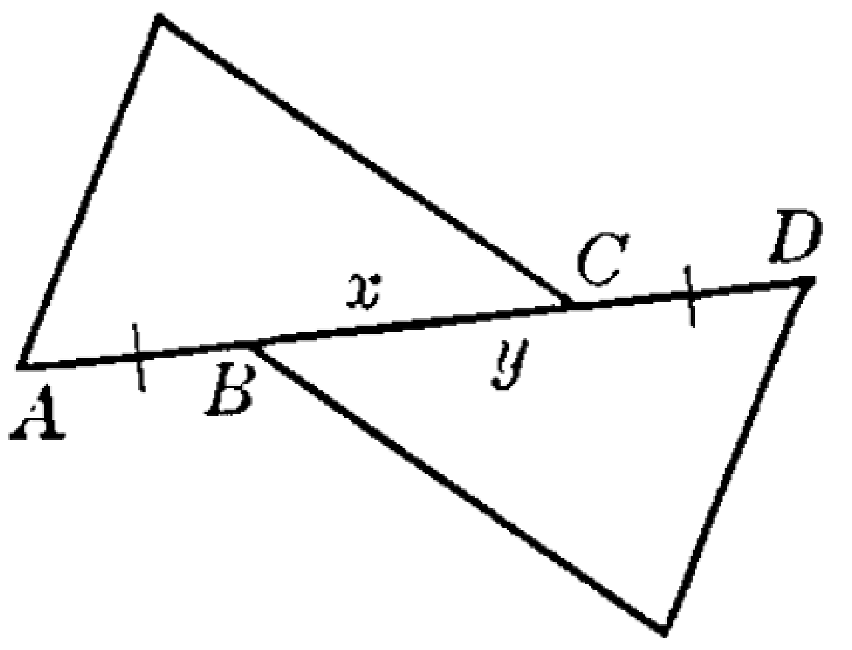
Given:
$AB = ED$
$BC = AD$
Prove:
$AC = AE$

2.



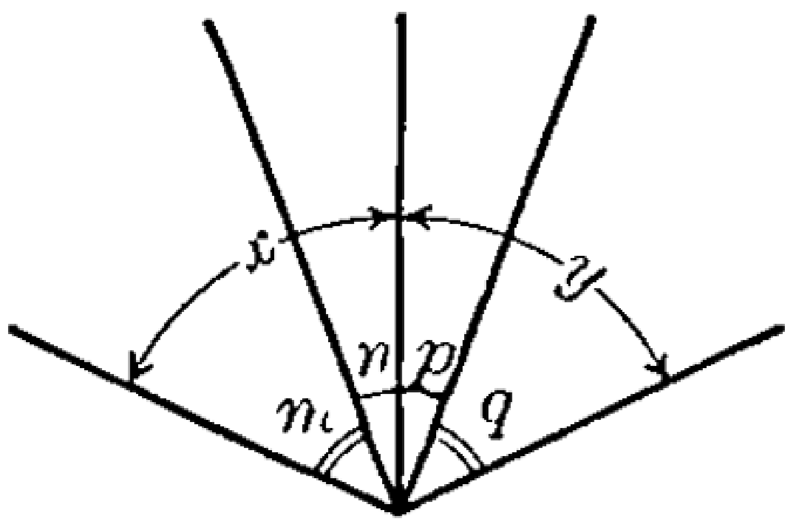
Given:
$\angle ABE = \angle DCE$
$\angle EBC = \angle ECB$
Prove:
$\angle ABC = \angle DCB$

3.



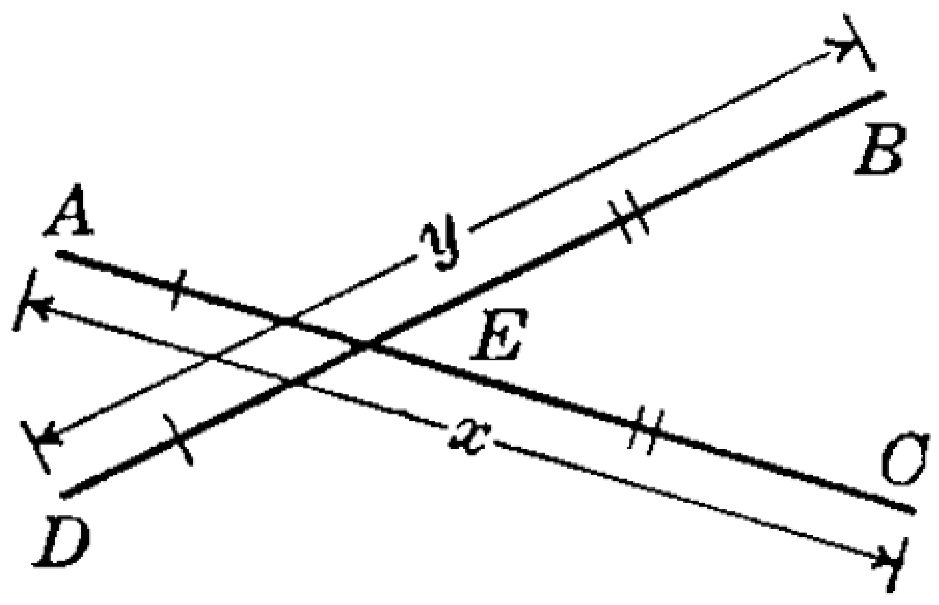
Given:
$AB = CD$
Prove:
$AC = BD$

4.



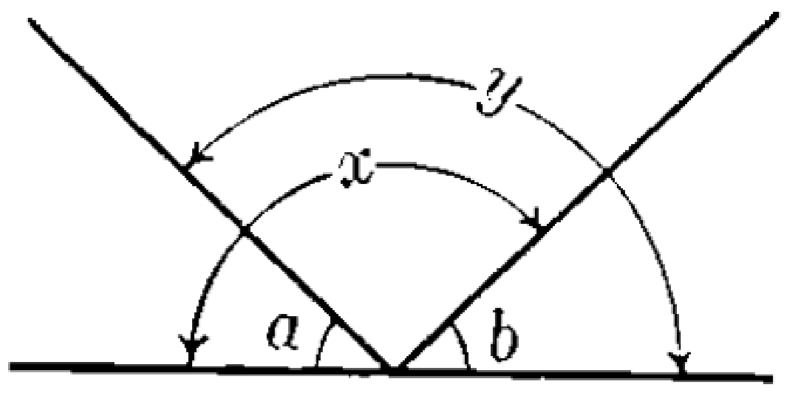
Given:
$\angle n = \angle p$
$\angle m = \angle q$
Prove:
$\angle x = \angle y$

5.



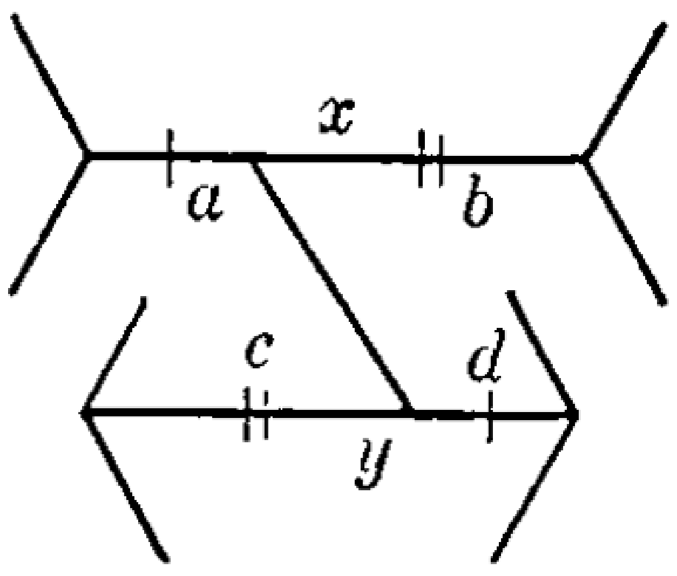
Given:
$AE = DE$
$EC = EB$
Prove:
$AC = DB$

6.



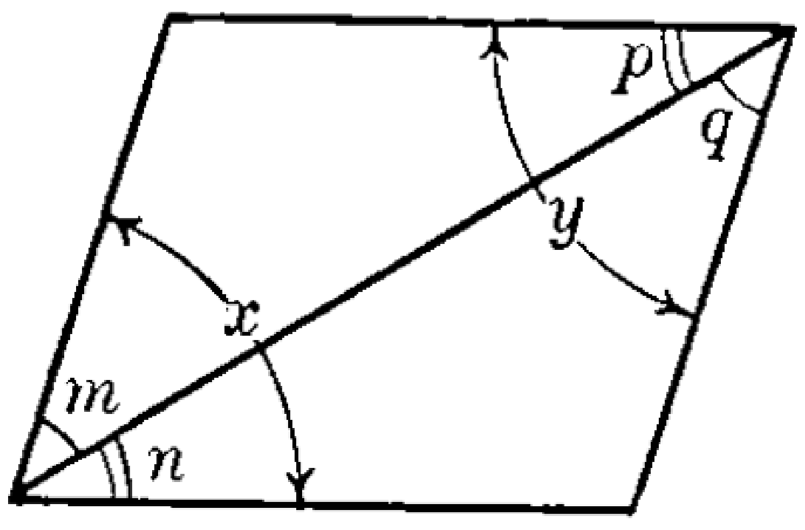
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

7.



Given:
$a = d$
$b = c$
Prove:
$x = y$

8.



Given:
$\angle m = \angle q$
$\angle n = \angle p$
Prove:
$\angle x = \angle y$

The Subtraction Axiom

In algebra we learned to use both positive and negative numbers. We learned how to subtract any two numbers and get a definite answer. This is just another way of saying that two numbers determine a difference, just as two points in space determine a straight line. As in the addition axiom, this means that if the members of one equation are subtracted from the members of another equation, the differences are equal.

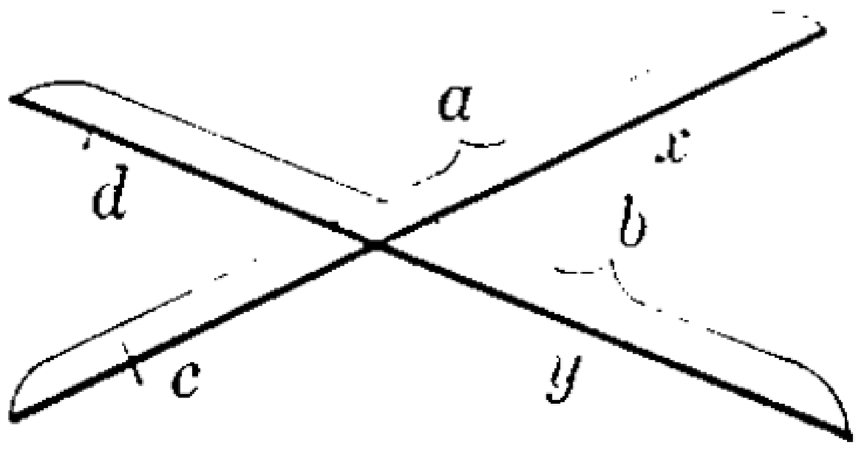
14. Subtraction Axiom. If equals are subtracted from equals, the remainders are equal.

In algebra the subtraction axiom is used in solving equations like $3x = x + 10$. The reasoning is as follows:

If from equals	$3x = x + 10$
we subtract equals,	$x = x$
then the remainders are equal.	$2x = 10$

The examples that follow show the application of the subtraction axiom to geometric reasoning. The axiom is referred to in shortened form as Subt. Ax.

Illustrative Example



Given:
$a = b$
$c = d$
Prove:
$x = y$

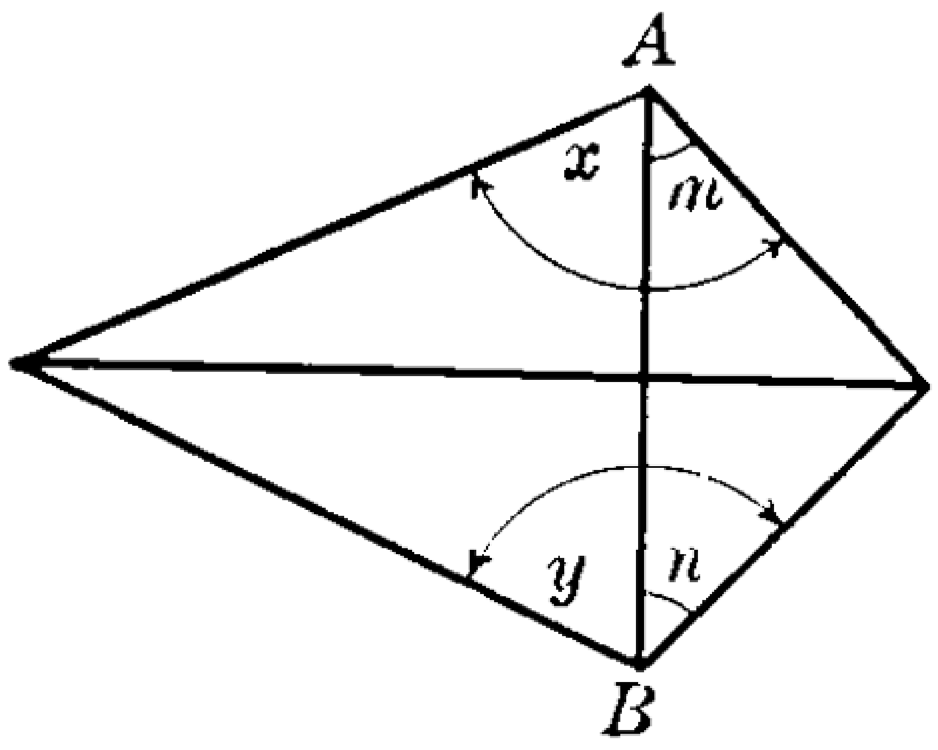
PROOF

Statements	*Reasons*
1. $a = b$ $c = d$	1. Given.
2. $x = y$	2. Subt. Ax.

When c is subtracted from a, the remainder is x; when d is subtracted from b, the remainder is y. From the subtraction axiom the conclusion is that $x = y$, because the quantities that were used in the subtraction were given equal.

As you go over the proof, repeat the subtraction axiom and point out the parts to which you refer, as follows: "If equals" (point out the equal line segments c and d) "are subtracted from equals" (point out the equal line segments a and b), "then the remainders" (point to the line segments x and y) "are equal."

Illustrative Example



Given:
$\angle A = \angle B$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

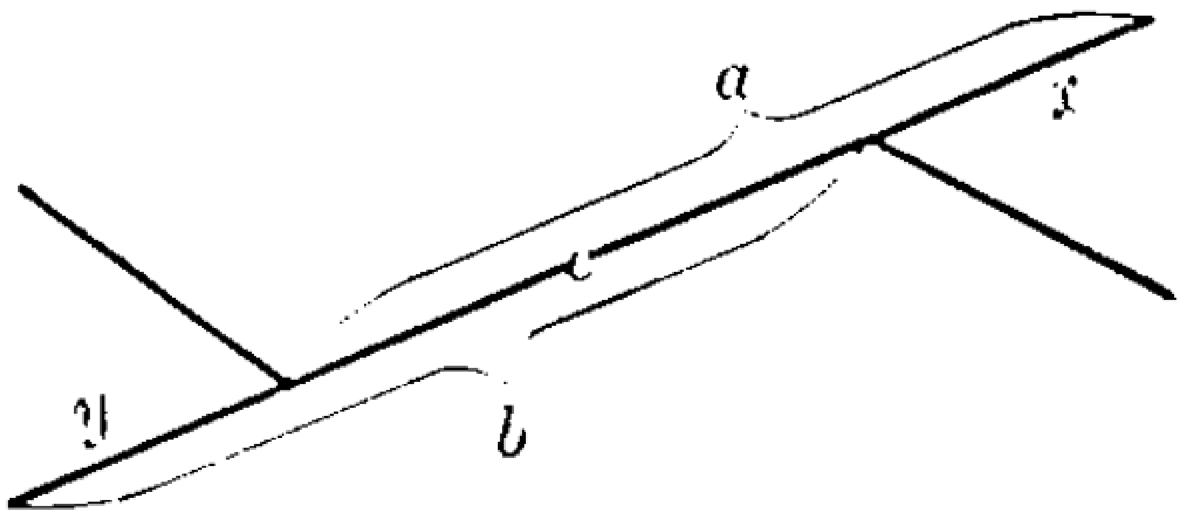
PROOF

Statements	*Reasons*
1. $\angle A = \angle B$ $\angle m = \angle n$	1. Given.
2. $\angle x = \angle y$	2. Subt. Ax.

As you go over the proof, repeat the subtraction axiom as follows: "If equals" (point out ∠m and ∠n) "are subtracted from equals" (point out the entire angles A and B), "then the remainders" (point to ∠x and ∠y) "are equal."

Illustrative Example

Given: $a = b$
Prove: $x = y$



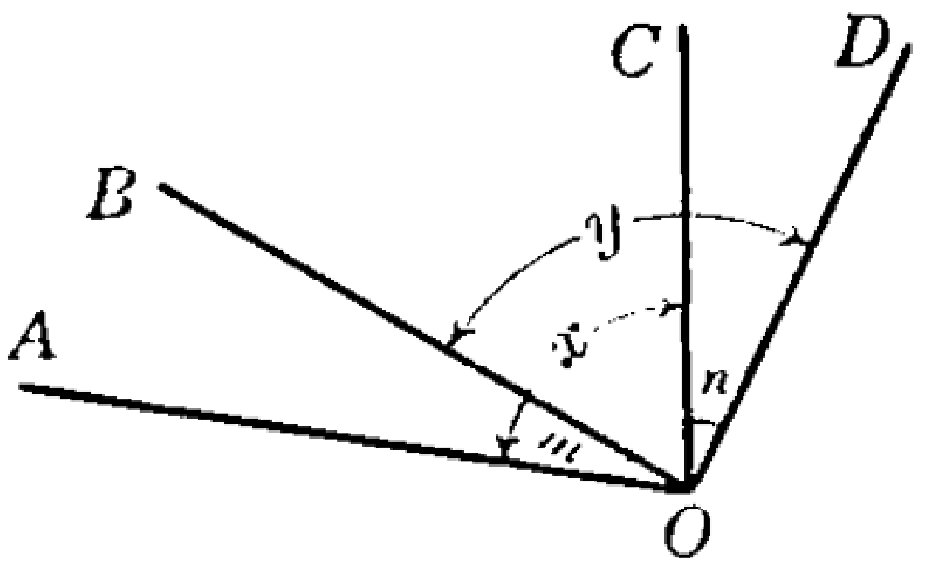
PROOF

Statements	*Reasons*
1. $a = b$	1. Given.
2. $c = c$	2. Identity.
3. $x = y$	3. Subt. Ax.

Notice that the identity is subtracted from the first equation and gives a new equation $x = y$.

Illustrative Example

Given: $\angle m = \angle n$
Prove: $\angle AOC = \angle BOD$



PROOF

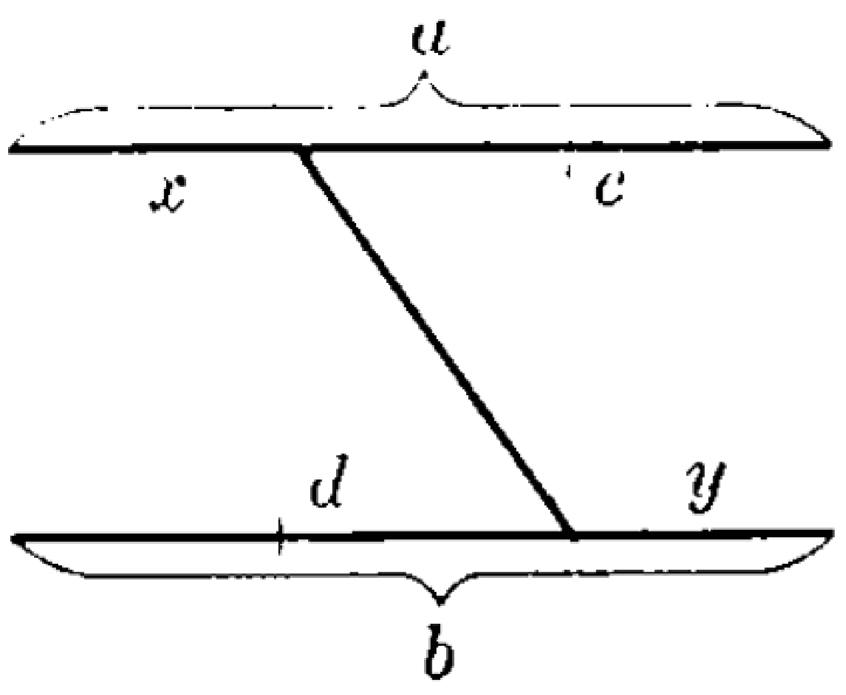
Statements	*Reasons*
1. $\angle AOD = \angle AOD$	1. Identity.
2. $\angle m = \angle n$	2. Given.
3. $\angle AOC = \angle BOD$	3. Subt. Ax.

Notice that this is the same proposition that was proved previously by the addition axiom.

EXERCISES

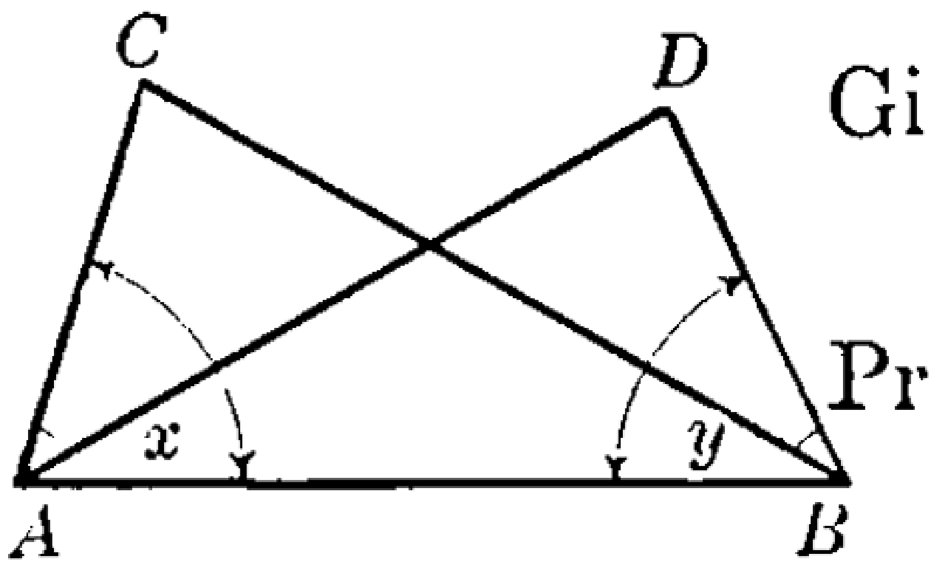
Write out the proof for each of the following exercises. Use the form given in the illustrative examples of the use of the subtraction axiom. Write down what is given and what is to be proved; then under "Proof" list statements and reasons.

1.



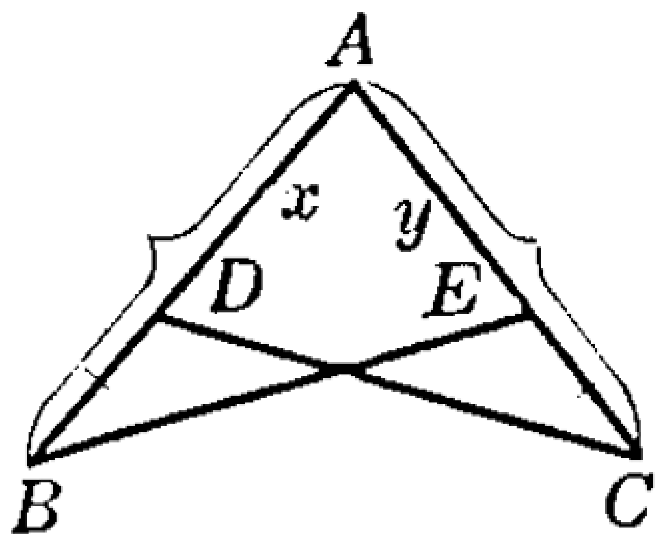
Given:
$a = b$
$c = d$
Prove:
$x = y$

2.



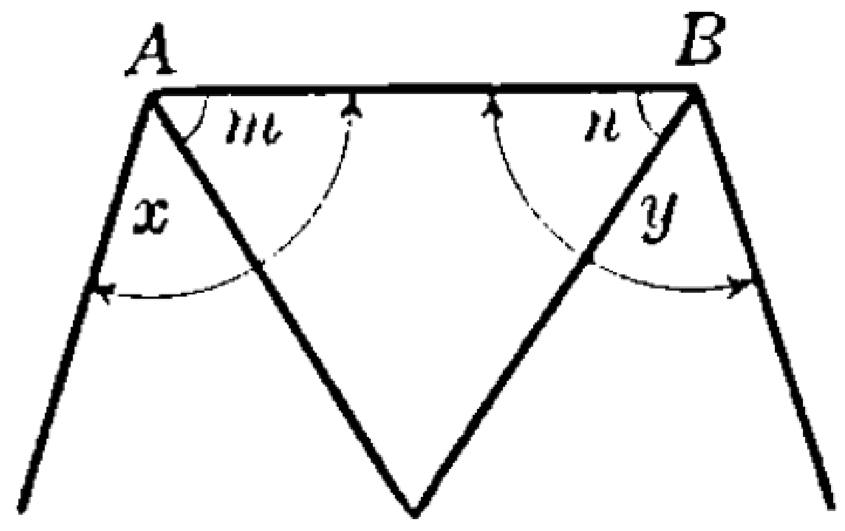
Given:
$\angle A = \angle B$
$\angle CAD = \angle DBC$
Prove:
$\angle x = \angle y$

3.



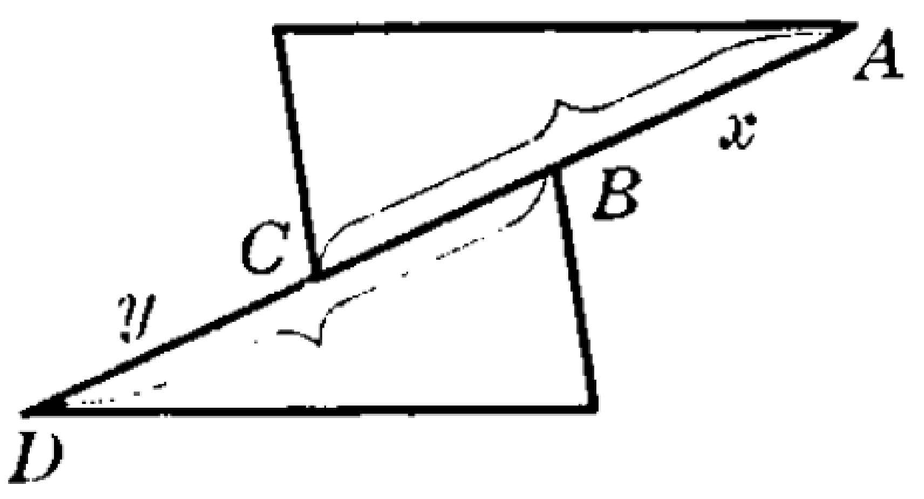
Given:
$AB = AC$
$DB = EC$
Prove:
$AD = AE$

4.



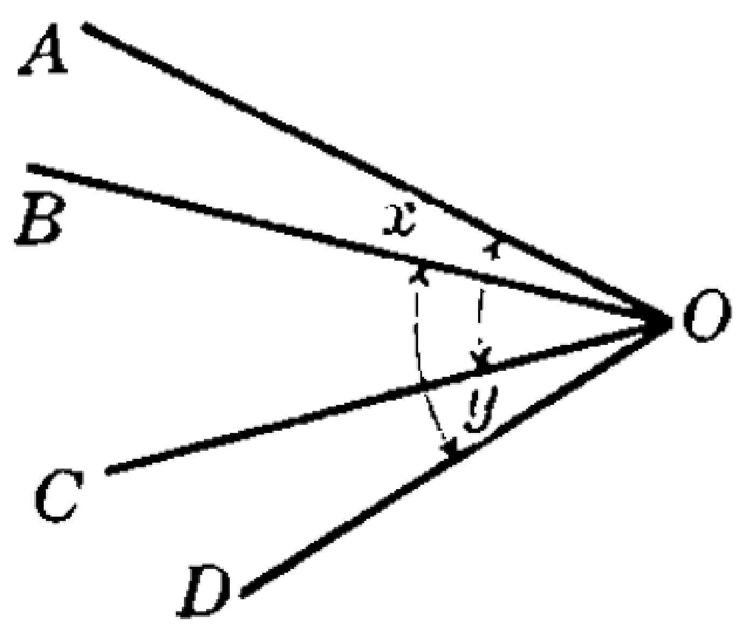
Given:
$\angle A = \angle B$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

5.



Given:
$AC = BD$
Prove:
$AB = CD$

6.



Given:
$\angle AOC = \angle BOD$
Prove:
$\angle x = \angle y$

7. Prove Ex. 5 again by using an identity different from that which you have already used.
8. Prove Ex. 6 again by using another identity.
9. Prove Ex. 3 in the previous set (page 12) using the subtraction axiom.
10. Prove Ex. 6 in the previous set (page 13) using the subtraction axiom.

The Multiplication Axiom

Since multiplication is repeated addition, we assume that if the members of one equation are multiplied by the members of another equation, the products are equal. This assumption is known as the *multiplication axiom.*

15. Multiplication Axiom. If equals are multiplied by equals, the products are equal.

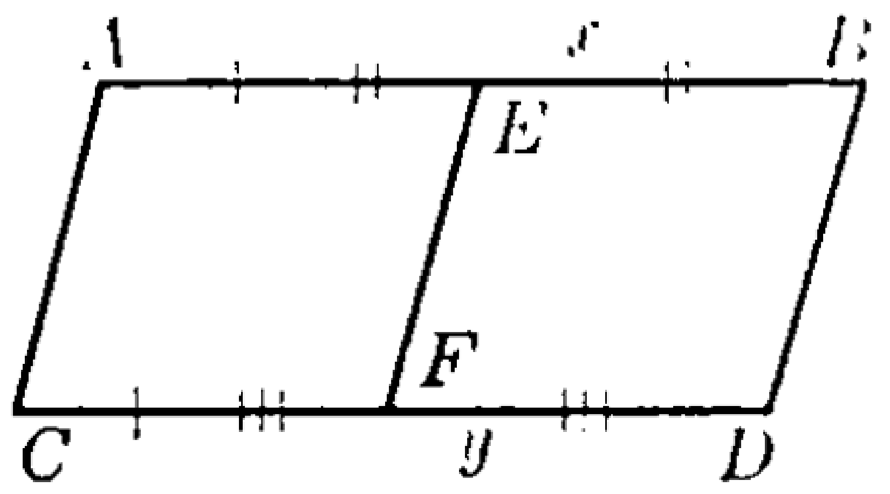
The multiplication axiom is used in solving equations like $\frac{2}{3} = \frac{4}{x}$. First the lowest common denominator $3x$ is found. Then the reasoning is as follows:

If equals	$\frac{2}{3} = \frac{4}{x}$
are multiplied by equals,	$3x = 3x$
then the products are equal.	$2x = 12$

The examples that follow show the application of the multiplication axiom to geometric reasoning. Notice that the axiom is referred to in abbreviated form as Mult. Ax.

When a quantity is made up of two equal parts, we shall use the principle that is illustrated in the following example: If two girls contribute equally to buy a book, then the cost of the book is twice the amount either girl contributes; and each girl owns half the book.

Illustrative Example



Given:
$AE = CF$
$AE = EB$
$CF = FD$
Prove:
$AB = CD$

PROOF

Statements	*Reasons*
1. $AE = CF$	1. Given.
2. $AB = 2 \cdot AE$ $CD = 2 \cdot CF$	2. Given.
3. $AB = CD$	3. Mult. Ax.

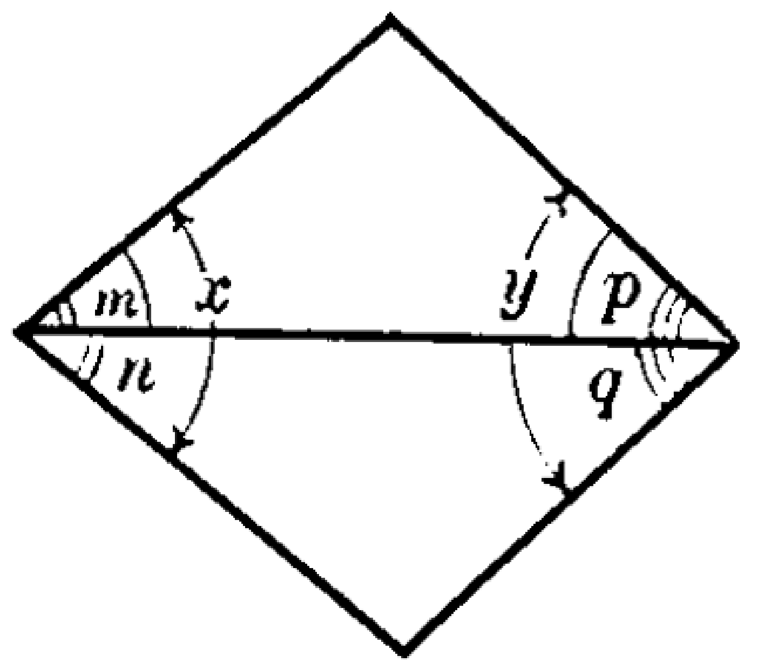
The statement that $AE = EB$ means that $AB = 2 \cdot AE$. The statement that $CF = FD$ means that $CD = 2 \cdot CF$. If we multiply the members of the first equation by 2, we shall get AB and CD. Finally, the multiplication axiom is the reason why the two products AB and CD are equal. When the common multiplier is 2, the multiplication axiom is usually written in this form:

Doubles of equals are equal.

As you see, this is a special case of the multiplication axiom.

In reciting the proof, explain the reason for the last step as follows: "If equals" (point out the two equal line segments AE and CF) "are doubled, then the doubles" (point out AB and CD) "are equal."

Illustrative Example



Given:
$\angle m = \angle p$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $\angle m = \angle p$	1. Given.
2. $\angle x = 2 \cdot \angle m$ $\angle y = 2 \cdot \angle p$	2. Given.
3. $\angle x = \angle y$	3. Mult. Ax.

The Division Axiom

With one exception we can always divide any two numbers and get a definite quotient. The exception is when the divisor is zero.

To explain the idea that division by zero does not determine a quotient, let us ask, "What is 5 divided by 0?" The answer to this question must be a number that will give us 5 when we multiply it by 0. But there is no such number, because every number multiplied by 0 gives 0 as the product. Again, let us ask, "What is 0 divided by 0?" Here we are looking for a number that multiplied by 0 will give us 0. Any number will do that. It is evident, then, that if we attempt to divide by zero we get no answer at all or any answer we please. In either case, a definite answer does not result if we try to divide by zero. Our conclusion therefore is that division by zero is a problem with no meaning.

If the dividend alone is zero, the quotient is determined. Thus, $\frac{0}{5} = 0$.

16. Division Axiom. If equals are divided by equals (except 0), the quotients are equal.

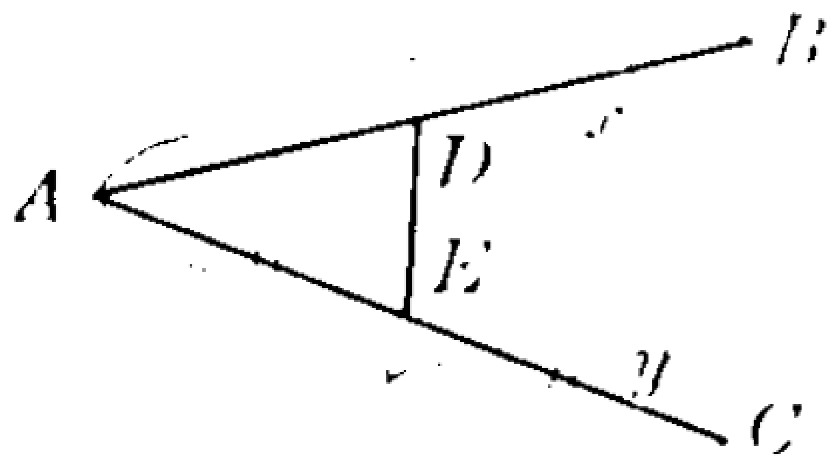
In algebra the division axiom has already been used in solving equations like $3x = 15$. The reasoning is as follows:

If equals	$3x = 15$
are divided by equals,	$3 = 3$
then the quotients are equal.	$x = 5$

The following examples show that in geometric reasoning the division axiom is used in the same way. The axiom is referred to in abbreviated form as Div. Ax.

Again, we shall use the principle illustrated on page 15 by the two girls who are equal owners of a book.

Illustrative Example



Given:
$AB = AC$
$AD = DB$
$AE = EC$
Prove:
$DB = EC$

PROOF

Statements	*Reasons*
1. $AB = AC$	1. Given.
2. $DB = AB \div 2$ $EC = AC \div 2$	2. Given.
3. $DB = EC$	3. Div. Ax.

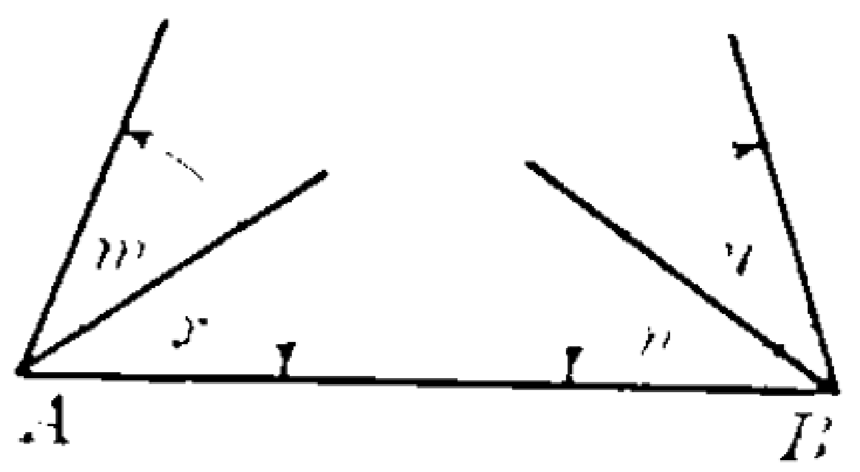
From the figure it can be seen that the given equation $AD = DB$ can be stated in the form that DB is one-half of AB, or $DB = AB \div 2$. Each member of the first equation is divided by 2 to get the line segments DB and EC. From the division axiom the conclusion is that the two quotients DB and EC are equal.

In the special case where the common divisor is 2, the division axiom may be given as follows:

Halves of equals are equal.

Explain the reason for the last step as follows: "If equals" (point out the equal line segments AB and AC) "are divided into halves, then the halves" (point out DB and EC) "are equal."

Illustrative Example



Given:
$\angle A = \angle B$
$\angle x = \angle m$
$\angle y = \angle n$
Prove:
$\angle x = \angle y$

PROOF

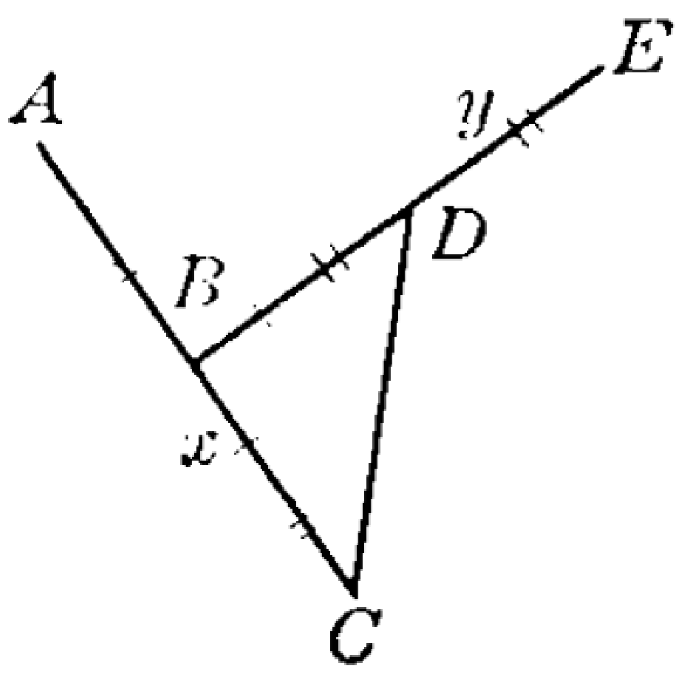
Statements	*Reasons*
1. $\angle A = \angle B$	1. Given.
2. $\angle x = \frac{1}{2}\angle A$ $\angle y = \frac{1}{2}\angle B$	2. Given
3. $\angle x = \angle y$	3. Div. Ax.

Sometimes pupils try to use the subtraction axiom to prove this proposition. Show that the conclusion $\angle x = \angle y$ cannot be obtained by subtracting the equations that are given in this example.

EXERCISES

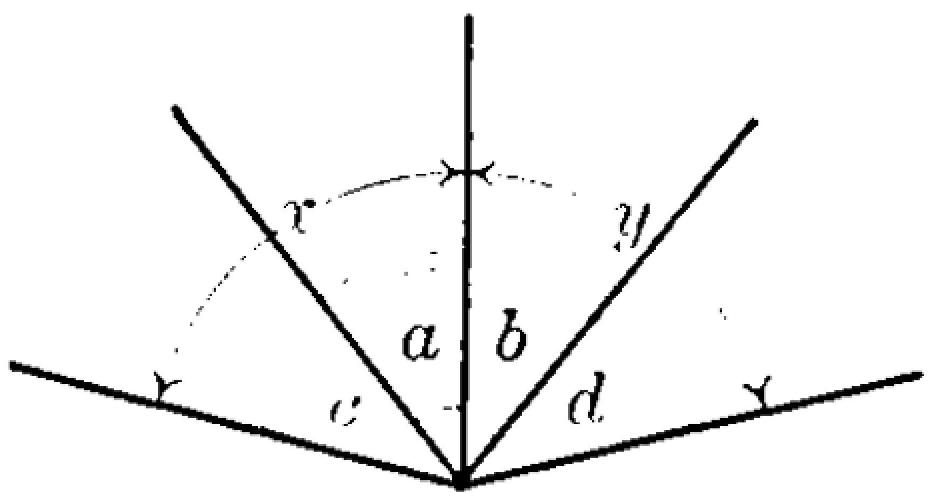
Write out the proof for each of the following exercises. Use the form given in the illustrative examples.

1.



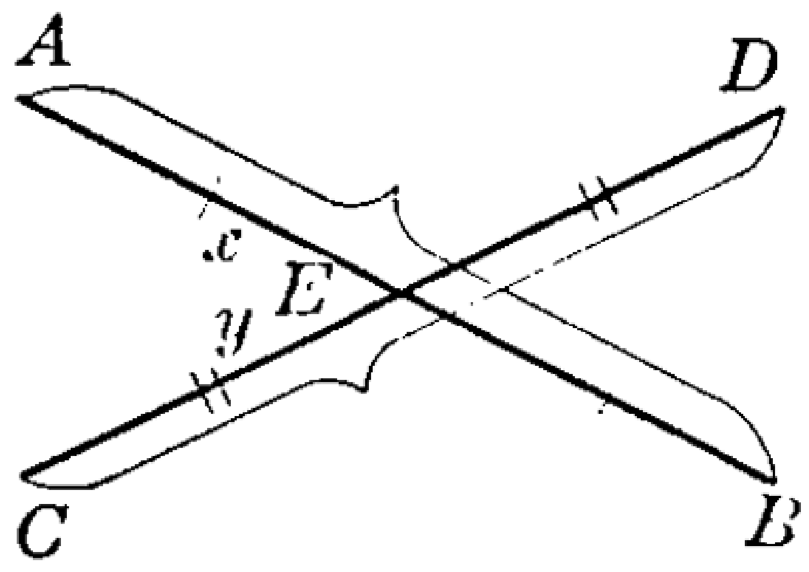
Given:
$BC = BD$
$AB = BC$
$BD = DE$
Prove:
$AC = BE$

2.



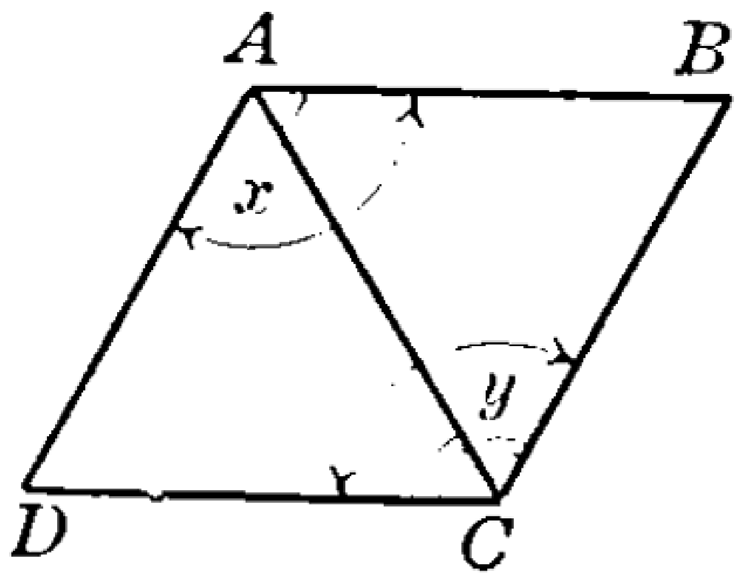
Given:
$\angle a = \angle b$
$\angle a = \angle c$
$\angle b = \angle d$
Prove:
$\angle x = \angle y$

3.



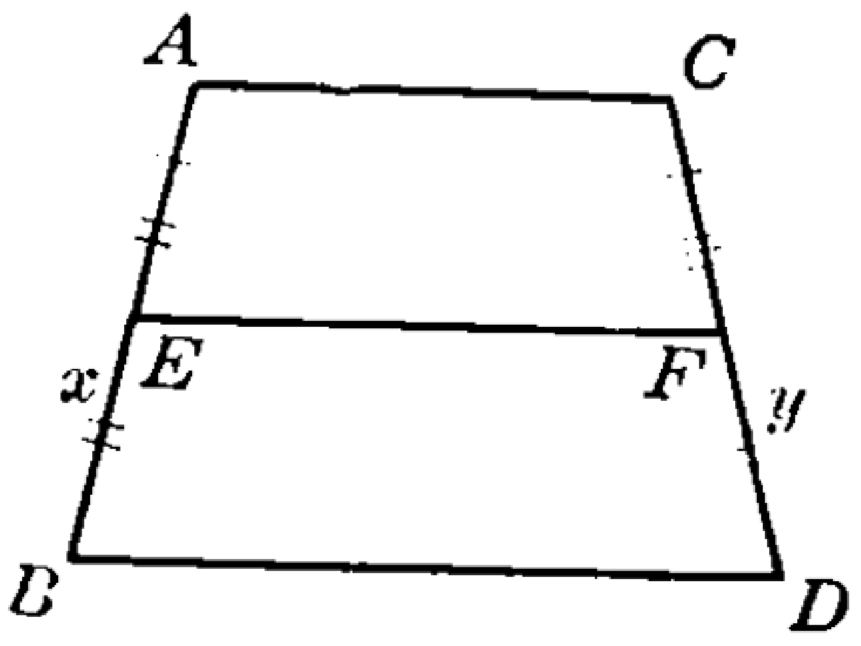
Given:
$AB = CD$
$AE = EB$
$CE = ED$
Prove:
$AE = CE$

4.



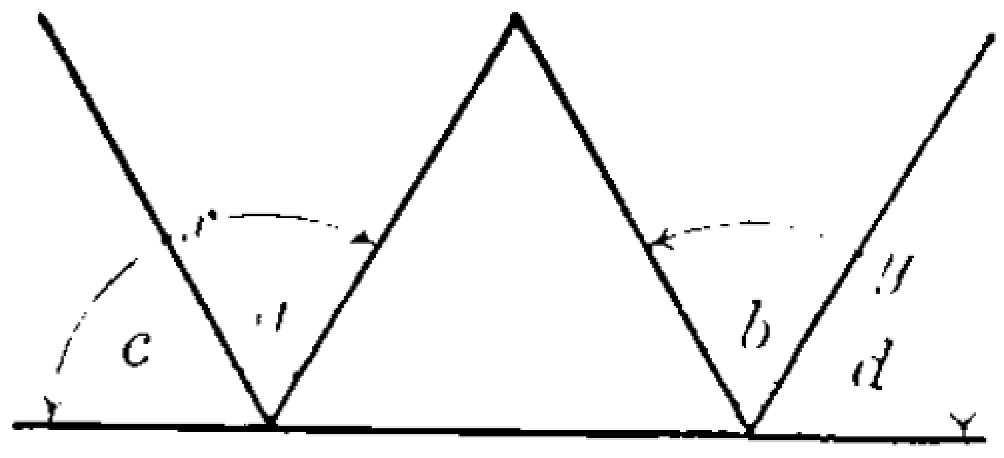
Given:
$\angle A = \angle C$
$\angle DAC = \angle CAB$
$\angle BCA = \angle ACD$
Prove:
$\angle DAC = \angle BCA$

5.



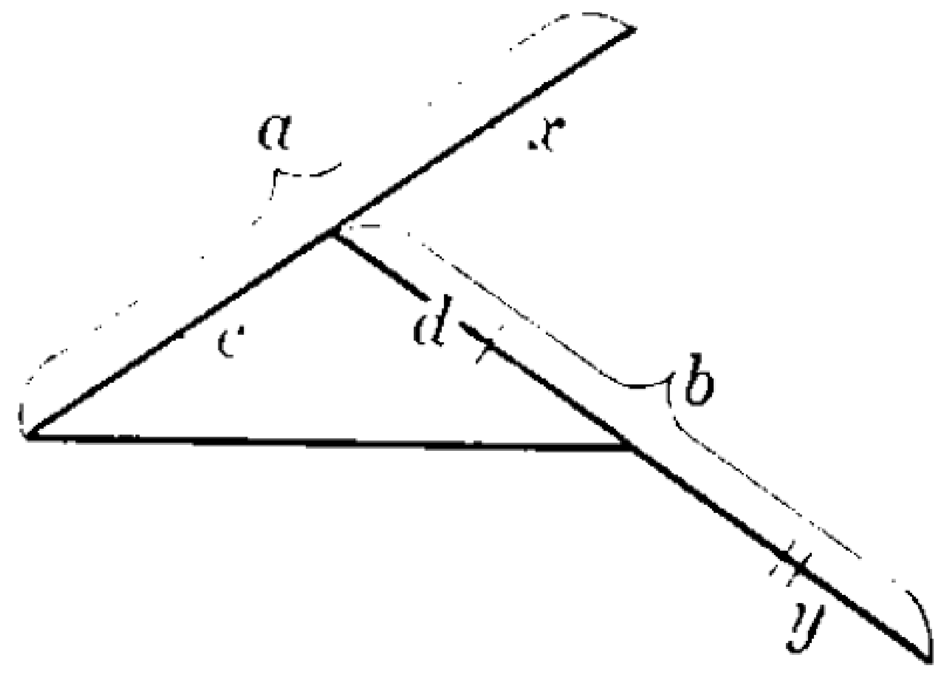
Given:
$AE = CF$
$AE = EB$
$CF = FD$
Prove:
$AB = CD$

6.



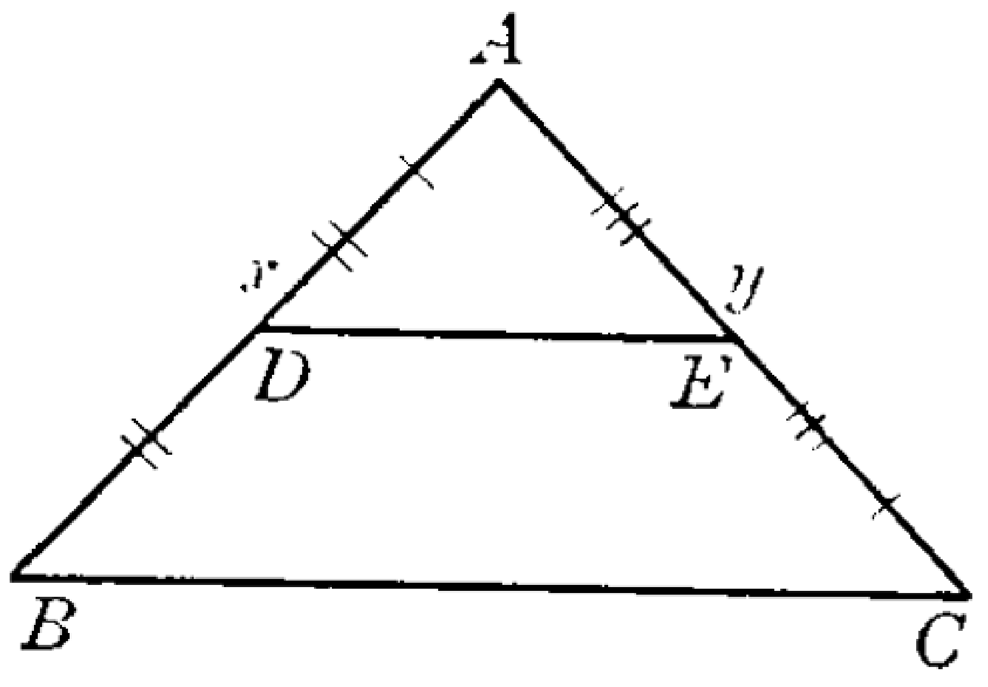
Given:
$\angle a = \angle b$
$\angle a = \angle c$
$\angle b = \angle d$
Prove:
$\angle x = \angle y$

7.



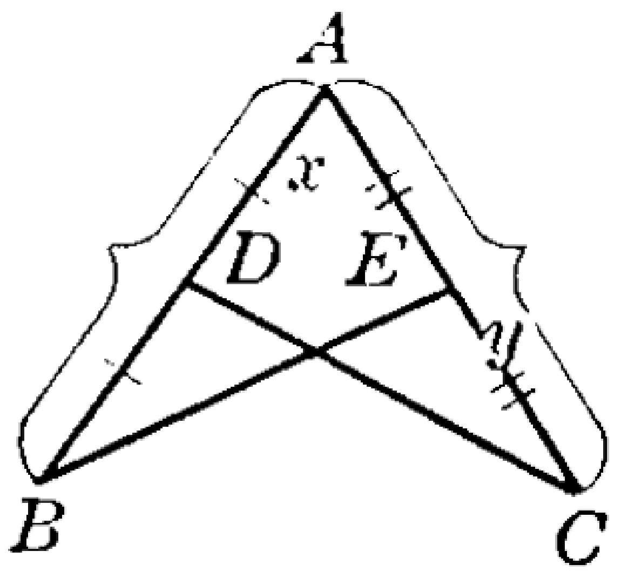
Given:
$a = b$
$x = c$
$y = d$
Prove:
$x = y$

8.



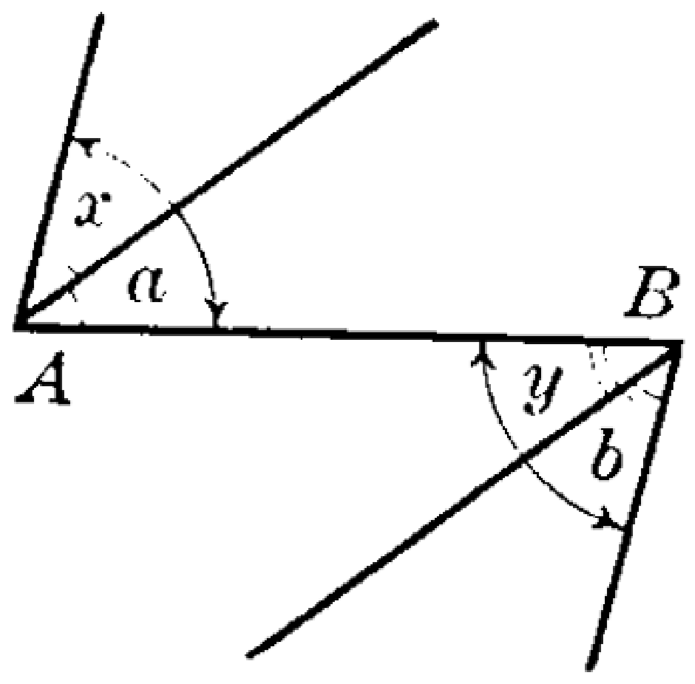
Given:
$AD = CE$
$AD = DB$
$CE = EA$
Prove:
$AB = AC$

9.



Given:
$AB = AC$
$AD = DB$
$AE = EC$
Prove:
$AD = EC$

10.



Given:
$\angle A = \angle B$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

The Substitution Axiom

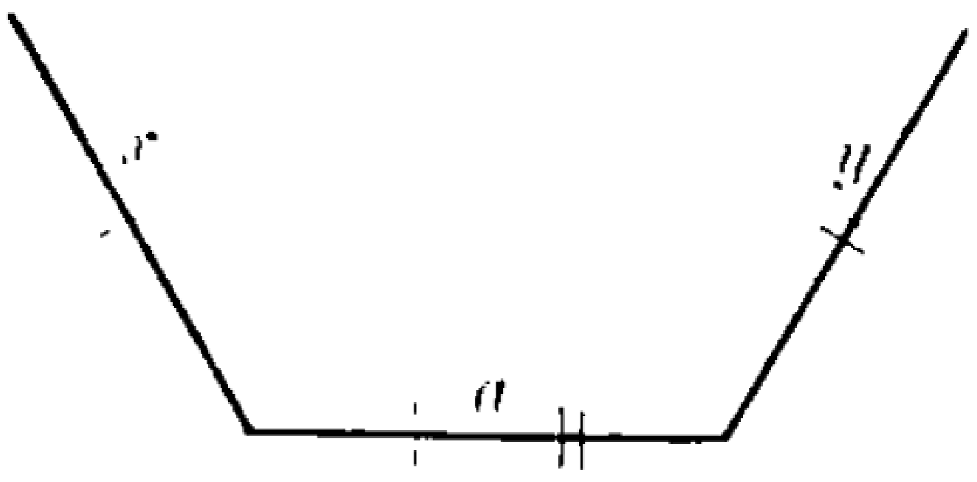
In algebra if $x = 3$, the numerical value of y in the equation $x + y = 10$ is easily determined by substituting 3 for x

in the equation. From the new equation $3 + y = 10$, y is easily found. The assumption that a quantity may be substituted for its equal without changing the value of the expression is called the *substitution axiom.*

17. Substitution Axiom. A quantity may be substituted for its equal in an expression without changing the value of the expression.

The application of the substitution axiom in geometry is shown in the following examples. The axiom is referred to in abbreviated form as Subs. Ax.

Illustrative Example



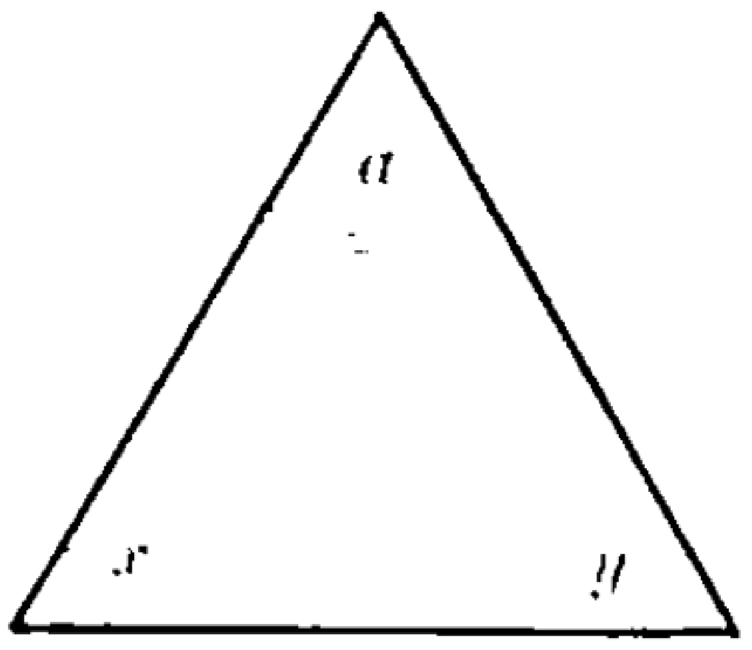
Given:
$x = a$
$y = a$
Prove:
$x = y$

PROOF

Statements	*Reasons*
1. $x = a$ $y = a$	1. Given.
2. $x = \overset{y}{(a)}$	2. Subs. Ax.

In the first equation, y is substituted for its equal a; x could also have replaced its equal a in the second equation.

Illustrative Example

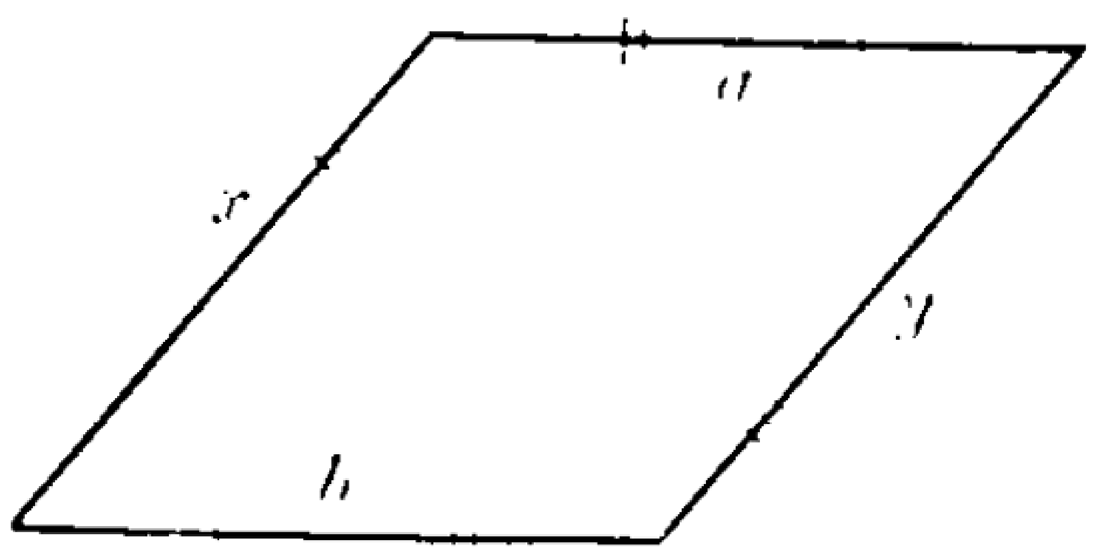


Given:
$\angle x = \angle a$
$\angle y = \angle a$
Prove:
$\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $\angle x = \angle a$ $\angle y = \angle a$	1. Given.
2. $\angle x = \overset{\angle y}{(\angle a)}$	2. Subs. Ax.

Illustrative Example



Given:
$a = b$
$x = a$
$y = b$
Prove:
$x = y$

PROOF

Statements	*Reasons*
1. $a = b$ $x = a$ $y = b$	1. Given.
2. $\overset{x}{(a)} = \overset{y}{(b)}$	2. Subs. Ax.

Substitution occurs twice in the first equation, where x replaces a and y replaces b.

We could have substituted in the second equation. Thus, $x = \overset{\overset{y}{(b)}}{(a)}$, where b would first replace a and then y would replace b. How would the substitution be made in the third equation?

After you have had enough practice with this axiom, you may simply write $x = y$, and explain orally how you substitute.

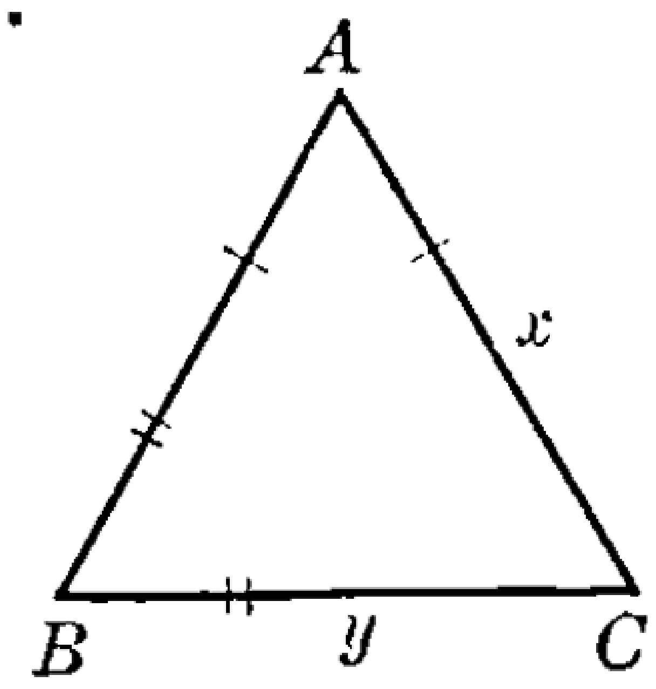
To make sure that we are using the substitution axiom correctly, we must be prepared to do three things: (1) point to a known equation (an equation that is given or one that has already been proved) in which we are substituting, (2) tell which quantities replace each other; and (3) point out known equations telling that the quantities replacing each other are equal.

For a simple substitution two equations are needed: one equation in which to substitute, and another to tell that the things that replace each other are equal. For a double substitution, three equations are needed: one equation in which to substitute, and two equations to tell that the terms that change places are equal.

EXERCISES

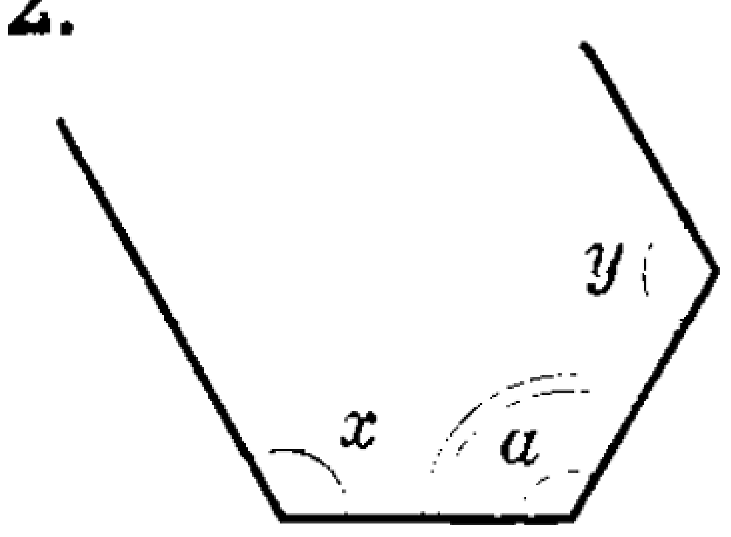
Write out the proof for each of the following. Use the form given in the illustrative examples.

1.



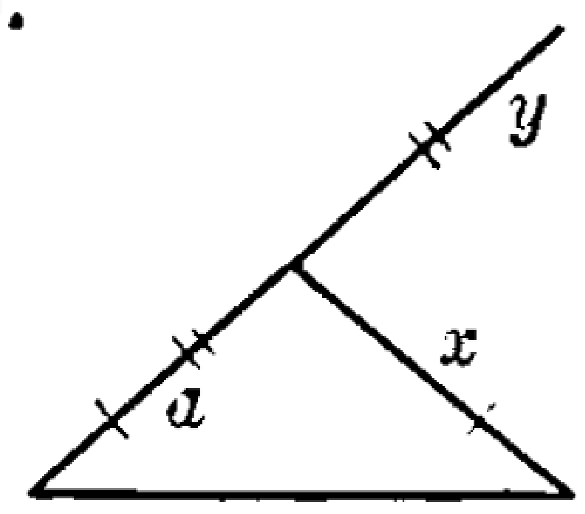
Given:
$AC = AB$
$BC = AB$
Prove:
$AC = BC$

2.



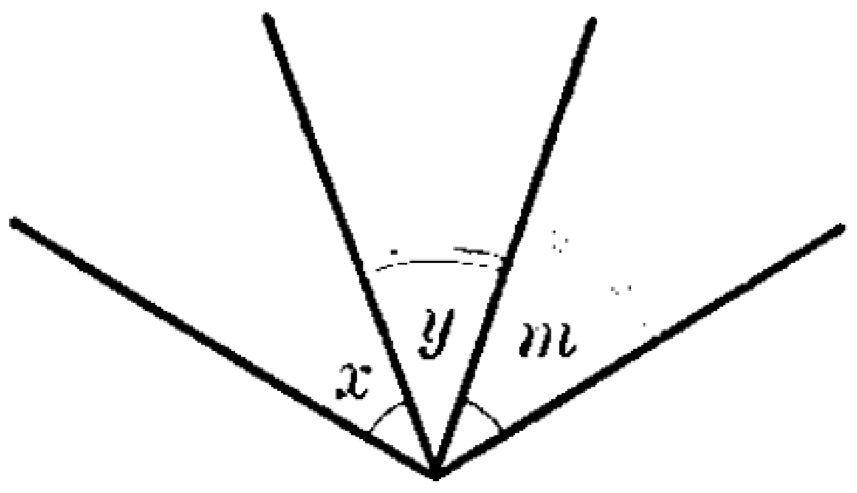
Given:
$\angle x = \angle a$
$\angle y = \angle a$
Prove:
$\angle x = \angle y$

3.



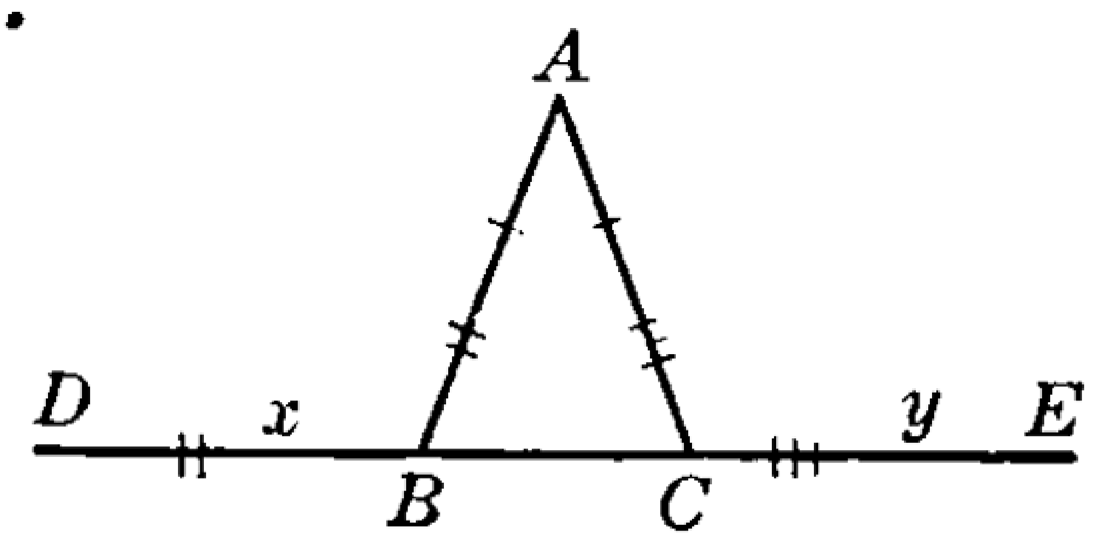
Given:
$x = a$
$y = a$
Prove:
$x = y$

4.



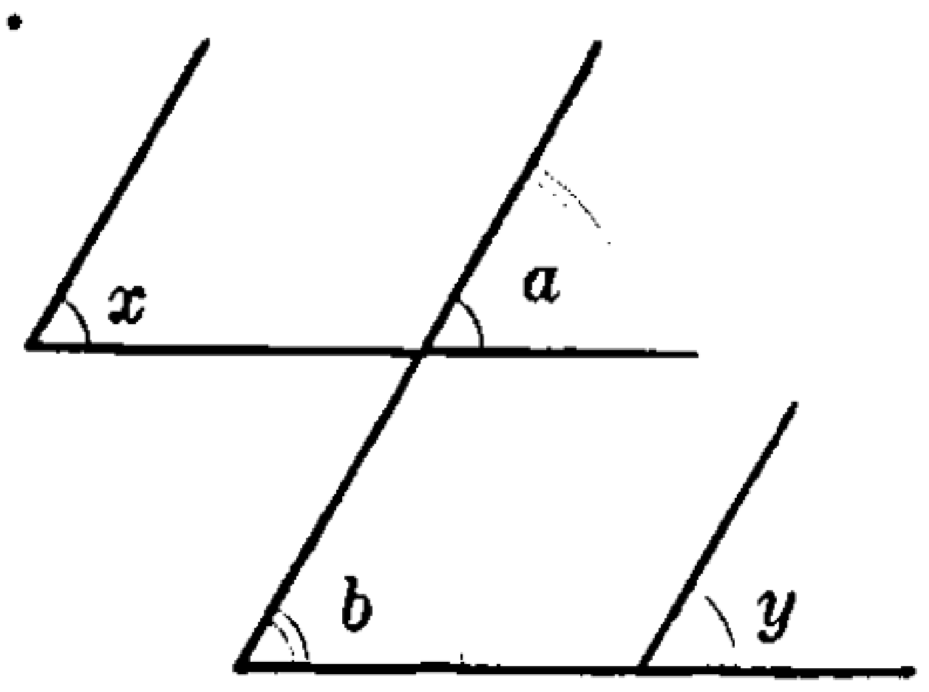
Given:
$\angle x = \angle m$
$\angle y = \angle m$
Prove:
$\angle x = \angle y$

5.



Given:
$AB = AC$
$DB = AB$
$CE = CA$
Prove:
$DB = CE$

6.



Given:
$\angle x = \angle a$
$\angle a = \angle b$
$\angle b = \angle y$
Prove:
$\angle x = \angle y$

SUPERIOR WORK

S1. Solve the equation $\frac{x}{9} + \frac{5}{2} = \frac{x}{4}$, pointing out all the axioms you use in the solution.

S2. Solve the following simultaneous equations for x and y, pointing out all the axioms that you use.

$$2x + 3y = 15$$
$$5x - 4y = 26$$

S3. (*a*) Answer the following questions:

1. If A is a brother of B, and B is a brother of C, then is A a brother of C?

2. If a is greater than b, and b is greater than c, then is a greater than c?

3. If a is included in b, and b is included in c, then is a included in c?

4. If A is far from B, and B is far from C, then is A far from C?

5. If A is different from B, and B is different from C, then is A different from C?

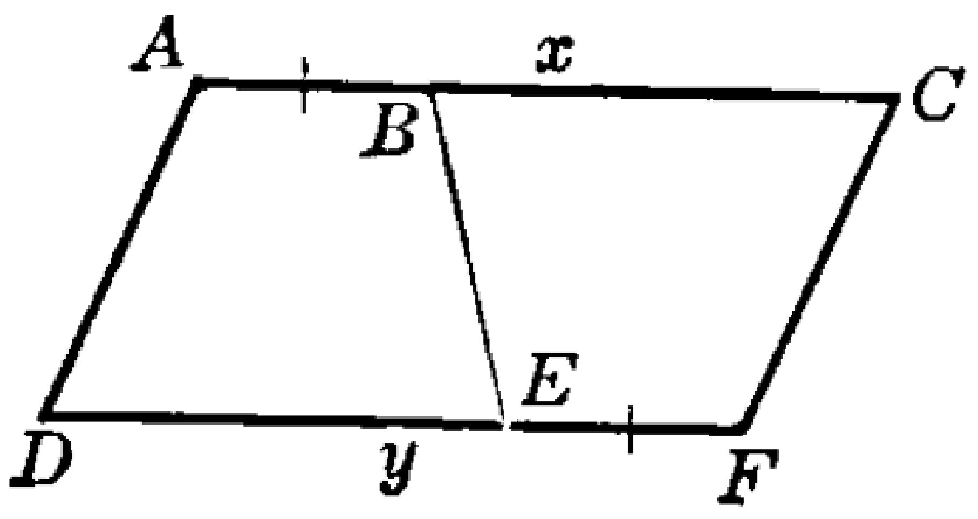
6. If a equals b, and b equals c, then does a equal c?

From these examples you can see that in some cases the relationship that is true between a and b, and between b and c, carries over from a to c. In other cases the relationship does not carry over.

(*b*) The substitution axiom really says that the relationship of equality carries over. Write two other relationships that carry over and two that do not.

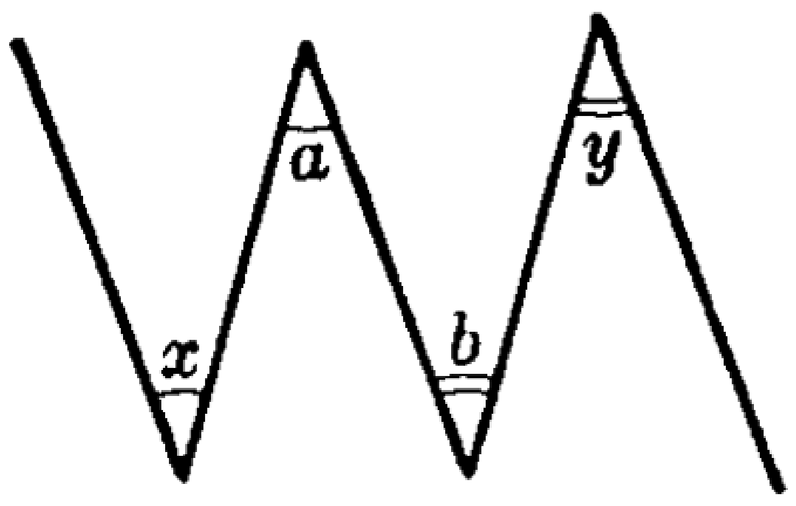
In each of the following exercises, supply one additional equation that will make the conclusion valid. Then prove it.

S4.



Given:
$AB = EF$
and — ? —
Prove:
$AC = DF$

S5.



Given:
$\angle x = \angle a$
$\angle y = \angle b$
and — ? —
Prove:
$\angle x = \angle y$

Defined and Undefined Terms

18. In order to study or discuss a subject, we must have a clear idea of the meaning of the words or terms used. When we describe these words or terms, we are *defining* them or giving *definitions.*

Here are three definitions that you can find in the dictionary. Read them carefully.

1. An *island* is a tract of land surrounded by water and smaller than a continent.

2. A *scythe* is a farm tool consisting of a long bent handle and a long curved blade, attached at an angle, for mowing grass or grain by hand.

3. A *prime number* is an integer that is divisible by no other integer except itself and one.

A definition states the meaning of a word or term by using other words or terms. Unless we understand these other words, the definition is of no use to us. These other words must be looked up until we find their meaning given in words that we do understand. All definitions depend on the fact that we do know the meanings of some words without having to look them up.

We have already seen that in geometry it is necessary to assume certain propositions without proof. In the same way we must accept certain terms without defining them; that is, we shall agree that we know their meanings. Those terms that we take for granted are called *undefined terms.* Having agreed on the undefined terms, we then define the others using the terms that we had agreed to accept without definition.

An *undefined term* is one whose meaning we take for granted without describing it in words.

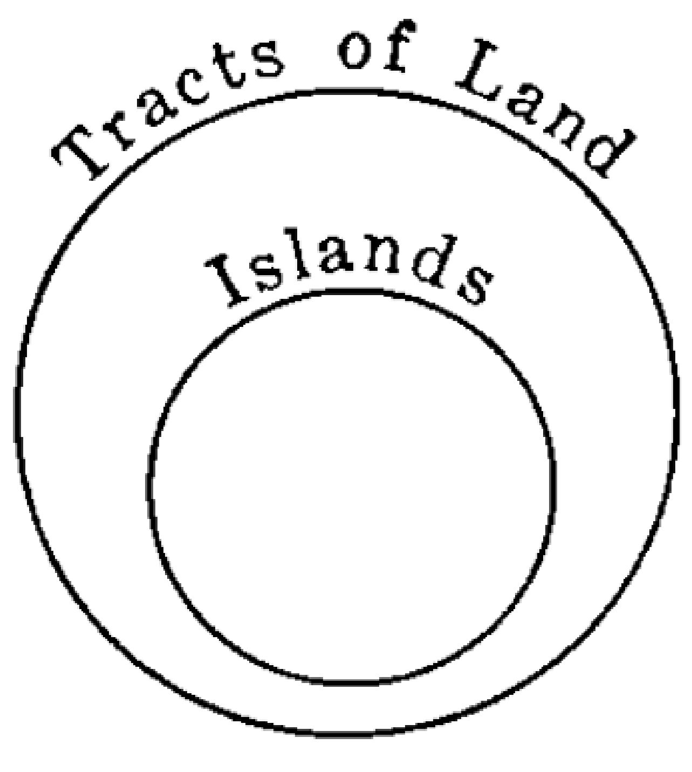
A *defined term* is one which we describe in words.

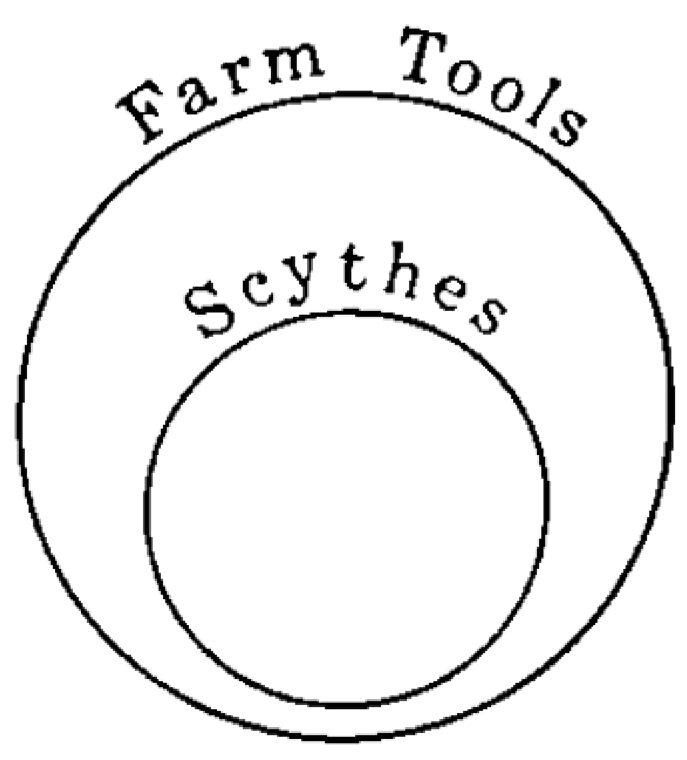
Properties of a Definition

19. When we describe the meaning of a defined term in words, the words we use must be those that we defined earlier or those that we agreed to accept without definition. We accept without definition words like *is, draw, extend, figure, between, relationship,* and so on. These words have the same meaning in geometry as in ordinary language. The first property of a definition is this:

Every definition must be expressed in terms that have been accepted.

It is customary to define a term as part of the next larger group. Thus, in § 18 an island is defined as part of a group of tracts of land. In all the tracts of land, there are some that will be islands. The second definition implies that if you look over all the farm tools, then you find scythes among them. The idea of a term as part of a larger group may be pictured by a circle within a circle.





The second property of a definition is that it must include all the things that it defines but exclude all others. A definition must be limiting. When something is defined, it must be described well

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enough to distinguish it from the other things in the group that are not being defined. For example, it must be stated definitely how an island is different from all other tracts of land. Suppose it was said that an island is a tract of land surrounded by water. This statement would be correct as a fact, but it could not be used as a definition because it would include other tracts of land such as the continent of North America.

There is a simple test that enables us to tell whether or not we have excluded all the things in the larger group from the inner group. In this test we first write our definition in the "if–then" form, and then turn it around. Thus, we might test the poor definition in the last paragraph by writing it first as follows: If a tract of land is an island, then it is surrounded by water. Then, turning this around, we write: If a tract of land is surrounded by water, then it is an island. The first of these statements is true. But the second proposition, which is called the *converse* of the first, is not true. In a definition, both must be true. A definition must be *reversible*. This means that the converse of a definition must always be true.

The *converse* of a proposition is a new proposition formed by interchanging the hypothesis and the conclusion.

A *reversible proposition* is a proposition whose converse is equally valid.

A definition is reversible.

A good definition should be clear and concise. It should not tell more than is necessary to locate the boundary separating the smaller from the larger group. A definition that tells more is *redundant*. For example, if an *offense* were defined as a dishonorable sin, the word "dishonorable" would be redundant. Explain.

Requirements for EVERY definition:

1. It must be expressed in terms that have been accepted.
2. It must be reversible.

Properties of a GOOD definition:

1. It should be clear and concise.
2. It should not be redundant.

EXERCISES

1. Each of the following statements defines the *italicized* word in terms of a larger group. Name the larger group.

(*a*) *Suffrage* is the right to vote.

(*b*) A *policeman* is a public servant who enforces the law and maintains order.

(*c*) *Mathematics* is the science of exact reasoning.

(*d*) An *integer* is a whole number.

(*e*) An *odd integer* is an integer that is not divisible by two.

2. Each of the following again defines the *italicized* word in terms of a larger group, but the name of the larger group is omitted. Tell what it is.

(*a*) A *strait* is a narrow —?—.

(*b*) A *Doric column* is the simplest type of Greek —?—.

(*c*) In chemistry an *element* is a(n) —?— that cannot be separated into substances different from itself.

(*d*) An *atom* is the smallest particle of a(n) —?— that can exist.

(*e*) A *barometer* is a(n) —?— for determining the pressure of the atmosphere.

3. Write the converse of each proposition and tell whether or not the proposition is reversible.

(*a*) If a person is born in the United States, then he has right to American citizenship.

(*b*) If a person is President of the United States, then he is more than 35 years old.

(*c*) If a person is a patriot, then he loves his country.

(*d*) An honest person may be trusted.

(*e*) A good companion is cheerful.

4. Write the two "if-then" propositions contained in each of the following definitions.
   - (*a*) *Autumn* is the season of the year between summer and winter.
   - (*b*) A *passport* is a document permitting a person to travel.
   - (*c*) *Loyalty* is the quality of being faithful.
   - (*d*) A *triangle* is a figure bounded by three straight lines.
   - (*e*) An *imaginary number* is an indicated even root of a negative number.
5. Criticize the five definitions that follow, pointing out which properties of a definition they have and which they do not have.
   - (*a*) A *democracy* is a form of government that is headed by a president.
   - (*b*) An *umpire* is an arbiter.
   - (*c*) An *airplane* is a conveyance designed to travel in the air.
   - (*d*) *Pleasure* is a gratifying sensation caused by favorable circumstances.
   - (*e*) *Mathematics* is the study of arithmetic, algebra, geometry, factoring, percentage, and all problems about number and space.
6. Define the following: (*a*) an exponent (in algebra), (*b*) a molecule, (*c*) a skiff, (*d*) a siphon, and (*e*) atomic energy.

SUPERIOR WORK

S1. Draw one circle inside another circle. Label one circle to represent all the integers (whole numbers) and the other circle to represent only the odd integers. Define the group in the smaller circle.

In the same diagram, draw another circle to represent all the prime integers except 2. Define this group.

S2. Define the following three terms in the proper order: the United States Congress, the United States government, and the United States Senate.

(*a*) Decide which term to define first.

(*b*) Having defined that term, use it to define the second. Then use the second to define the third.

(*c*) Draw and label three circles to picture your definitions.

HONOR WORK

H1. Draw and label circles to represent *government*, *democracy*, *dictatorship*, and *anarchy*.

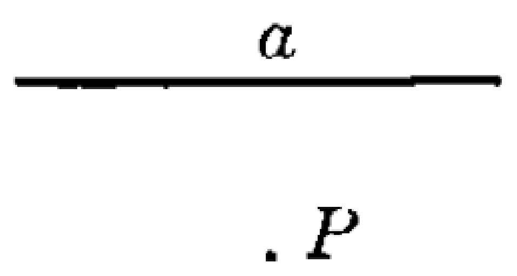
H2. Write out the definition of treason according to the dictionary and also according to Article III, Section 3 of the United States Constitution. Which definition is narrower? Why did the framers of our Constitution write their own definition?

Sequence of Definitions

Because a definition describes a class within a class, it is advisable to give definitions in a definite order or sequence. Each definition describes a class within a larger class that was described by a previous definition. Following is a sequence of definitions, mostly about angle relations. In each definition point out the undefined geometric terms and those terms that have previously been defined.

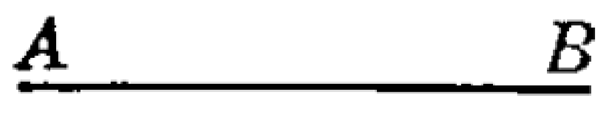
A definition describes a whole group of things. Often we choose one member of this group and point to it as typical of all the members. For example, after defining a *scythe* in § 18, it would have been helpful to have a picture or drawing of it. In the same way, after giving the definition of an angle (which describes every angle), we draw a figure of a *typical* angle. We then label it and point to it as an *illustration* of the definition. The definitions that follow have been illustrated.

20. Undefined Terms: a straight line; a point.



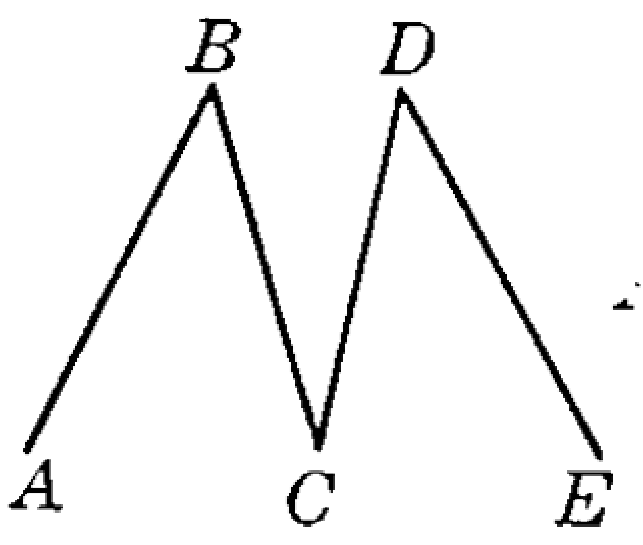
a is a st. line.
P is a point.

21. A line segment is the part of a straight line between two points.



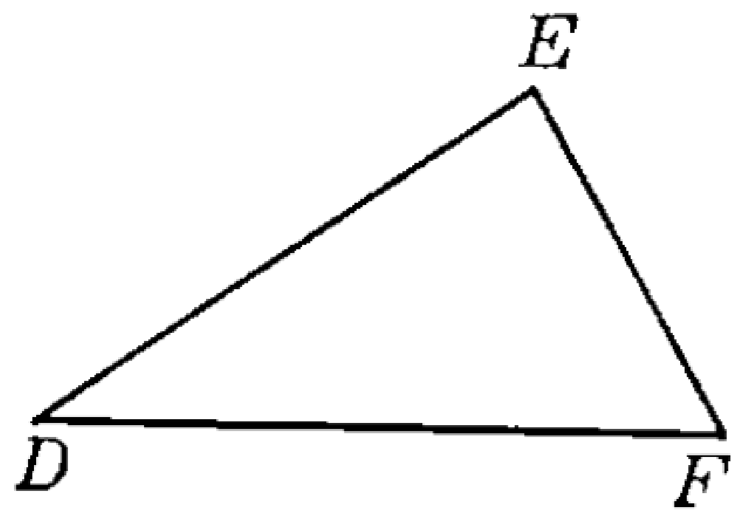
AB is a line segment.

22. **A broken line** is a figure in which a number of points not in a straight line are connected in succession by line segments.



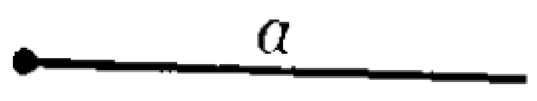
ABCDE is a broken line.

23. **A triangle** is a closed broken line having three line segments. (The sides of a triangle are its three line segments.)



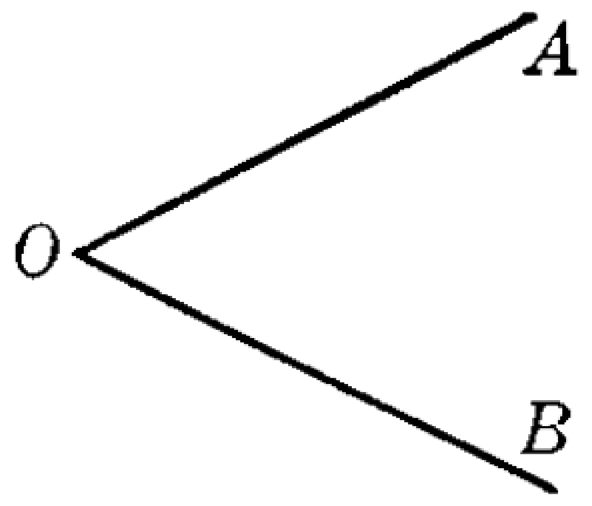
DEF is a triangle.

24. **A ray** is the part of a straight line to one side of a point.



a is a ray.

25. **An angle** is a figure that is formed by two rays from a common point.

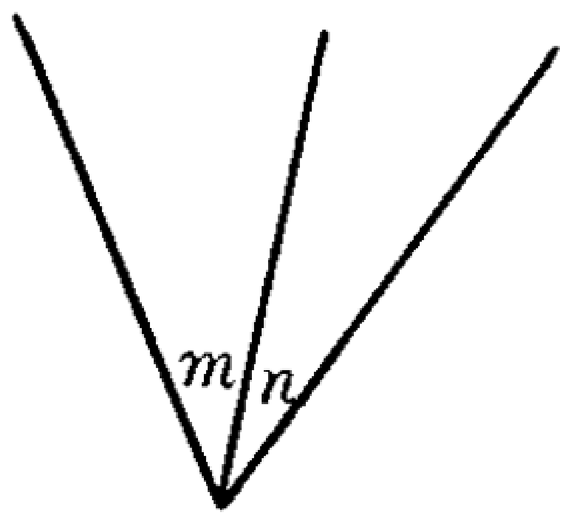


OA and *OB* are the sides and *O* is the vertex of $\angle O$.

26. **The sides of an angle** are the rays forming the angle.

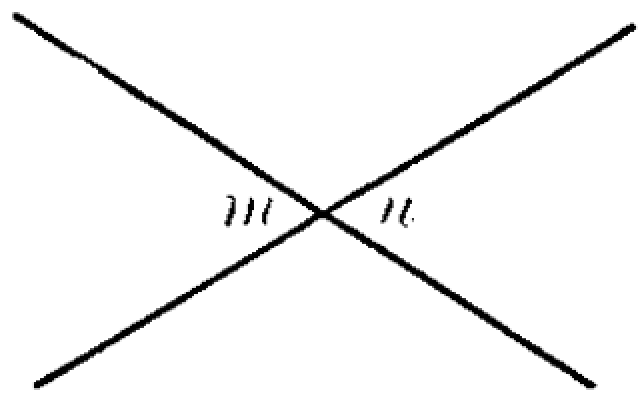
27. **The vertex of an angle** is the point where the sides of the angle meet.

28. **Adjacent angles** are two angles that have a common vertex and a common side between them.



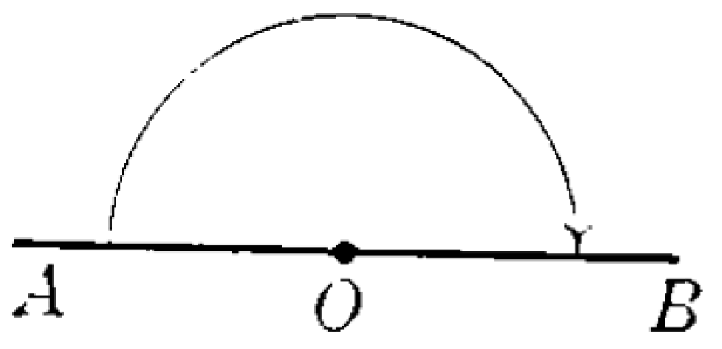
$\angle m$ and $\angle n$ are adjacent angles.

29. **Vertical angles** are two non-adjacent angles formed by two intersecting straight lines.



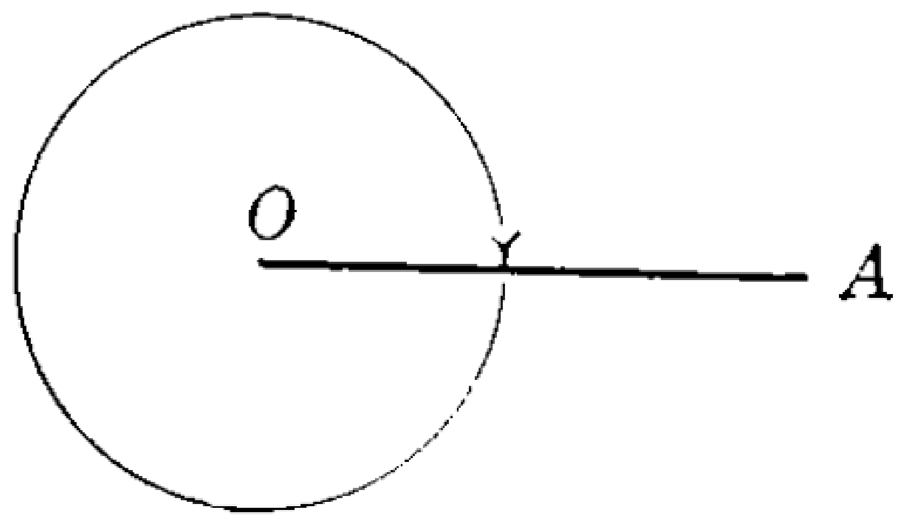
$\angle m$ and $\angle n$ are vertical angles.

30. **A straight angle** is an angle whose sides form a straight line.

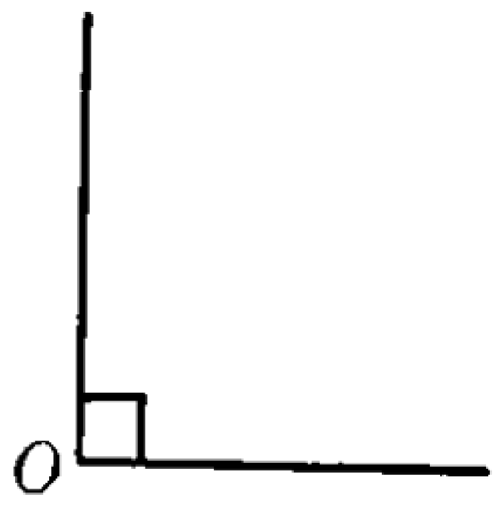


$\angle AOB$ is a st. angle.

An angle that is the result of a complete revolution is not a straight angle because its sides form a *ray* and not a *straight line*. Such an angle is sometimes called a round angle or a perigon.

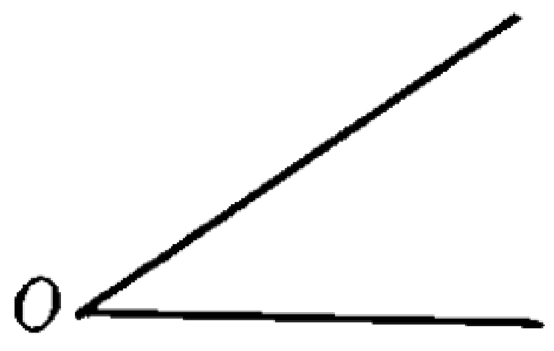


31. **A right angle** is an angle one-half as large as a straight angle.



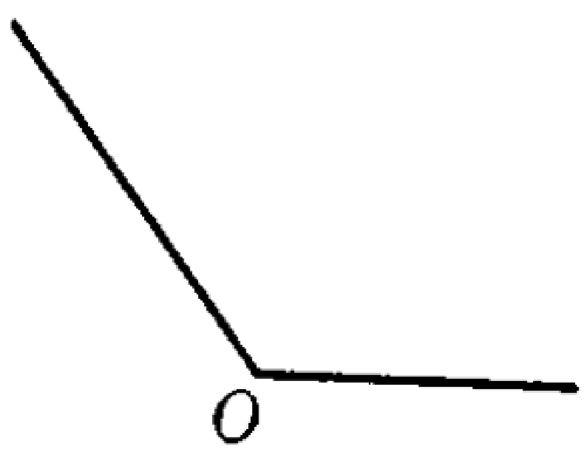
$\angle O$ is a right angle.

32. **An acute angle** is an angle less than a right angle.



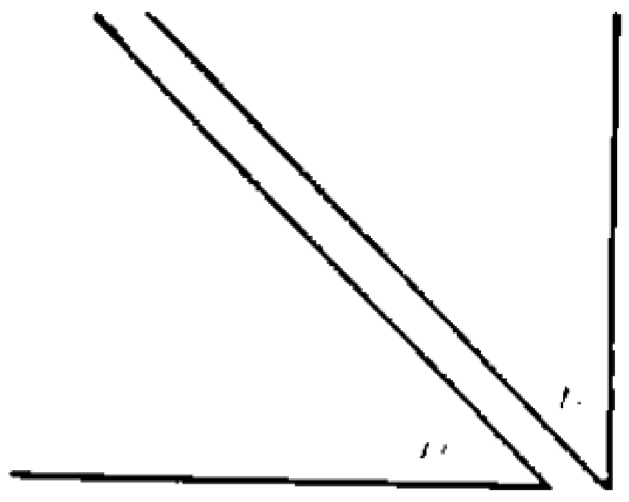
$\angle O$ is an acute angle.

33. **An obtuse angle** is an angle greater than a right angle but less than a straight angle.



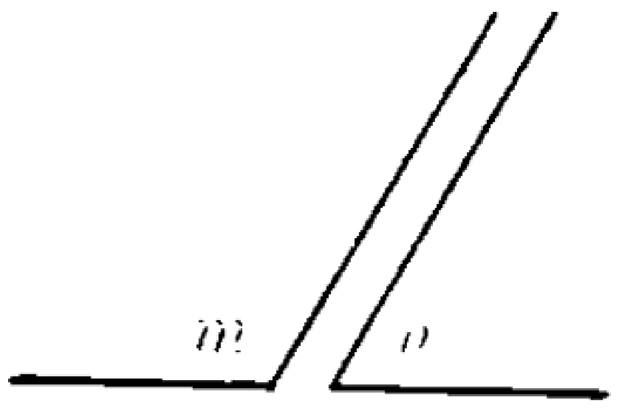
$\angle O$ is an obtuse angle.

34. **Complementary angles are two angles whose sum equals a right angle.**



∠*m* and ∠*n* are comp. angles.

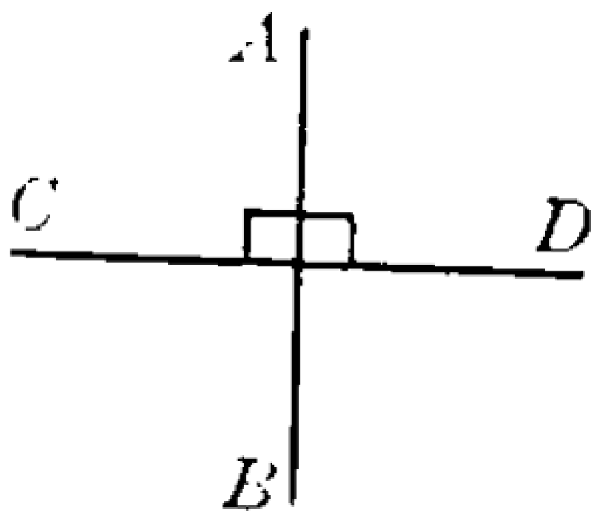
35. **Supplementary angles are two angles whose sum equals a straight angle.**



∠*m* and ∠*n* are sup. angles.

(To distinguish between complementary and supplementary angles remember the *s* sound: supplementary—straight angle.)

36. **Perpendicular lines are two straight lines that meet at right angles.**



$AB \perp CD$
(AB perpendicular to CD)

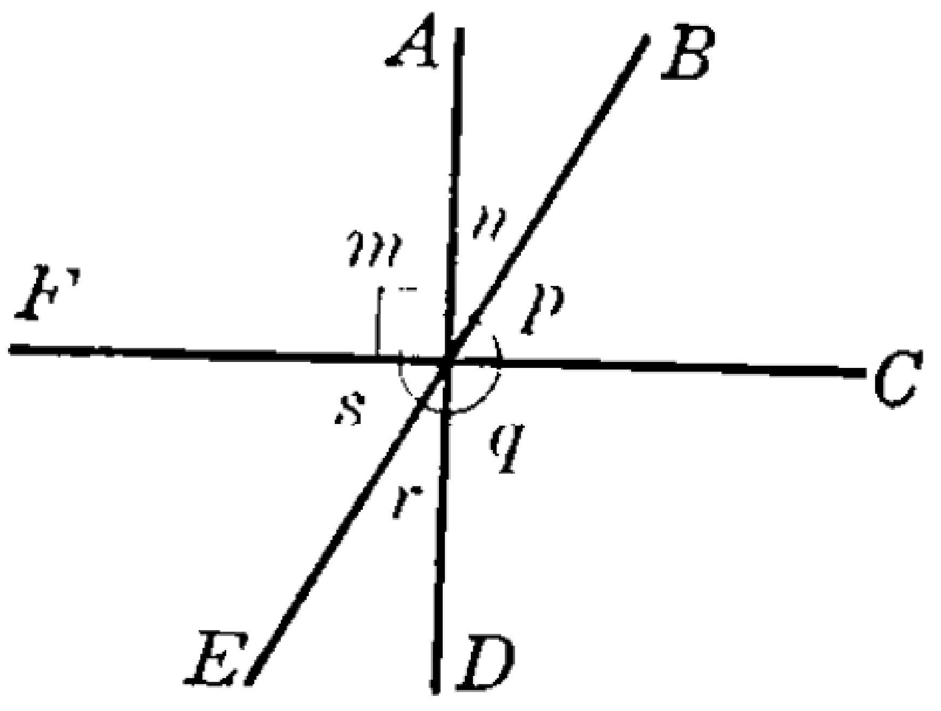
Definitions, Assumptions, and Theorems

37. Terms that remain undefined are similar to unproved propositions. Even definitions themselves are assumptions. For example, from our definition we know that if two angles are complementary, then their sum is a right angle. This is an assumption. Mathematicians the world over have merely agreed to call such angles complementary. Proof has nothing to do with it.

Assumptions consist of axioms, postulates, and definitions. The propositions that we shall prove—that is, the theorems—will all depend on axioms, postulates, and definitions.

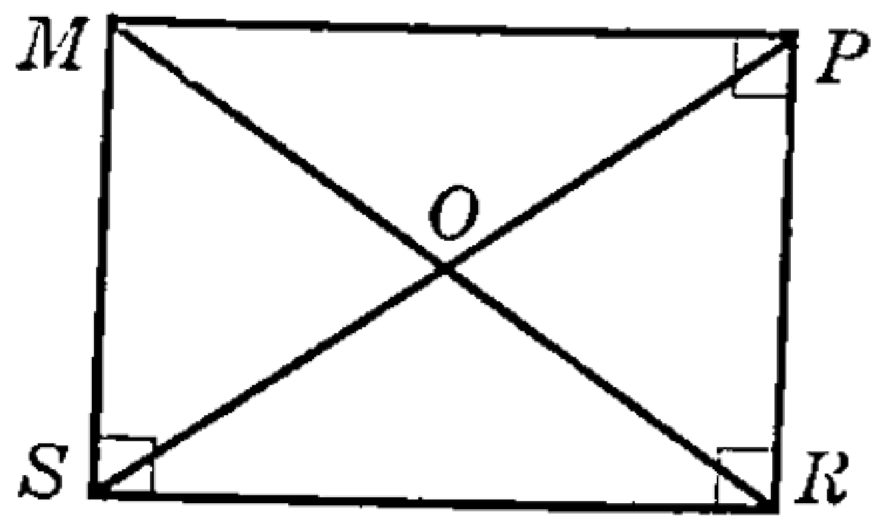
EXERCISES

1. In the figure, ∠*AOF* is a right angl



Name the following:
- (*a*) Three pairs of adjacent angles
- (*b*) Four pairs of vertical angles
- (*c*) Two right angles
- (*d*) Two acute angles
- (*e*) Two obtuse angles
- (*f*) One pair of adjacent complementary angles
- (*g*) One pair of adjacent supplementary angles
- (*h*) Two perpendicular lines

2. In the figure, ∠*P*, ∠*R*, and ∠*S* are right angles. Name the following:



- (*a*) One pair of vertical angles
- (*b*) One pair of adjacent complementary angles
- (*c*) One pair of adjacent supplementary angles
- (*d*) One pair of nonadjacent supplementary angles
- (*e*) Two perpendicular lines

3. Draw the following angles and label them:
- (*a*) Two adjacent acute angles
- (*b*) An acute angle adjacent to an obtuse angle
- (*c*) Two adjacent obtuse angles
- (*d*) Two adjacent complementary angles
- (*e*) Two nonadjacent complementary angles
- (*f*) Two adjacent supplementary angles
- (*g*) Two nonadjacent supplementary angles
- (*h*) Two vertical angles

SUPERIOR WORK

S1. Write a definition of a straight line, using the simplest words you can. Underline the key words. Are they simpler than the idea of a straight line itself?

S2. The following is one definition of *democracy:*

Democracy is a form of society in which (1) the general welfare is the concern of each citizen; (2) each citizen possesses important civil liberties; (3) the government and its powers exist by the consent of the governed; and (4) appeal to reason is a guiding principle.

(*a*) Is this definition reversible? In other words, can every society that satisfies the four conditions given be called a democracy?

(*b*) Is it stated in terms that are easily understood, or do some of the terms themselves require definition?

(*c*) Can you find any redundancy in the definition?

(*d*) Find some other definition of democracy and compare it with this one.

HONOR WORK

H1. The Constitution of the United States defines our form of government.

(*a*) Read the Constitution and list some of the undefined terms in it like "interstate commerce" and "republican form of government."

(*b*) In case of dispute, who decides the meaning of these terms?

(*c*) Could the framers of the Constitution have avoided undefined terms? Could they have narrowed their meanings? What would have been some advantages and disadvantages of doing so?

H2. Suppose that someone has seen a few football games and knows in a general way that when his team advances the ball it is time to cheer. However, he does not know what a "first down" means. For his benefit, write out a sequence of definitions leading up to the definition of a "first down."

SUMMARY

38. In geometry certain propositions are accepted or assumed without proof and certain terms without definition. Propositions accepted without proof are called *axioms* or *postulates;* those accepted only after they are proved are called *theorems.* Terms accepted without definition are called *undefined terms;* the others are called *defined terms.*

It is assumed that two points determine a straight line. With any two points given, one and only one straight line can be drawn through them. The axioms about the four fundamental operations with numbers are assumed and applied to geometric reasoning. The substitution axiom which allows us to substitute a quantity for its equal in an equation is also assumed.

The converse of a proposition is another proposition in which the original hypothesis and conclusion have been interchanged. If a proposition is just as valid as its converse, either proposition is called reversible.

Every definition must be reversible and must be expressed in terms that have been accepted. A good definition is also clear and concise, and is not redundant.

Definitions are stated so that each term that is defined is part of a larger group that has been defined earlier, or of a group that has been accepted without definition.

To illustrate a definition means to point out (usually by a diagram) a member of the group that the definition describes.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

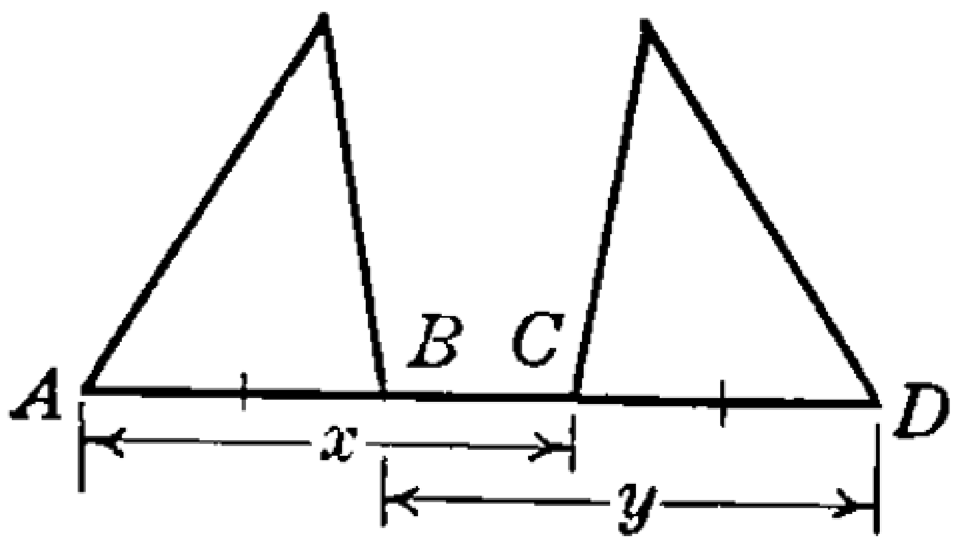
axiom (ax.) § 10
postulate (post.) § 10
theorem § 10
definition (def.) § 18
defined and undefined terms § 18
converse § 19
reversible proposition § 19
redundant § 19
broken line § 22
triangle (△) § 23
adjacent angles (adj. ∠s) § 28
vertical angles (vert. ∠s) § 29

straight angle (st. ∠) § 30
right angle (rt. ∠) § 31
acute angle § 32
obtuse angle § 33
c lementary angles (comp. ∠s) § 34
s ementary angles (sup. ∠s) § 35
p ndicular lines (⊥s) § 36

REVIEW EXERCISES

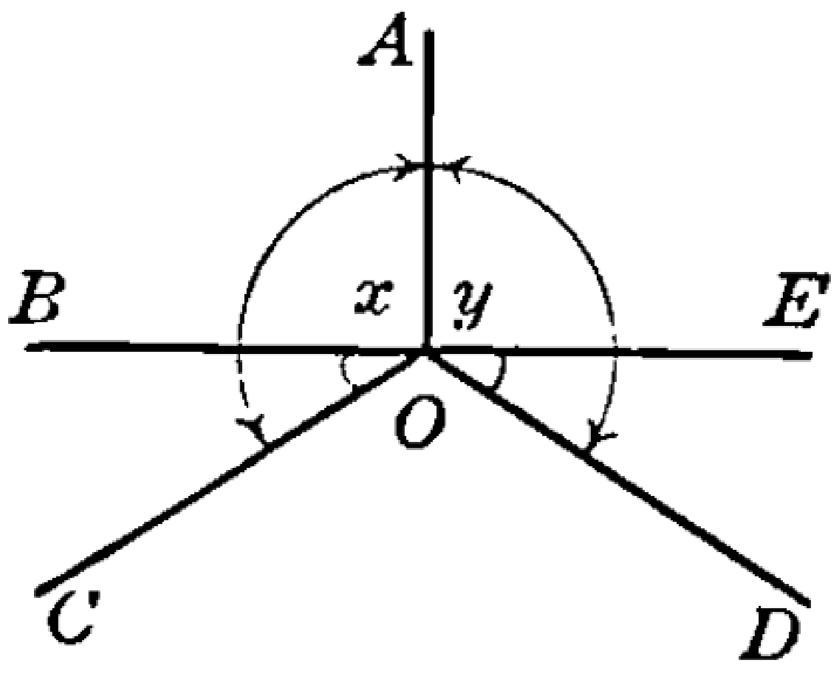
Write a proof for each of the following exercises. If in some cases the conclusion does not follow, write "not valid" beside the conclusion.

1.



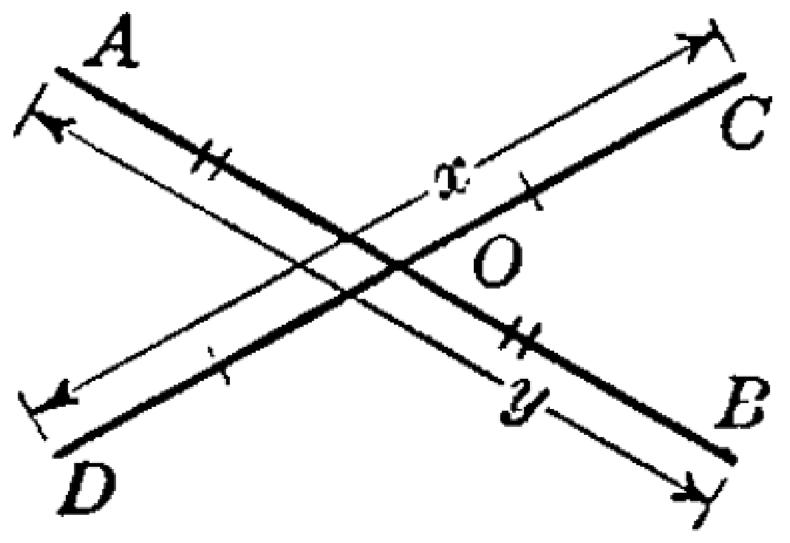
Given:
$AB = CD$
Prove:
$AC = BD$

2.



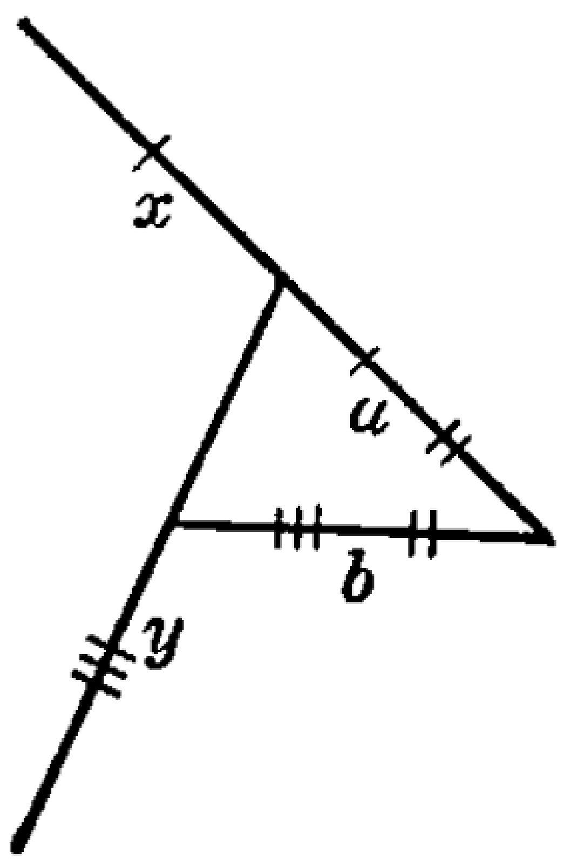
Given:
$\angle AOC = \angle AOD$
$\angle BOC = \angle EOD$
Prove:
$\angle AOB = \angle AOE$

3.



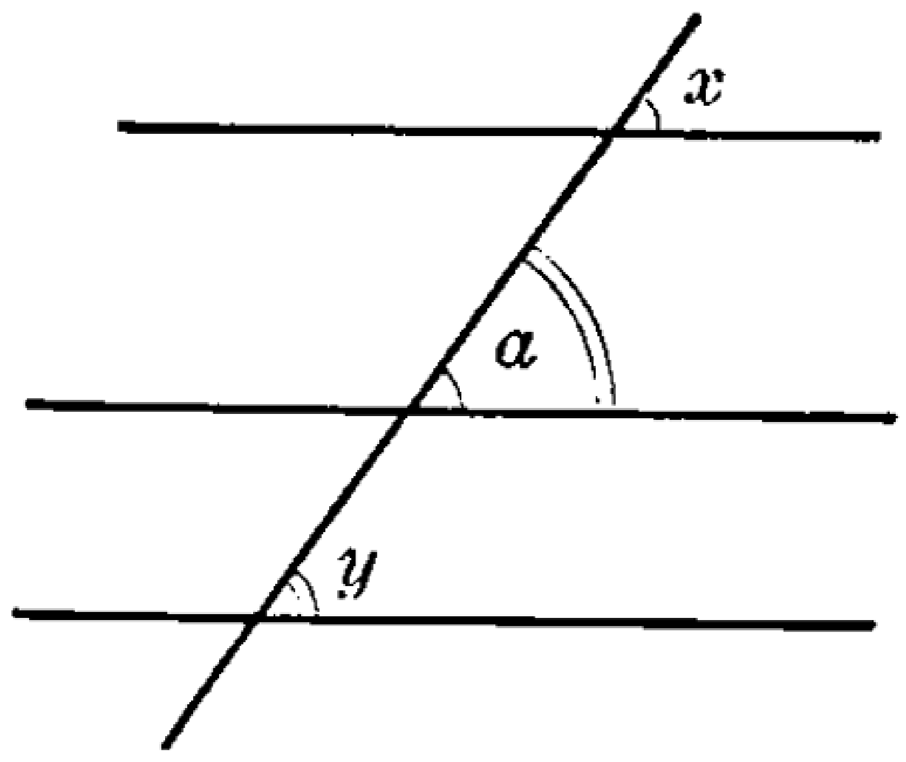
Given:
$AO = OB$
$CO = OD$
Prove:
$AB = CD$

4.



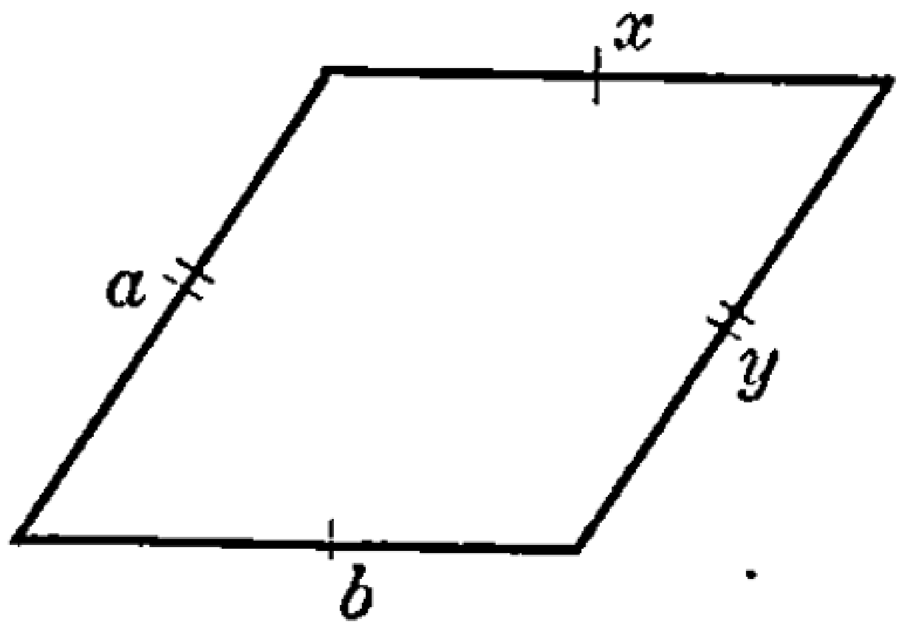
Given:
$x = a$
$a = b$
$b = y$
Prove:
$x = y$

5.



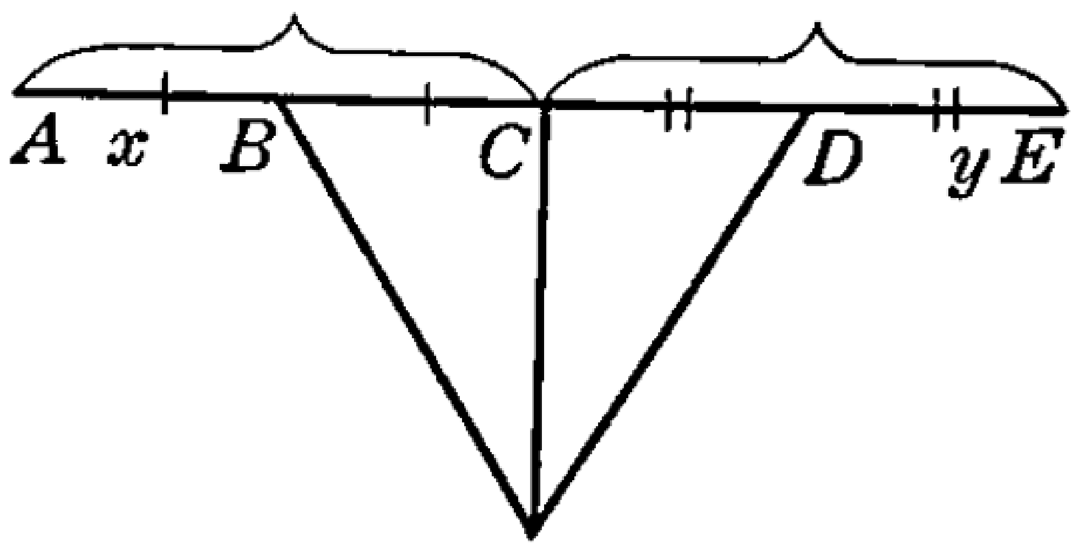
Given:
$\angle x = \angle a$
$\angle a = \angle y$
Prove:
$\angle x = \angle y$

6.

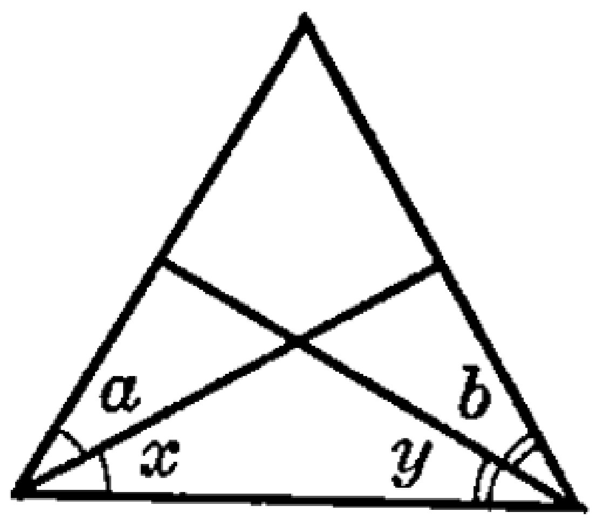


Given:
$x = b$
$y = a$
Prove:
$x = y$

7. Given: $AC = CE$
$AB = BC$
$CD = DE$
Prove: $x = y$

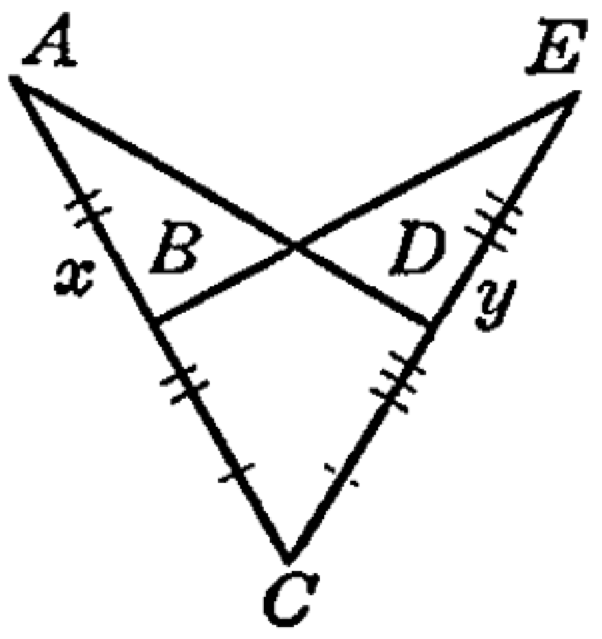


8.



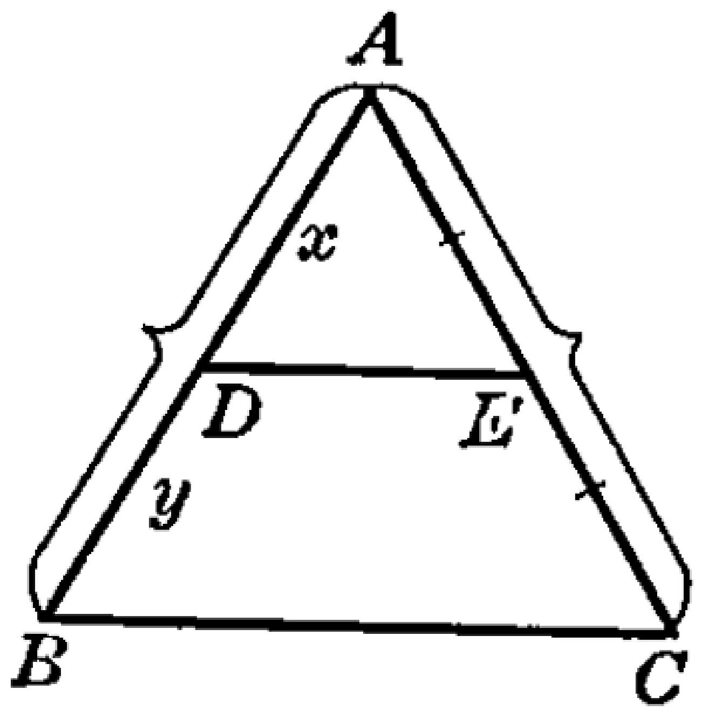
Given:
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

9.



Given:
$BC = DC$
$AB = BC$
$ED = DC$
Prove:
$AC = CE$

10.



Given:
$AB = AC$
$AE = EC$
Prove:
$AD = DB$

11. Indicate whether each of the following statements is true or false.

(*a*) A straight angle is a straight line.

(*b*) If the sides of one angle are the prolongations of the sides of another angle through the vertex, then the two angles are vertical angles.

(*c*) Perpendicular lines are two lines that meet so that the adjacent angles are equal.

(*d*) If two perpendicular lines are drawn then all the angles about the point of intersection are right angles.

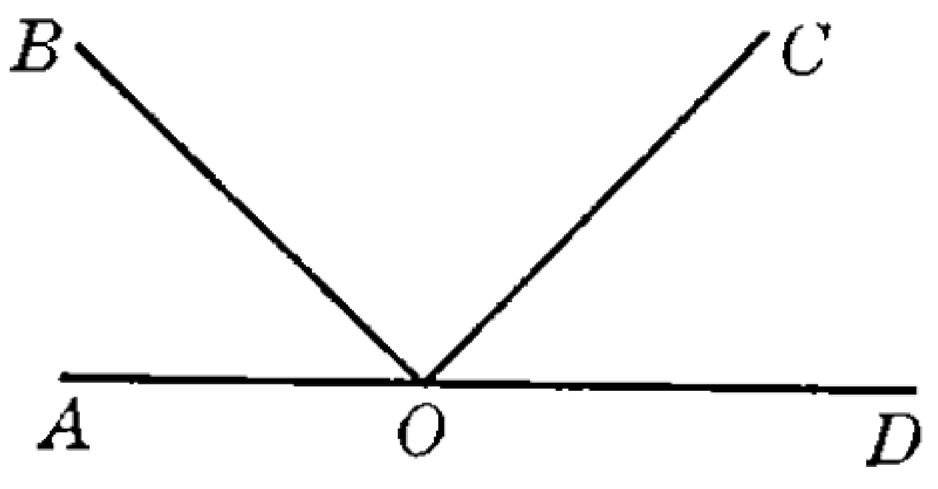
(*e*) An obtuse angle is greater than a right angle.

12. Supply the one word necessary to complete each statement.

(*a*) A side of an angle is a(n) —?—.

(*b*) If the noncommon sides of two adjacent angles are in a straight line, the angles are —?—.

(*c*) If AOD is a straight line then the supplement of $\angle AOB$ is $\angle$ —?—.

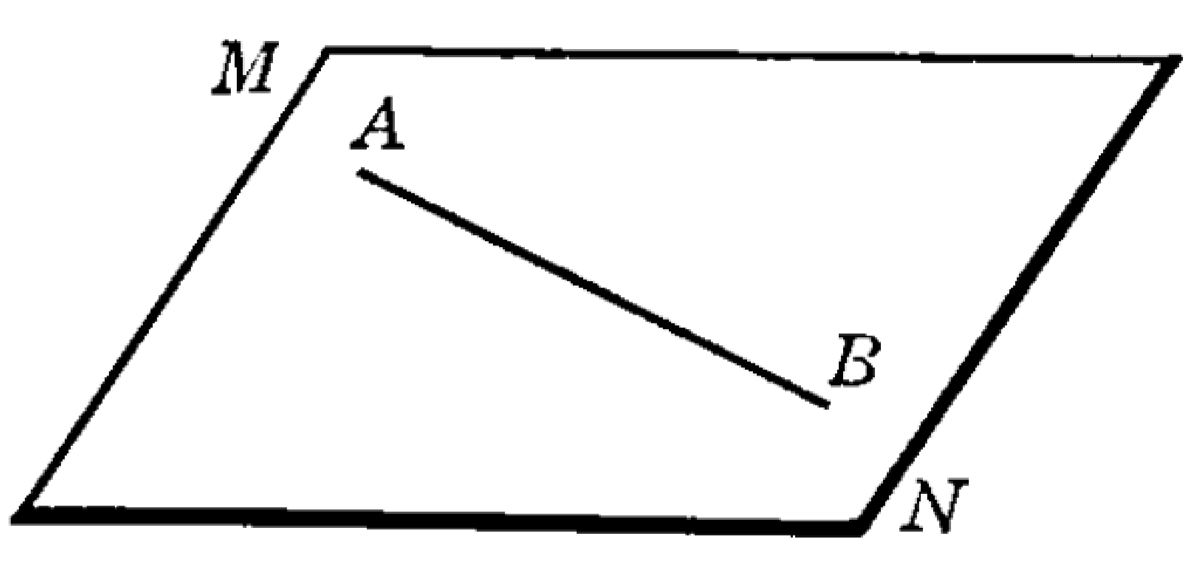


(*d*) At 3:20 the hands of a clock form a(n) —?— angle.

(*e*) At 3:40 the hands of a clock form a(n) —?— angle.

SUPERIOR WORK

S1. (*a*) A plane is often represented by a four-sided figure MN. If a line is drawn through any two points A and B in plane MN, does that line lie entirely in the plane?



(*b*) Write this as a proposition.

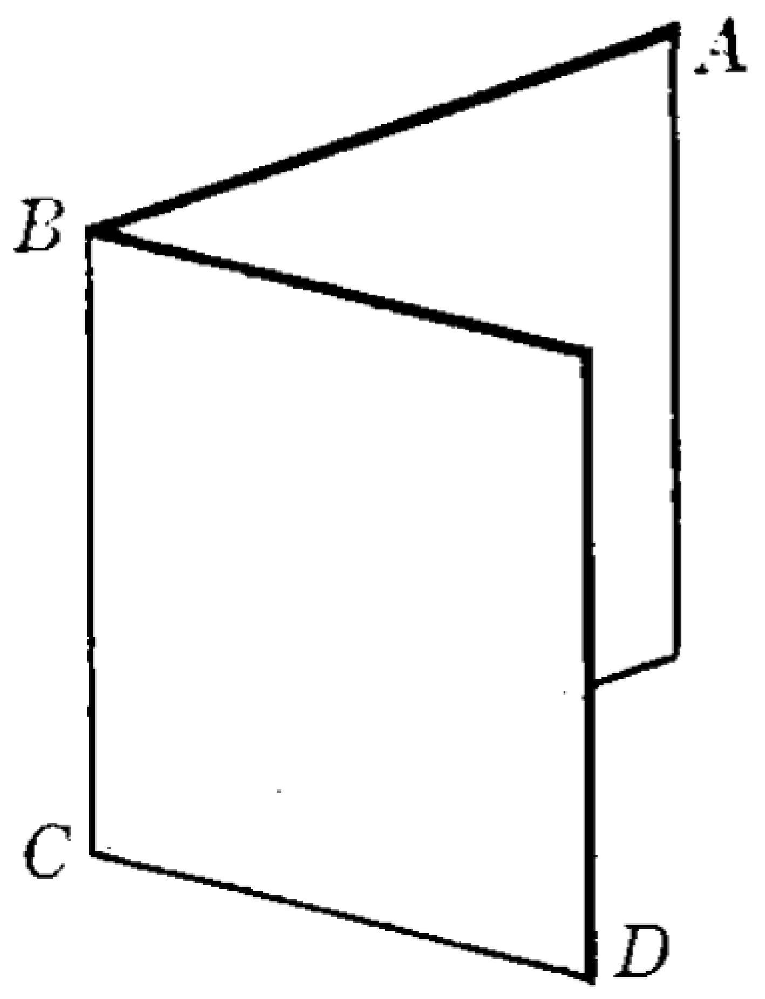
(*c*) Write the converse of the proposition in (*b*).

(*d*) Using your propositions in (*b*) and (*c*), define a plane.

S2. A straight line can be divided into two half-lines (or rays) by a point. What kind of figure would divide a plane into two half-planes?

S3. In a plane, an angle has been defined as a figure formed by two half-lines (or rays) from a common point.

In general space, a *dihedral angle* can be defined as a figure formed by two half-planes from a common edge (line). The dihedral angle in the figure is read $\angle A-BC-D$.



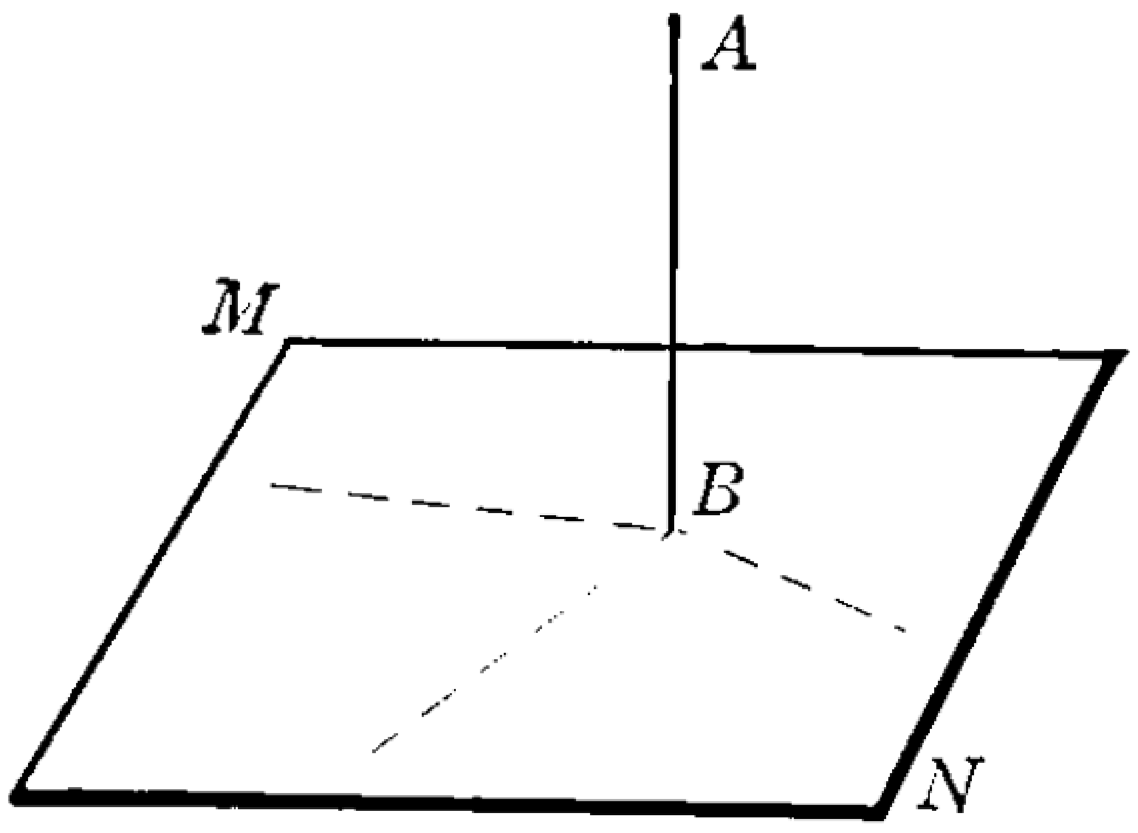
(*a*) Recall the definition of two adjacent angles in a plane. Now define two adjacent dihedral angles. Draw a figure to illustrate your definition.

(*b*) Define a straight dihedral angle.

(*c*) Define a right dihedral angle.

(*d*) Define two perpendicular planes.

S4. In the figure, line AB is perpendicular to the plane MN. The point B where the line meets the plane is called the *foot* of AB.



(*a*) In the plane MN, lines are drawn through B. How is AB related to each of these lines?

(*b*) Using your answer to (*a*), write a definition of a line perpendicular to a plane.

HONOR WORK

H1. Consider the following nonmathematical assumptions on which many of our beliefs and actions are based. Try to determine whether these are generally accepted or are being rejected, with the passing of time, in the United States and in other parts of the world.

(*a*) Honesty is the best policy.

(*b*) The pen is mighter than the sword.

(*c*) The majority rules.

(*d*) A free people cannot tolerate taxation without representation.

(*e*) Freedom of thought and conscience is as necessary to man as food and drink.

H2. List the "truths" that are held to be "self-evident" in the first part of our Declaration of Independence. Discuss these propositions as in Ex. H1.

H3. You know that multiplication is repeated addition. In the same way, division can be defined as repeated subtraction. For example, $12 \div 3 = 4$ means: if 3 is subtracted from 12 four times, the remainder is 0.

Using this definition of division, explain why $12 \div 0$ and also $0 \div 0$ cannot have definite answers and are therefore not determined.

Explain the meaning of $0 \div 5$ in terms of repeated subtraction.

H4. You have seen how conclusions in geometry depend on certain assumptions. Point out how our form of government depends on certain assumptions in the Declaration of Independence.

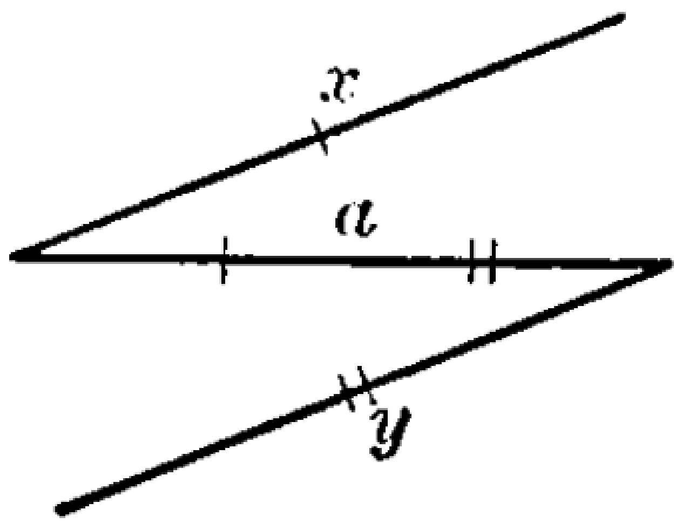
H5. When the French, following their defeat by the Germans in 1940, were forced to redefine their government they changed the assumptions of *liberté*, *egalité*, and *fraternité* (liberty, equality, and brotherhood) on which their democracy was founded to *travail*, *famille*, and *patrie* (work, family, and country). Explain why a fascist society must be based on the second set of assumptions rather than on the first set.

TEST

Part I Axioms

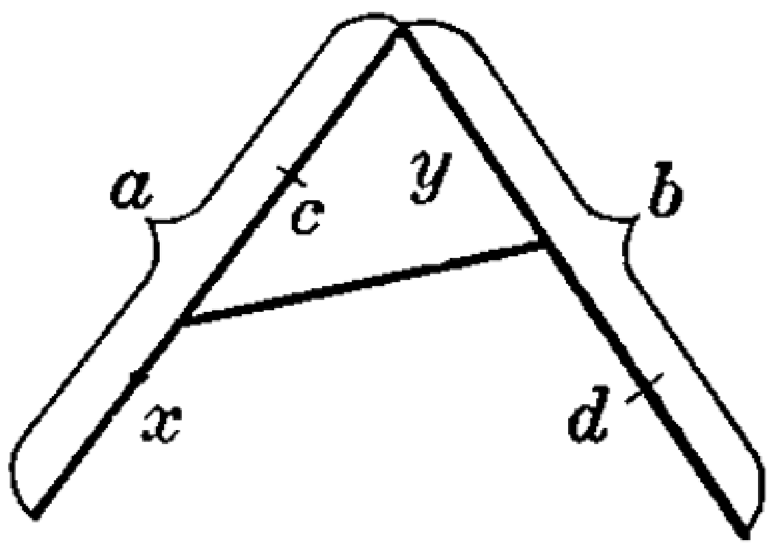
Copy the following exercises. Prove each or state that it is not valid.

1.



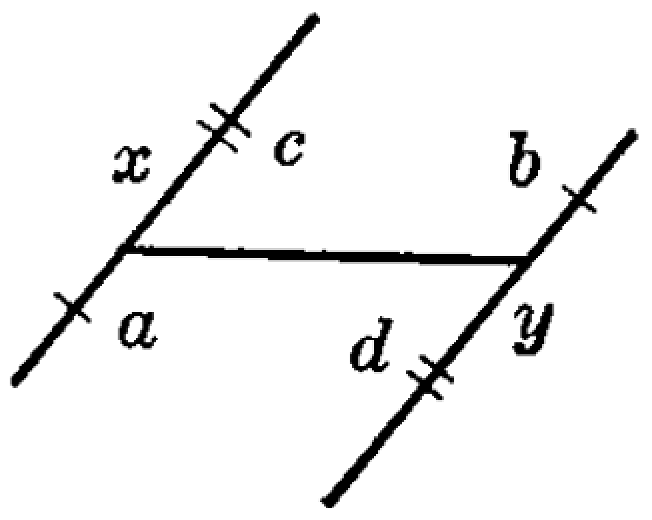
Given:
$x = a$
$a = y$
Prove:
$x = y$

2.



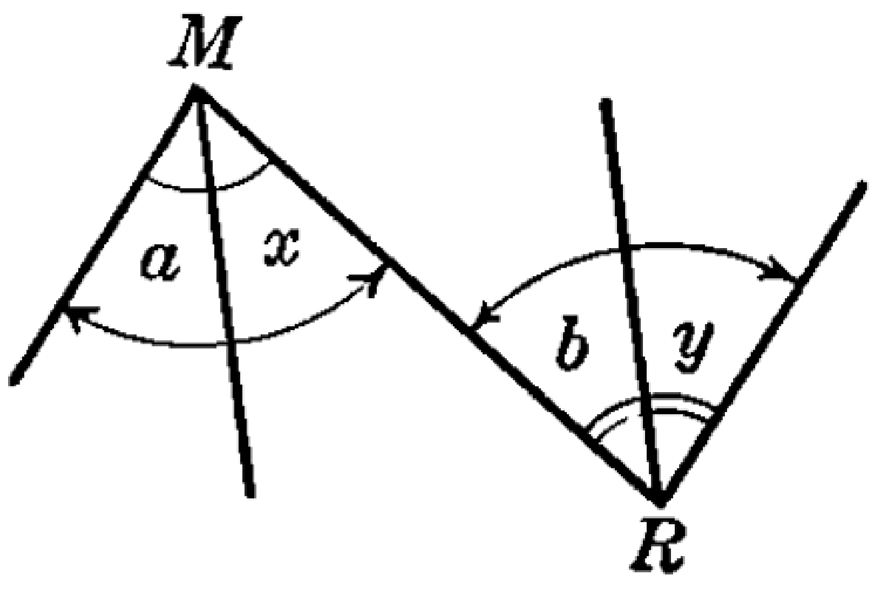
Given:
$a = b$
$c = d$
Prove:
$x = y$

3.



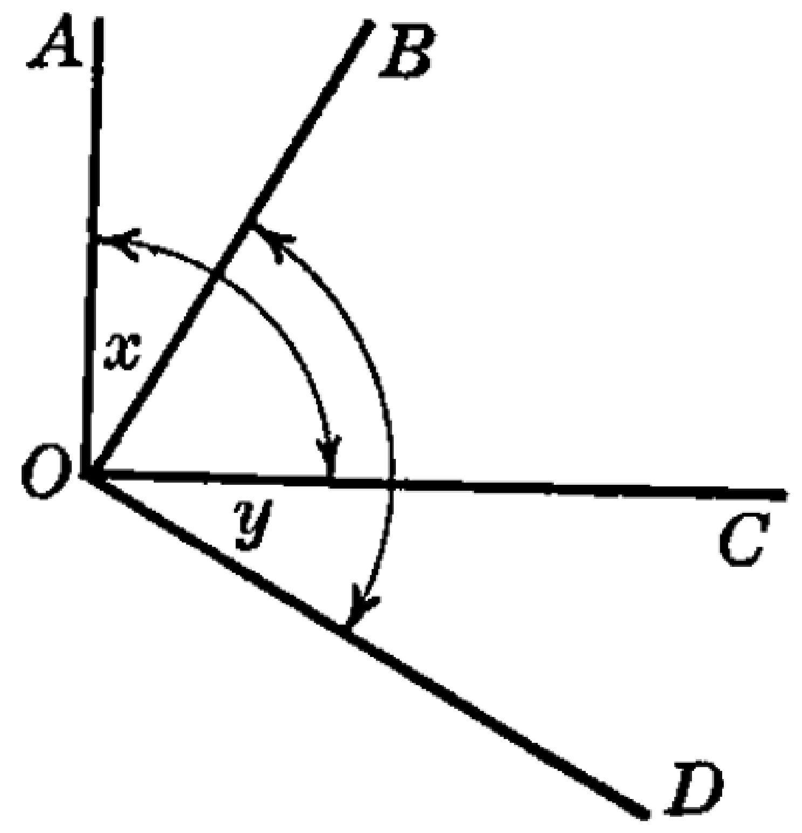
Given:
$a = b$
$c = d$
Prove:
$x = y$

4.



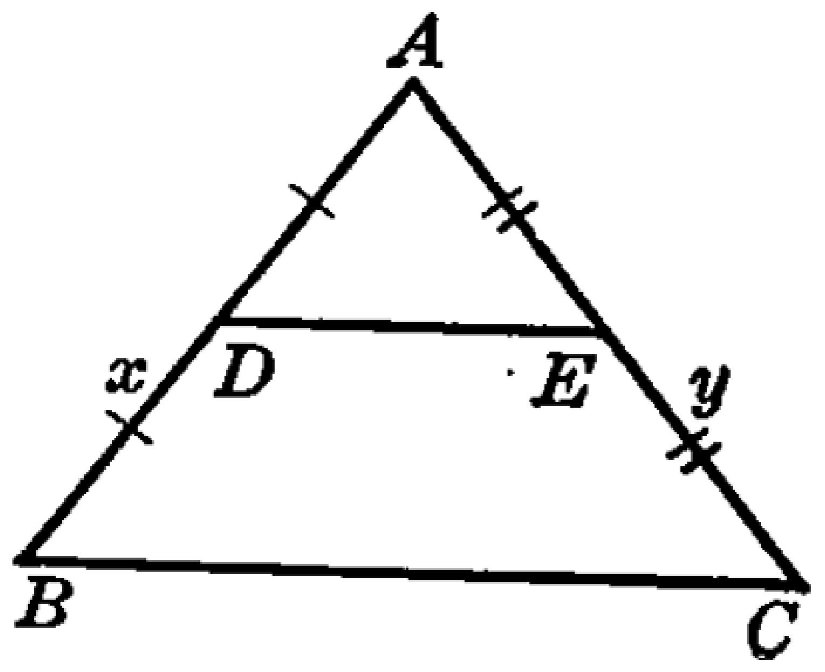
Given:
$\angle M = \angle R$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

5.



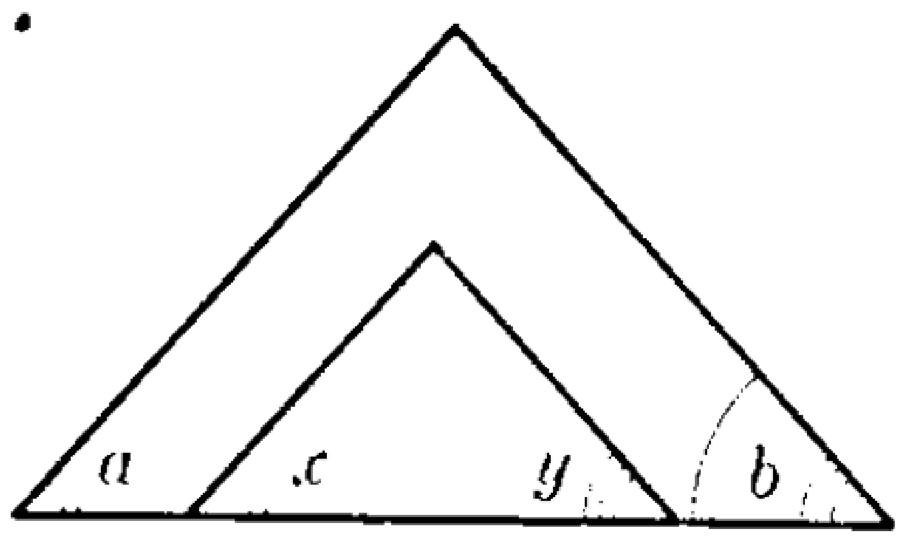
Given:
$\angle AOC = \angle BOD$
Prove:
$\angle x = \angle y$

6.



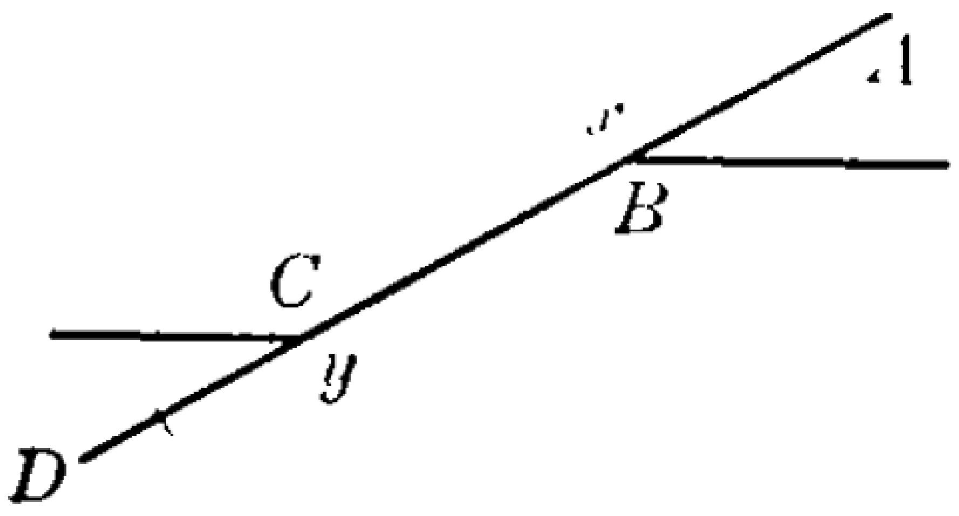
Given:
$AD = DB$
$AE = EC$
Prove:
$AB = AC$

7.



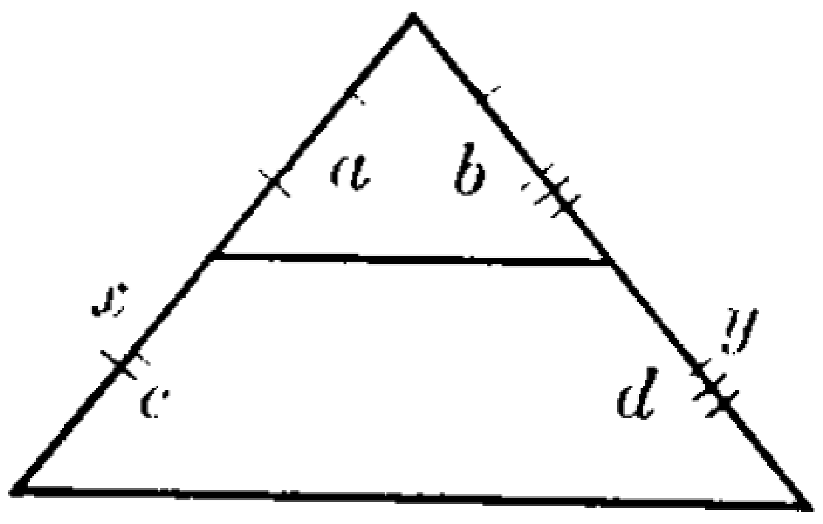
Given:
$\angle a = \angle b$
$\angle a = \angle x$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

8.



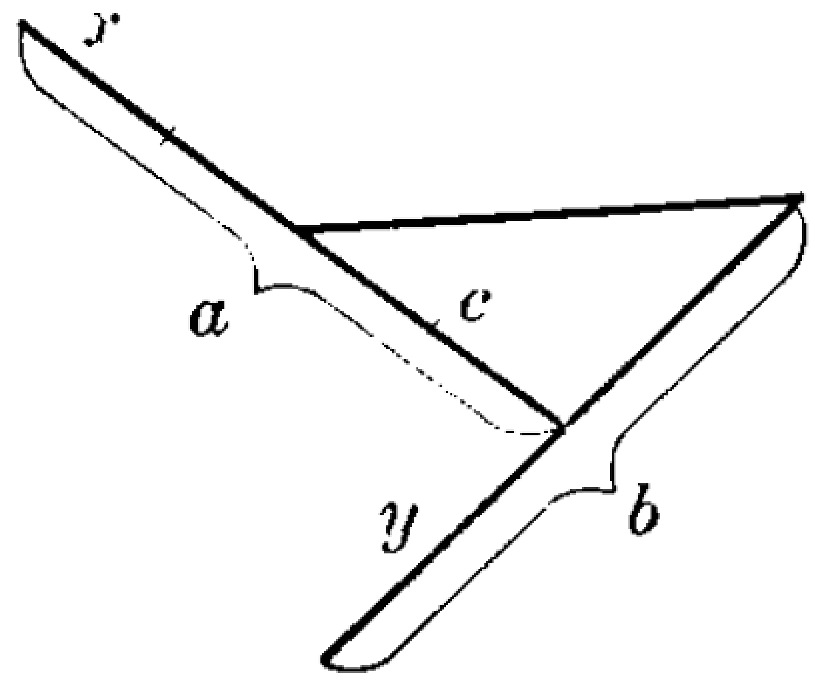
Given:
$AB = DC$
Prove:
$AC = BD$

9.



Given:
$a = b$
$a = c$
$b = d$
Prove:
$x = y$

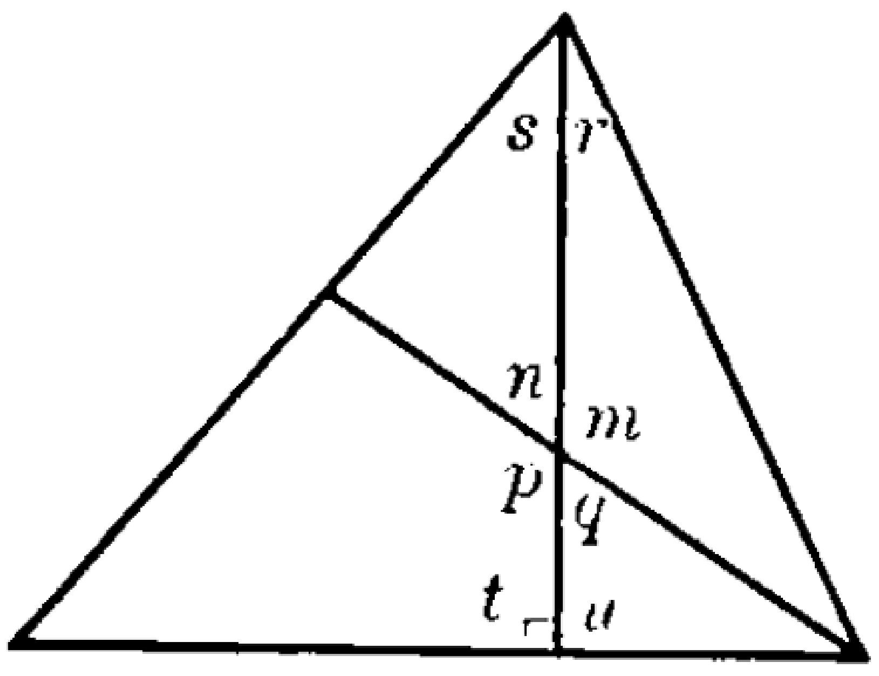
10.



Given:
$a = b$
$x = c$
Prove:
$x = y$

Part II Definitions

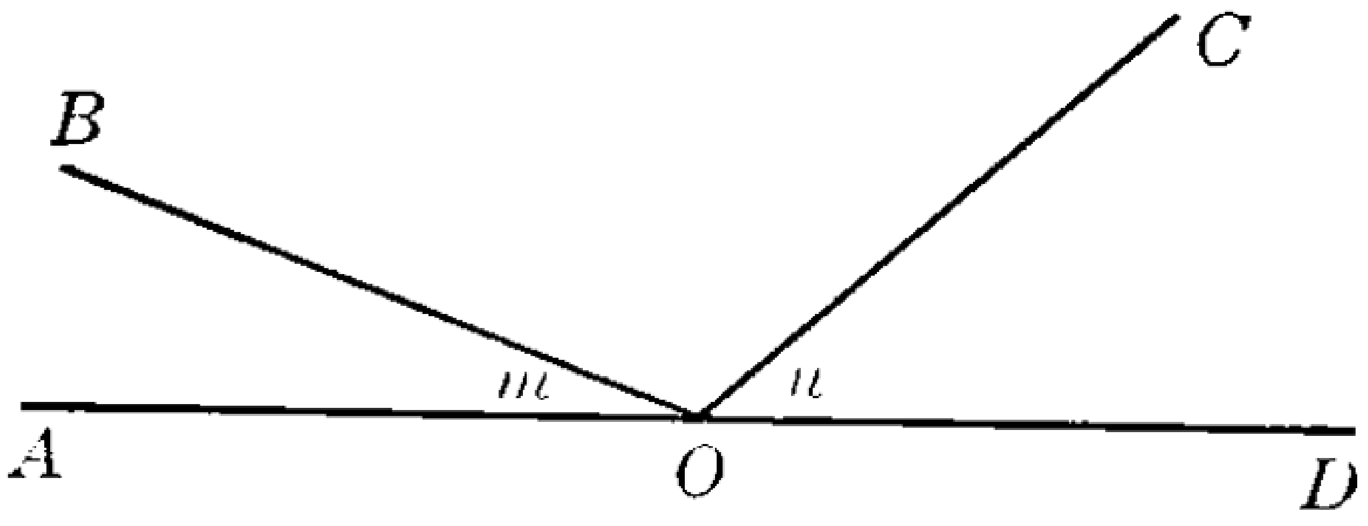
In the figure below, name the following:



1. One pair of adjacent acute angles
2. One pair of adjacent equal angles
3. One pair of unequal adjacent supplementary angles
4. One pair of vertical obtuse angles

In the preceding figure, tell how many of each of the following there are:

5. Pairs of adjacent angles
6. Right angles
7. Obtuse angles
8. Straight angles
9. In the figure below, what is the supplement of $\angle m$?



10. In the above figure, what is the supplement of $\angle AOC$?
11. Draw two adjacent complementary angles.
12. Draw two adjacent obtuse angles.
13. Draw an angle that is *about* one-half of a right angle.
14. Draw two adjacent angles whose noncommon sides form a straight line.
15. Draw two adjacent complementary angles and label them $\angle m$ and $\angle n$. Now draw a new angle p that is adjacent and complementary to $\angle n$.
16. Draw three angles m, n, and p, so that angle m is the adjacent supplement of angle n, and angle p is also the adjacent supplement of angle n.
17. Draw three angles a, b, and c, so that each angle is adjacent to the other two angles.
18. Draw two unequal angles a and b that have a common vertex but no common side.
19. Draw two angles a and b so that the sides of angle a and the sides of angle b form two straight lines.
20. Draw two angles that have a common vertex and a common side but are not adjacent. Name the two angles.

2

The Theorem

Proving Angle Relations

SUPPLEMENTARY AND COMPLEMENTARY ANGLES

The Straight Angle

39. In Chapter 1 we learned that a theorem is a proposition proved from assumptions or from other theorems. In this chapter, we shall prove theorems about the size of certain angles and the direction of their sides.

To compare the sizes of angles we need a standard unit of measurement. The standard unit for measuring an angle is the *degree*. In measuring an angle we find the number of times it contains an angle of 1°. An angle of 1° is $\frac{1}{360}$ of a complete rotation or $\frac{1}{180}$ of a straight angle. We have taken for granted that the size of a degree is the same on all protractors. Let us now state this clearly as an assumption. We shall first assume that a straight angle is the same size the world over. Then, since a straight angle is too large for measuring most angles, we shall divide it into 180 equal parts and call each part a *degree*.

40. Postulate. All straight angles are equal.

41. Definition. A degree is an angle that is $\frac{1}{180}$ of a straight angle.

Usually, the degree is subdivided into 60 equal parts, called *minutes;* and each minute is divided into 60 equal parts, called *seconds*. Thus, we could have an angle of 32° 20′ 30″—read 32 degrees, 20 minutes, and 30 seconds.

Supplementary Angles

42. Supplementary angles are two angles whose sum is a straight angle. It is assumed that all straight angles are equal. If two angles are equal, then the supplement of one angle must be equal to the supplement of the other angle. This proposition is a theorem because it depends on propositions accepted previously. To see exactly how this theorem depends on some earlier propositions, and also to learn how to write out a proof in geometry, study the form that follows.

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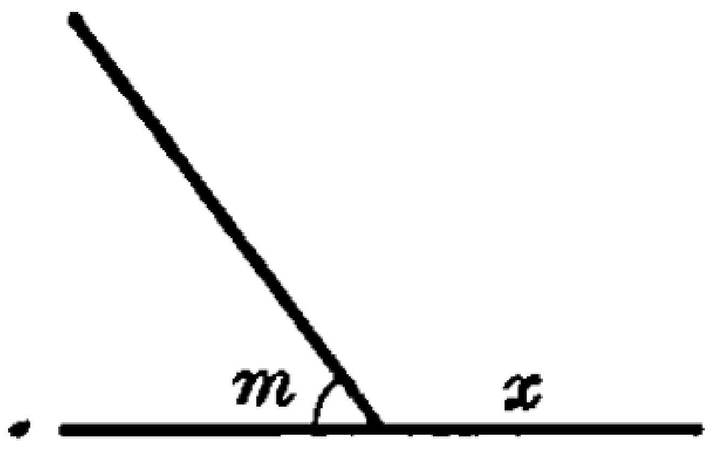
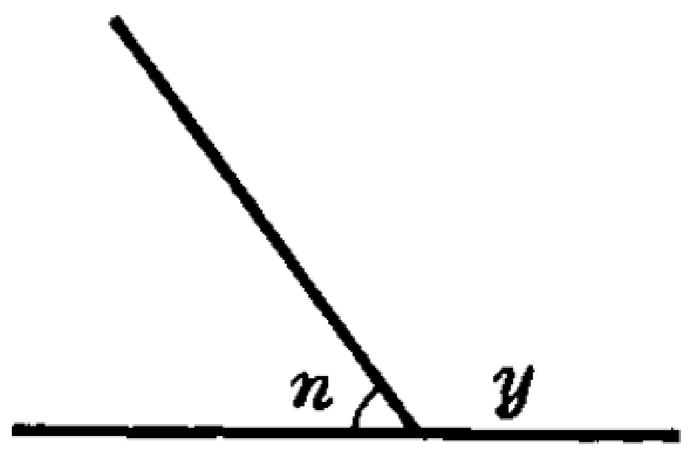
43. THEOREM. Supplements of equal angles are equal.





Given: $\angle m = \angle n$
$\angle x$ the sup. of $\angle m$
$\angle y$ the sup. of $\angle n$
Prove: $\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $\angle x$ the sup. of $\angle m$ $\angle y$ the sup. of $\angle n$	1. Given.
2. $\angle m + \angle x =$ a st. $\angle$ $\angle n + \angle y =$ a st. $\angle$	2. Def. sup. $\angle$s.
3. $\angle m + \angle x = \angle n + \angle y$	3. All st. $\angle$s are $=$.
4. $\angle m = \angle n$	4. Given.
5. $\angle x = \angle y$	5. Subt. Ax.

Note carefully how this theorem is proved so that you can follow it as a model in writing out your own proofs. You are asked to prove that if two angles are equal, then their supplements are equal. Think of the problem as follows: Two equal angles and their supplements are given. Any two equal angles, such as $\angle m$ and $\angle n$, are drawn. Then $\angle x$ is drawn so that it is the supplement of $\angle m$, and $\angle y$ so that it is the supplement of $\angle n$. These three conditions are written down as given. Then the conclusion is set down—what must be proved—$\angle x = \angle y$. Finally, using the given conditions and certain propositions that have been accepted earlier, the conclusion is proved.

The next proposition is a special case of the theorem just proved.

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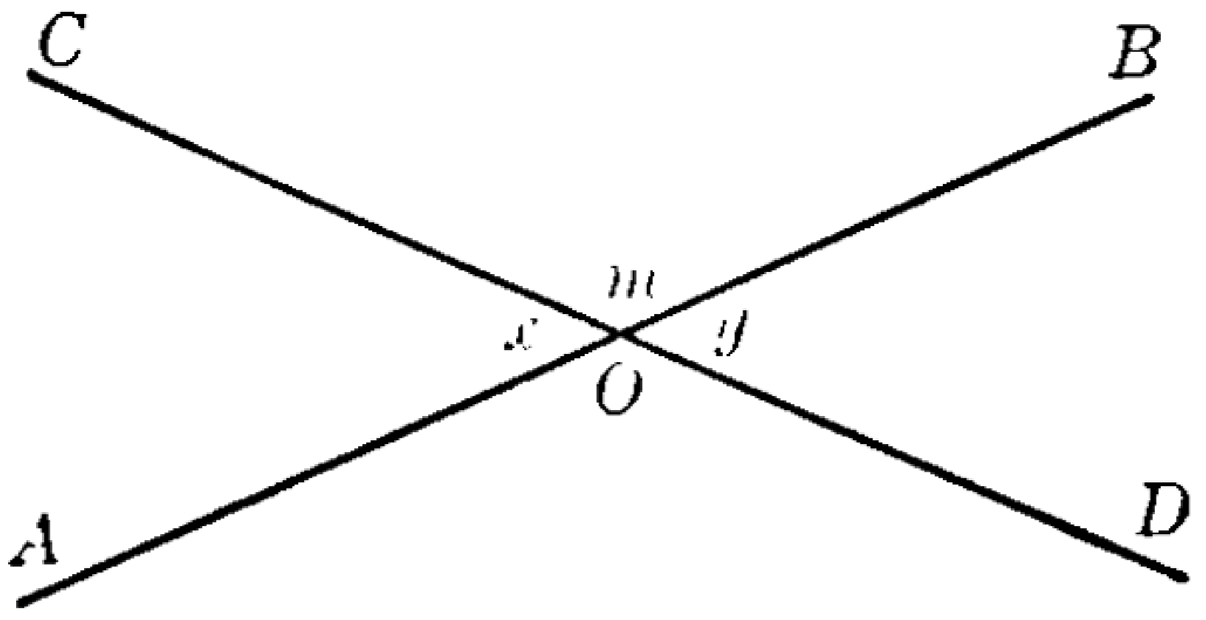
44. THEOREM. Supplements of the same angle are equal.



Given: $\angle x$ the sup. of $\angle m$
$\angle y$ the sup. of $\angle m$
Prove: $\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $\angle x$ the sup. of $\angle m$ $\angle y$ the sup. of $\angle m$	1. Given.
2. $\angle m + \angle x =$ a st. $\angle$ $\angle m + \angle y =$ a st. $\angle$	2. Def. sup. $\angle$s.
3. $\angle m + \angle x = \angle m + \angle y$	3. All st. $\angle$s are $=$.
4. $\angle m = \angle m$	4. Identity.
5. $\angle x = \angle y$	5. Subt. Ax.

Compare this proof with the proof of the theorem in § 43. You will see that the only difference is in Statement 4, where two identical angles instead of two equal angles are subtracted. If a proposition follows easily from another proposition, it is often called a *corollary* rather than a theorem, and the proof is not written out in detail.

45. Definition. A corollary is a theorem that is easily proved from another proposition.

In general, a corollary is either a special case or a simple extension of another proposition. We shall treat as corollaries most of the propositions whose proofs are so simple that there is no need to write them out in detail. You may, if you prefer, call the preceding theorem a corollary.

In the diagram of the preceding theorem, why are AOB and COD straight lines? How are the angles x and y related as to position? What is true about the sizes of two angles, such as x and y? Can you state this as a theorem? Is it easy enough for you so that you can call it a corollary?

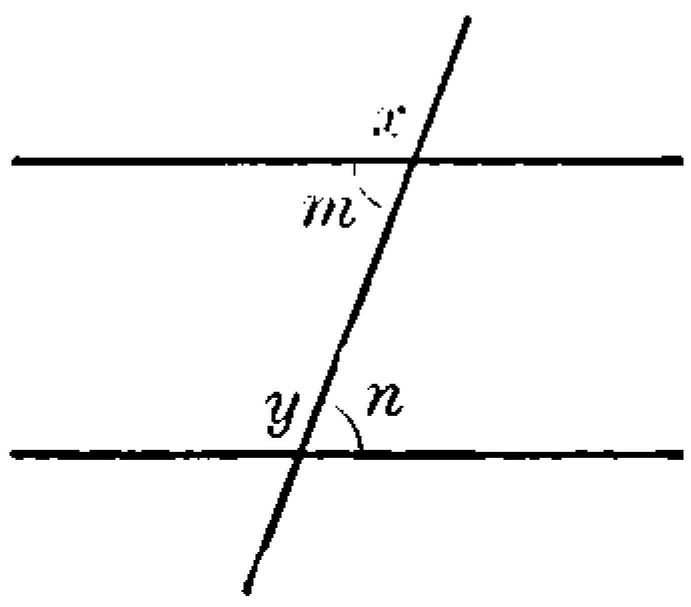
46. Corollary. Vertical angles are equal.

Theorems and corollaries may be used to prove other propositions. Make a practice of using the new theorems whenever possible, even though at first it may seem easier not to. Using the new theorems will often shorten the work and will always help you to understand the theorems better.

EXERCISES

In the following exercises, assume that lines given in the figure are straight lines.

Illustrative Example



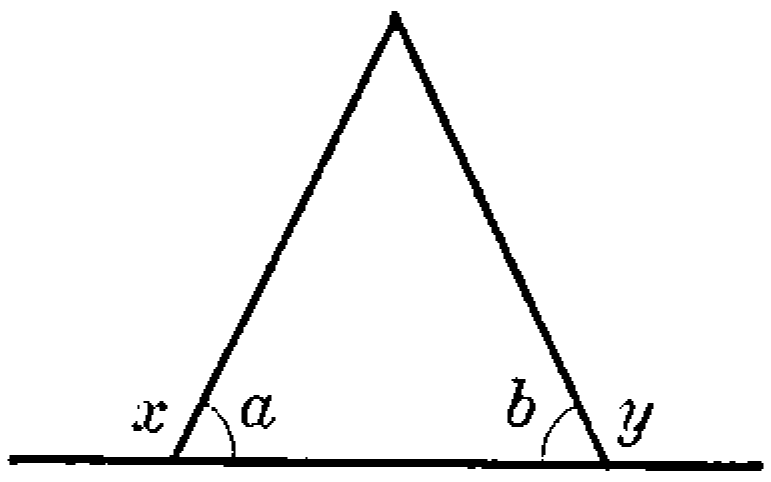
Given:
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $\angle x + \angle m =$ a st. $\angle$ $\angle y + \angle n =$ a st. $\angle$	1. Def. st. $\angle$.
2. $\angle x$ the sup. of $\angle m$ $\angle y$ the sup. of $\angle n$	2. Def. sup. $\measuredangle$.
3. $\angle m = \angle n$	3. Given.
4. $\angle x = \angle y$	4. Sups. of $= \measuredangle$ are $=$.

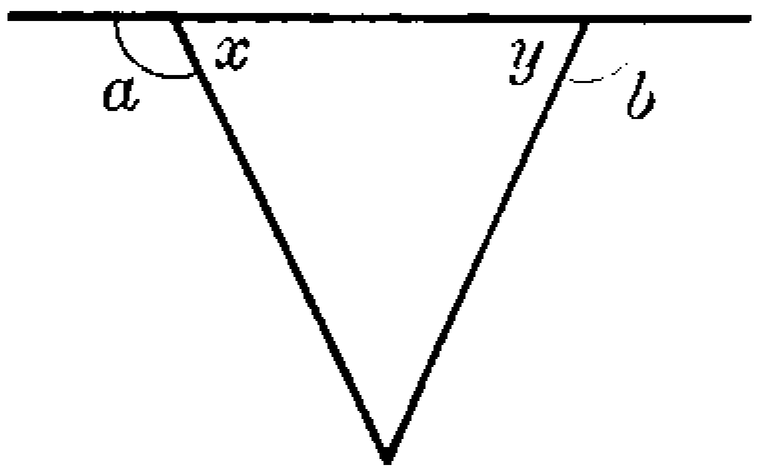
Explain the last statement of the proof orally as follows: "Supplements" (point to $\angle x$ and $\angle y$) "of equal angles" (point to $\angle m$ and $\angle n$; also point out that $\angle x$ and $\angle m$ form a straight angle and that $\angle y$ and $\angle n$ form a straight angle) "are equal."

1.



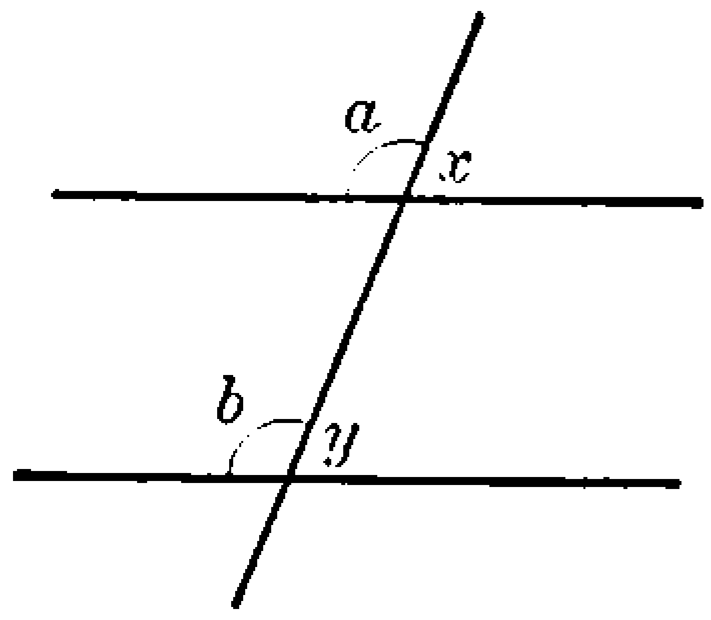
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

2.



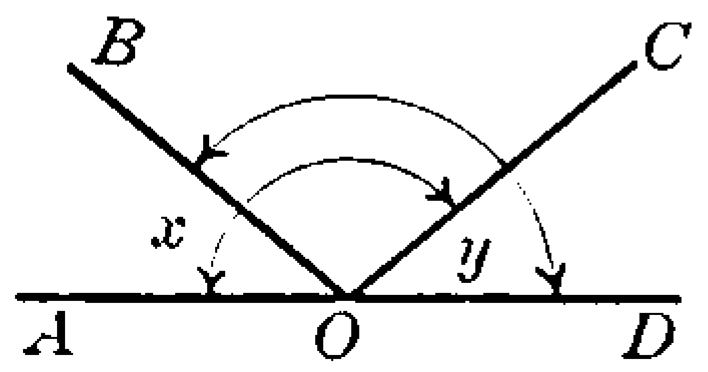
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

3.



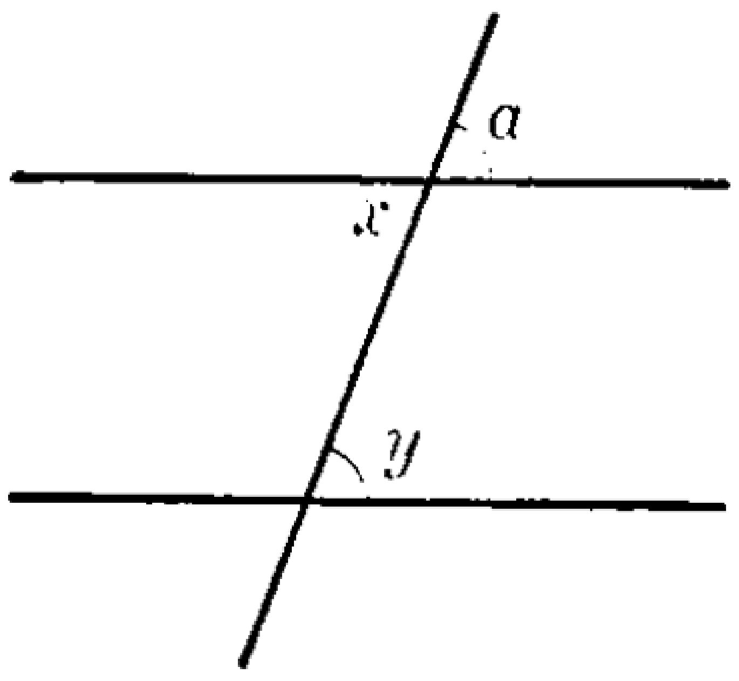
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

4.



Given:
$\angle AOC = \angle BOD$
Prove:
$\angle x = \angle y$

Illustrative Example

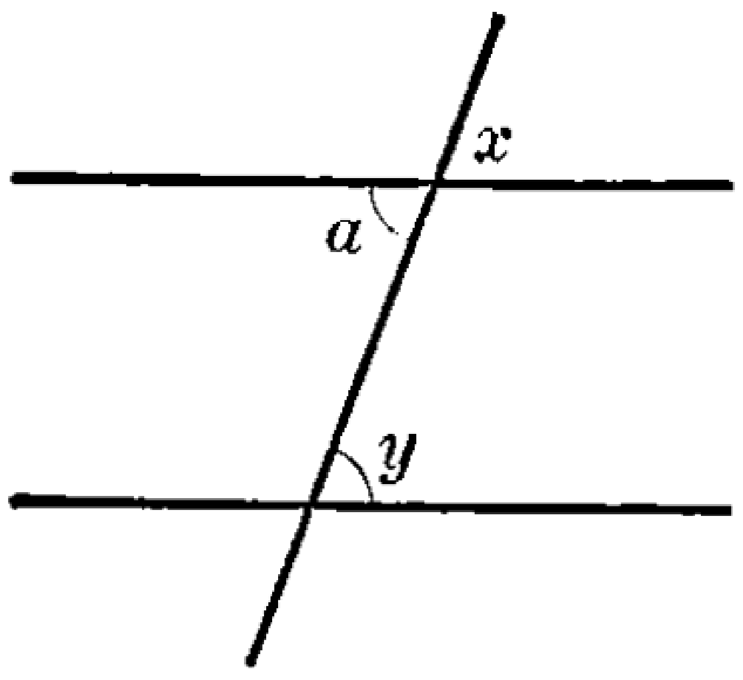


Given:
$\angle a = \angle y$
Prove:
$\angle x = \angle y$

PROOF

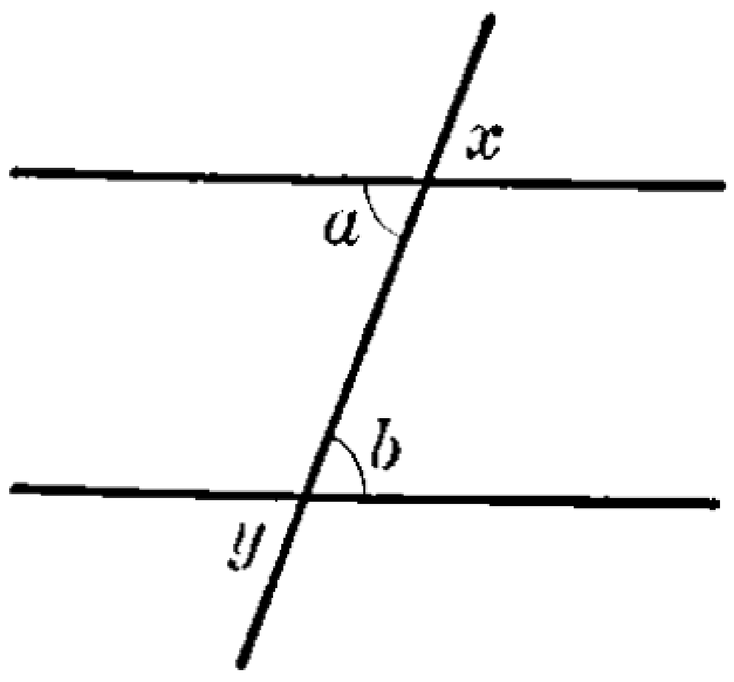
Statements	*Reasons*
1. $\angle a = \angle x$	1. Vert. ∡ are =.
2. $\angle a = \angle y$	2. Given.
3. $\overset{\angle x}{(\angle a)} = \angle y$	3. Subs. Ax.

5.



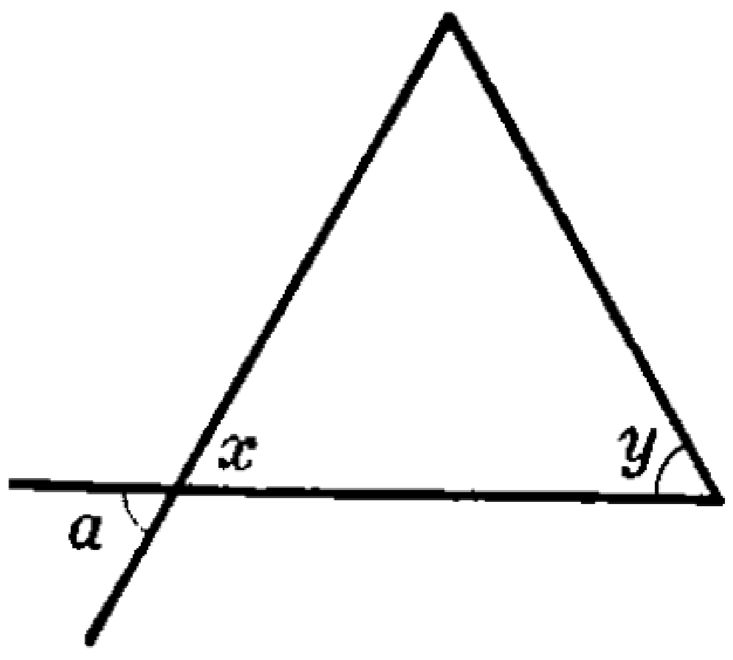
Given:
$\angle a = \angle y$
Prove:
$\angle x = \angle y$

6.



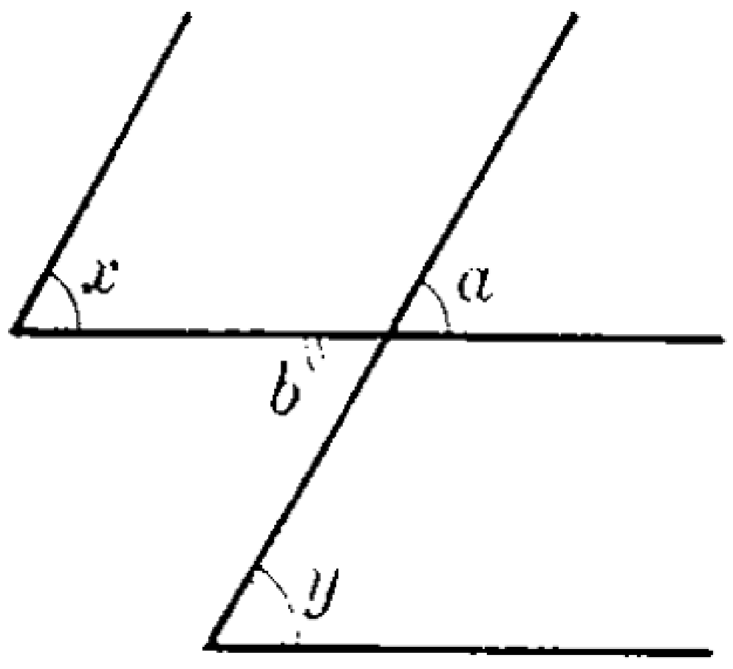
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

7.



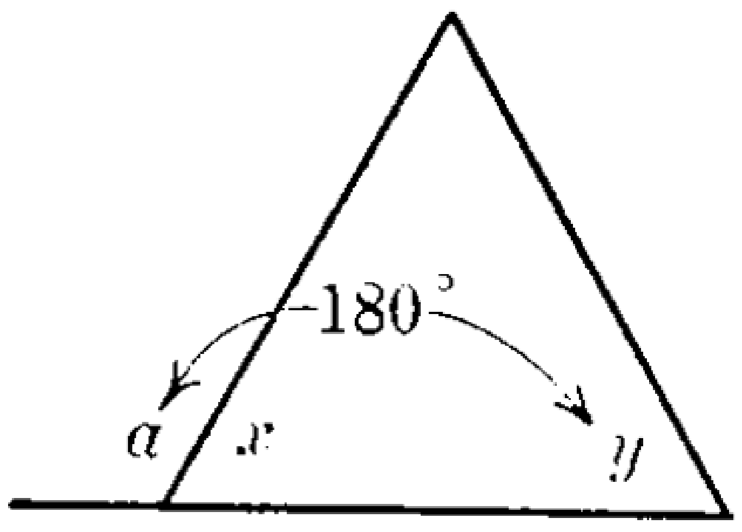
Given:
$\angle a = \angle y$
Prove:
$\angle x = \angle y$

8.



Given:
$\angle x = \angle a$
$\angle b = \angle y$
Prove:
$\angle x = \angle y$

Illustrative Example

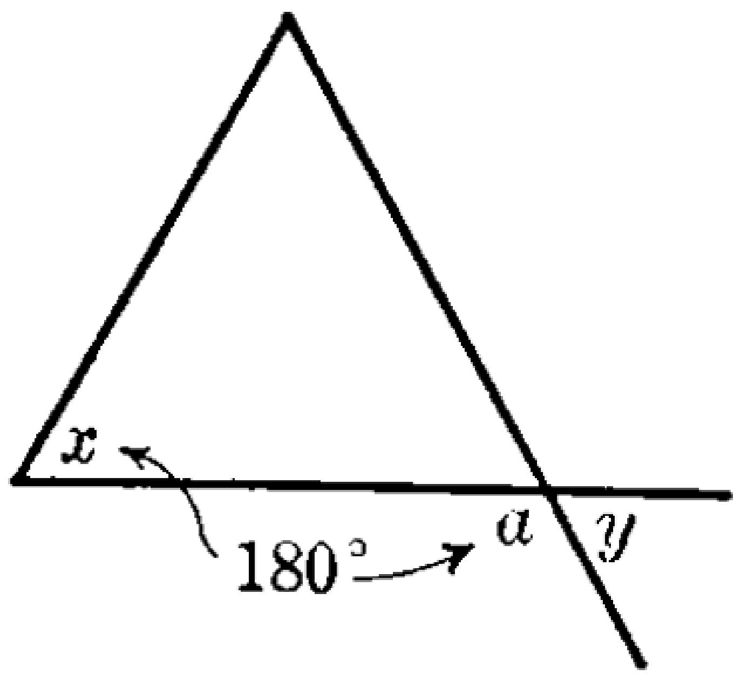


Given:
$\angle a + \angle y = 180°$
Prove:
$\angle x = \angle y$

PROOF

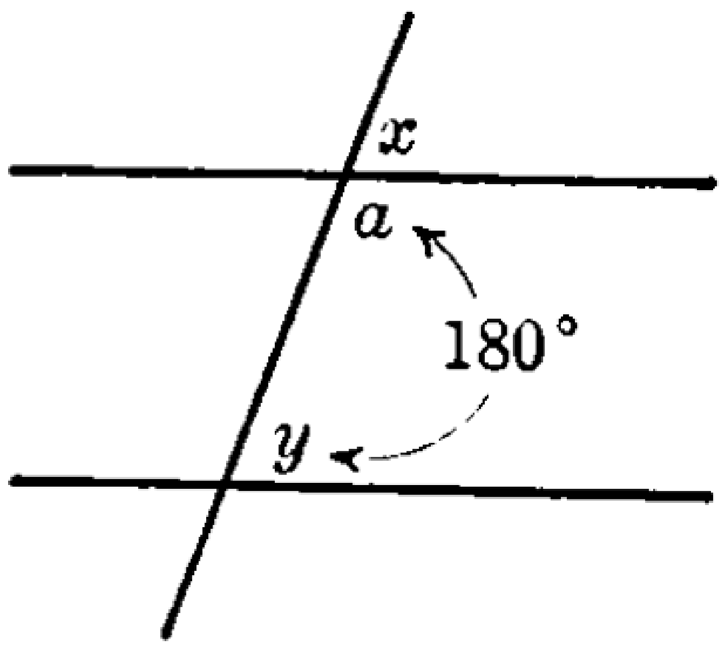
Statements	*Reasons*
1. $\angle a + \angle y = 180°$	1. Given.
2. $\angle a + \angle x = 180°$	2. Def. st. ∠.
3. $\angle y$ the sup. of $\angle a$ $\angle x$ the sup. of $\angle a$	3. Def. sup. ∡.
4. $\angle x = \angle y$	4. Sups. of the same ∠ are =.

9.



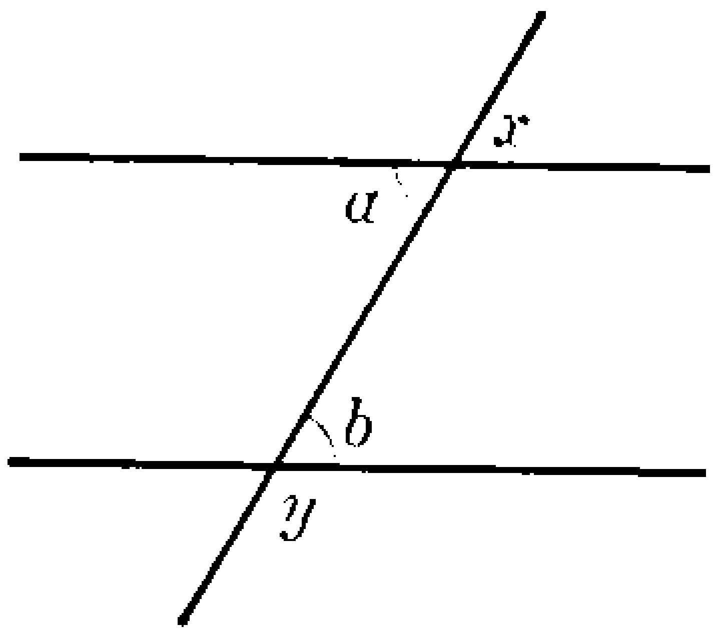
Given:
$\angle a + \angle x = 180°$
Prove:
$\angle x = \angle y$

10.



Given:
$\angle a + \angle y = 180°$
Prove:
$\angle x = \angle y$

Illustrative Example

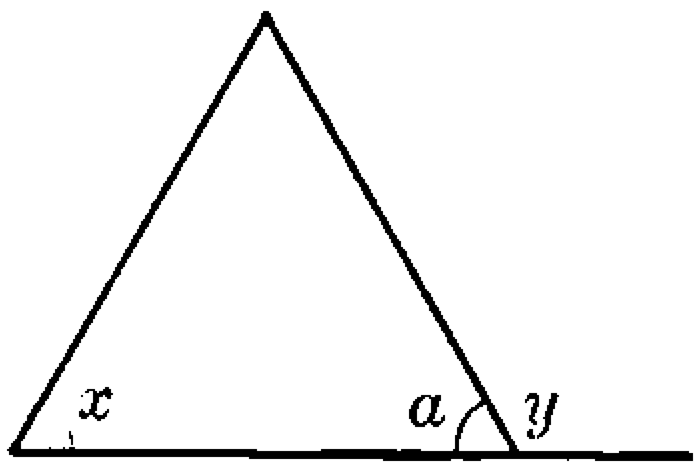


Given:
$\angle a = \angle b$
Prove:
$\angle x + \angle y = 180°$

PROOF

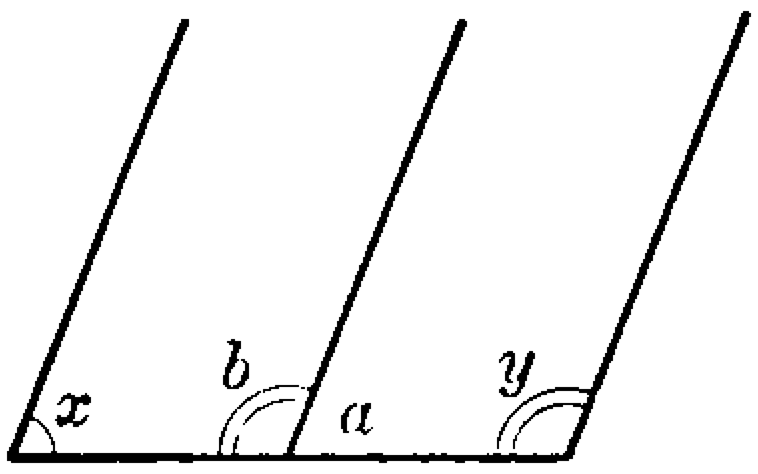
Statements	*Reasons*
1. $\angle b + \angle y = 180°$	1. Def. st. $\angle$.
2. $\angle a = \angle b$	2. Given.
3. $\angle x = \angle a$	3. Vert. $\measuredangle$ are =.
4. $\overset{\angle x}{\overset{(\angle a)}{(\angle b)}} + \angle y = 180°$	4. Subs. Ax.

11.



Given:
$\angle x = \angle a$
Prove:
$\angle x + \angle y = 180°$

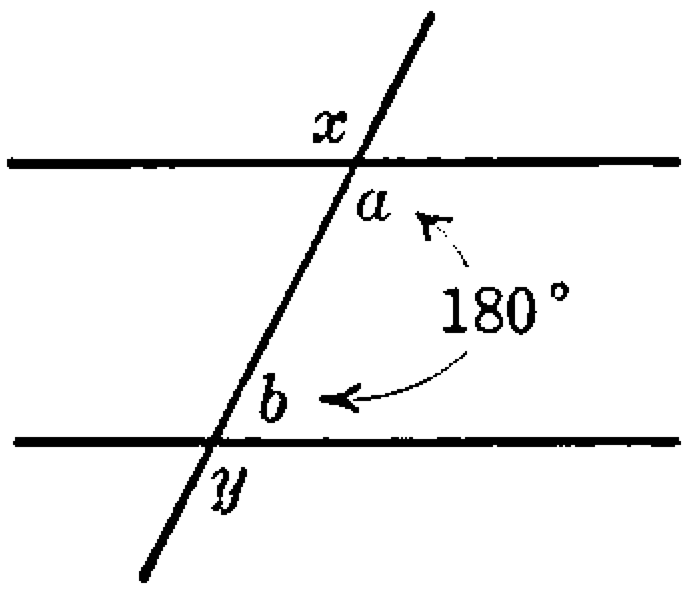
12.



Given:
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x + \angle y = 180°$

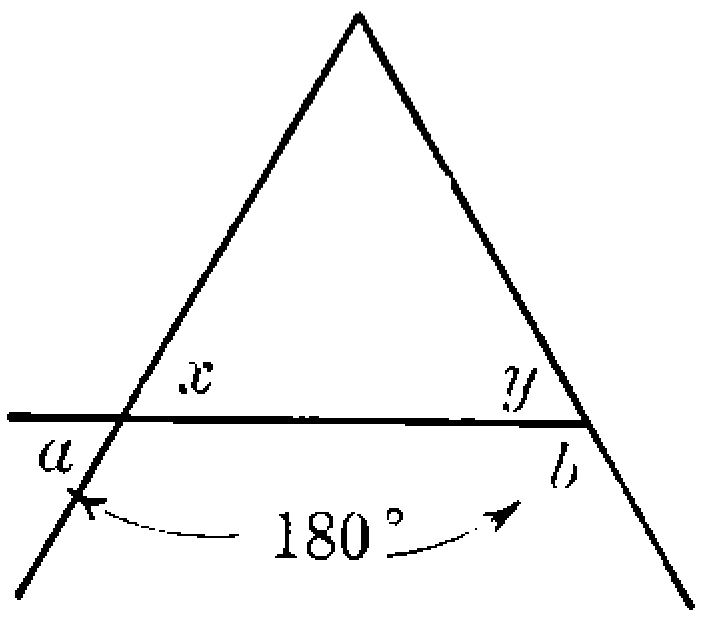
SUPERIOR WORK

S1.



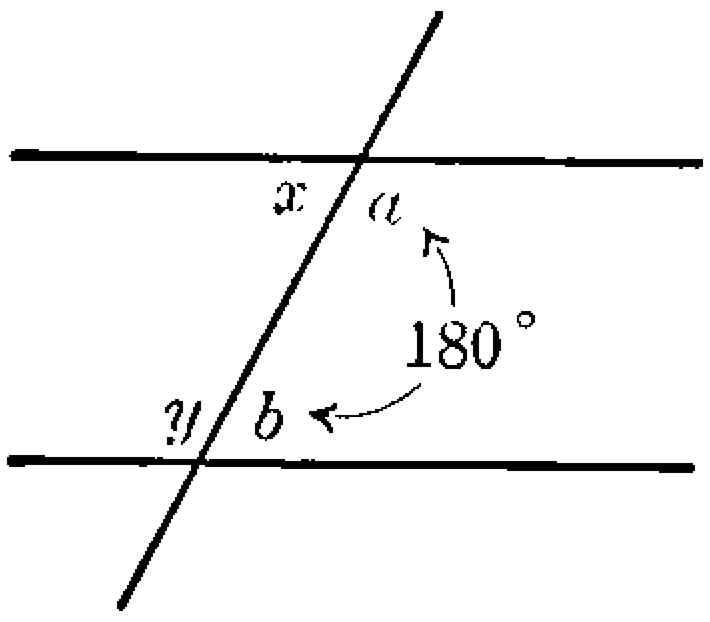
Given:
$\angle a + \angle b = 180°$
Prove:
$\angle x = \angle y$

S2.



Given:
$\angle a + \angle b = 180°$
Prove:
$\angle x = \angle y$

Illustrative Example



Given:
$\angle a + \angle b = 180°$
Prove:
$\angle x + \angle y = 180°$

PROOF

Statements	*Reasons*
1. $\angle a + \angle x = 180°$	1. Def. st. $\angle$.
2. $\angle a + \angle b = 180°$	2. Given.
3. $\angle x = \angle b$	3. Sups. of the same $\angle$ are =.
4. $\angle b + \angle y = 180°$	4. Def. st. $\angle$.
5. $\overset{\angle x}{(\angle b)} + \angle y = 180°$	5. Subs. Ax.

S3. Prove the Illustrative Example above by using a different method.

S4. Copy the figure for Ex. S2 above but change x from where it is to one of its adjacent angles. What can you prove about the angles x and y? Prove it.

S5. When a straight line cuts two other straight lines, it forms with them eight angles.

(*a*) Prove that if two of these angles with different vertices are either equal or supplementary, any other two angles are either equal or supplementary.

(*b*) Repeat (*a*), drawing a figure that is as different as you can make it.

Right Angles

We have assumed that all straight angles are equal. By the division axiom, we know that halves of all straight angles are equal. But half a straight angle is a right angle. We therefore have the following corollaries.

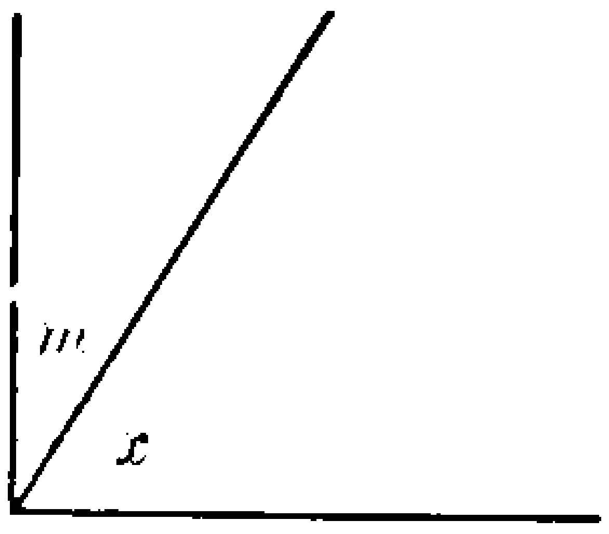
47. Corollary. **All right angles are equal.**

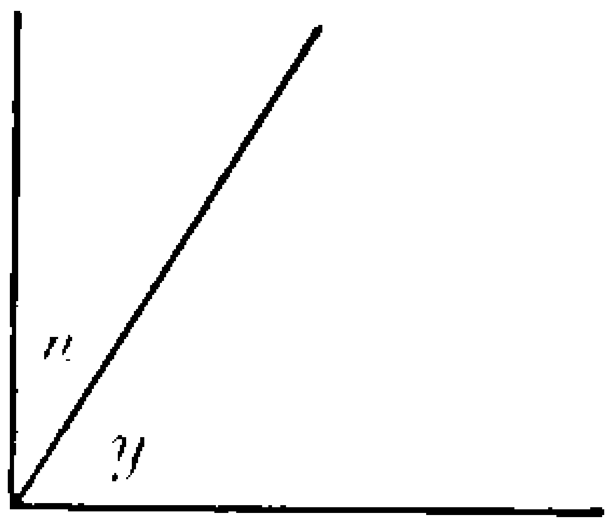
48. Corollary. **A right angle equals 90 degrees.**

Complementary Angles

From the postulate that all straight angles are equal, it has been proved that supplements of equal angles are equal. From the corollary that all right angles are equal, it is easy to prove that complements of equal angles are equal.

49. THEOREM. **Complements of equal angles are equal.**



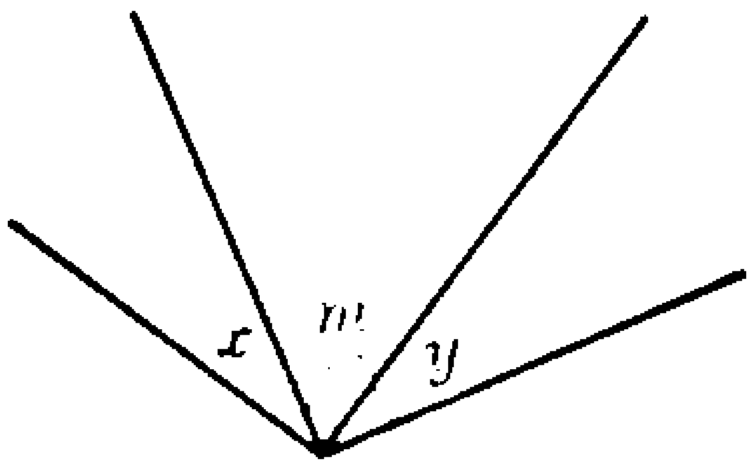


Given: $\angle m = \angle n$
$\angle x$ the comp. of $\angle m$
$\angle y$ the comp. of $\angle n$
Prove: $\angle x = \angle y$

Write out this proof, following the model in § 43.

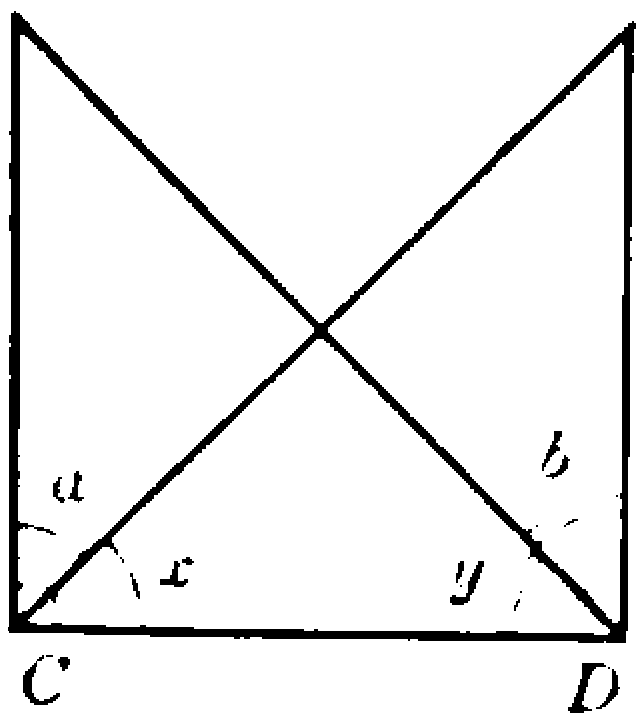
50. Corollary. **Complements of the same angle are equal.**

Given: $\angle x$ the comp. of $\angle m$
$\angle y$ the comp. of $\angle m$
Prove: $\angle x = \angle y$



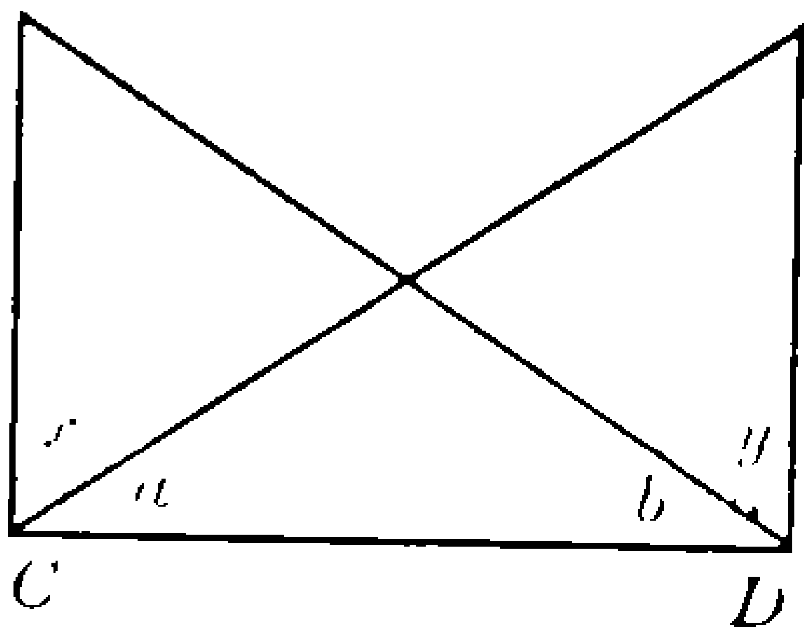
EXERCISES

1.



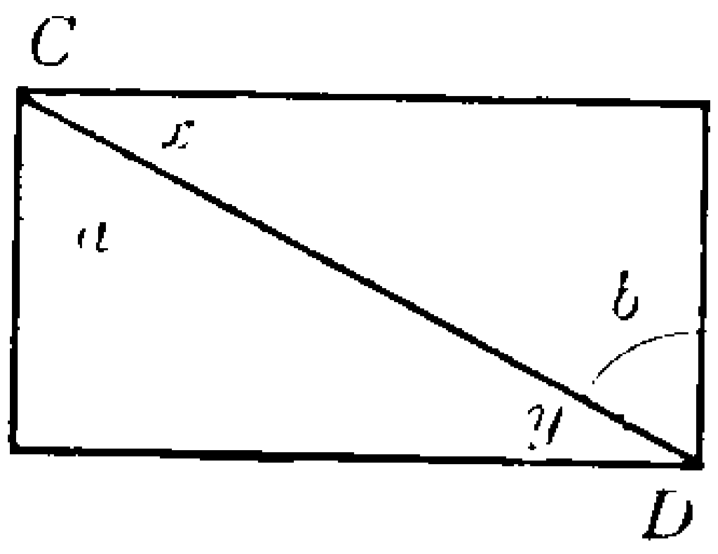
Given:
$\angle C$ a rt. $\angle$
$\angle D$ a rt. $\angle$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

2.



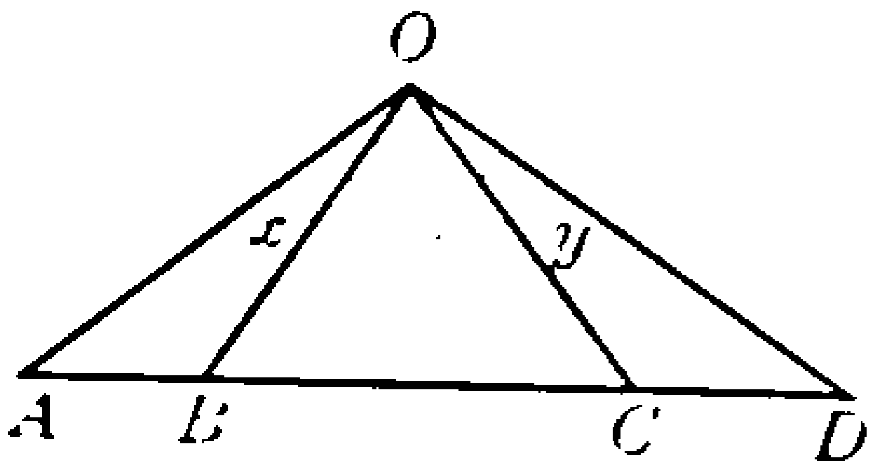
Given:
$\angle C$ a rt. $\angle$
$\angle D$ a rt. $\angle$
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

3.



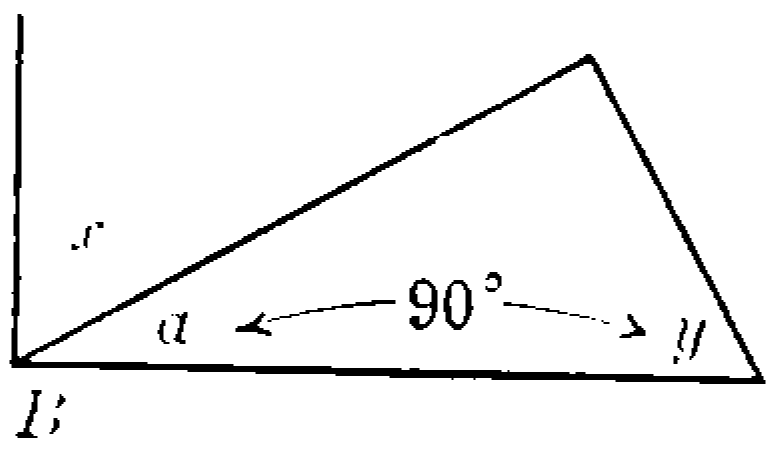
Given:
$\angle C$ a rt. $\angle$
$\angle D$ a rt. $\angle$
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

4.



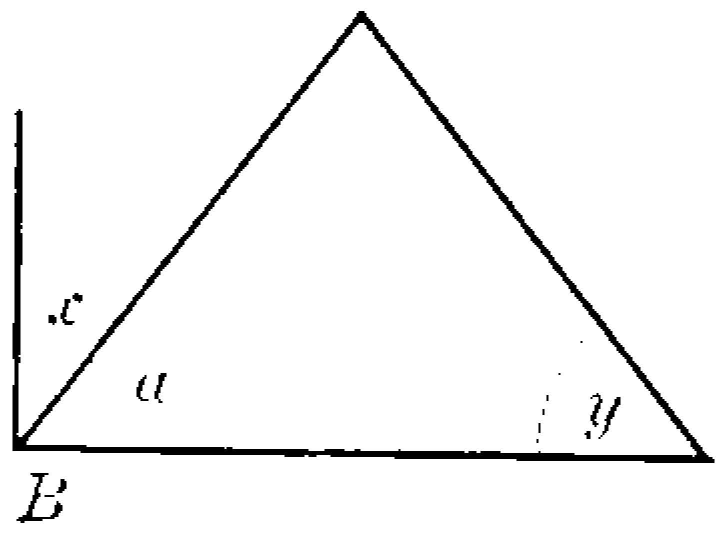
Given:
$\angle AOC$ a rt. $\angle$
$\angle BOD$ a rt. $\angle$
Prove:
$\angle x = \angle y$

5.



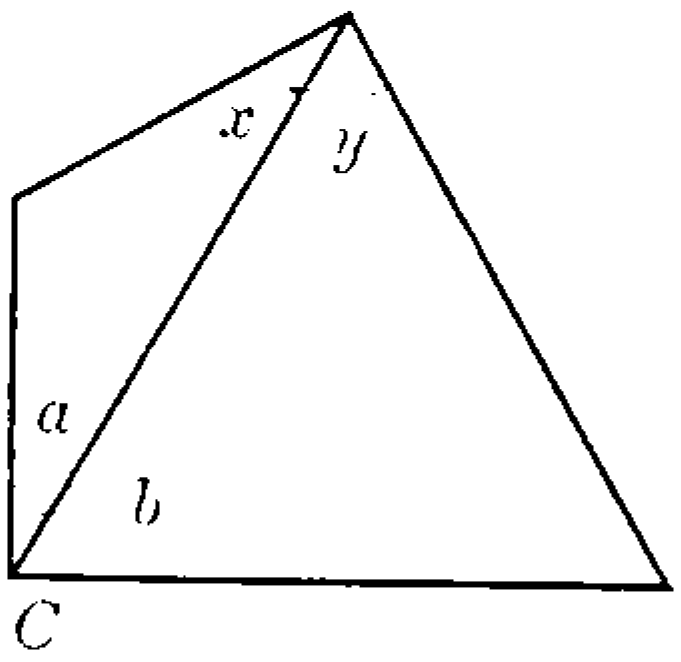
Given:
$\angle B$ a rt. $\angle$
$\angle a + \angle y = 90°$
Prove:
$\angle x = \angle y$

6.



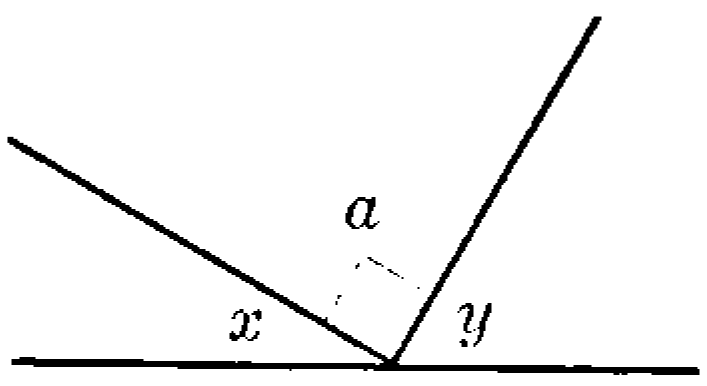
Given:
$\angle B$ a rt. $\angle$
$\angle a = \angle y$
Prove:
$\angle x + \angle y = 90°$

7.



Given:
$\angle C$ a rt. $\angle$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x + \angle y = 90°$

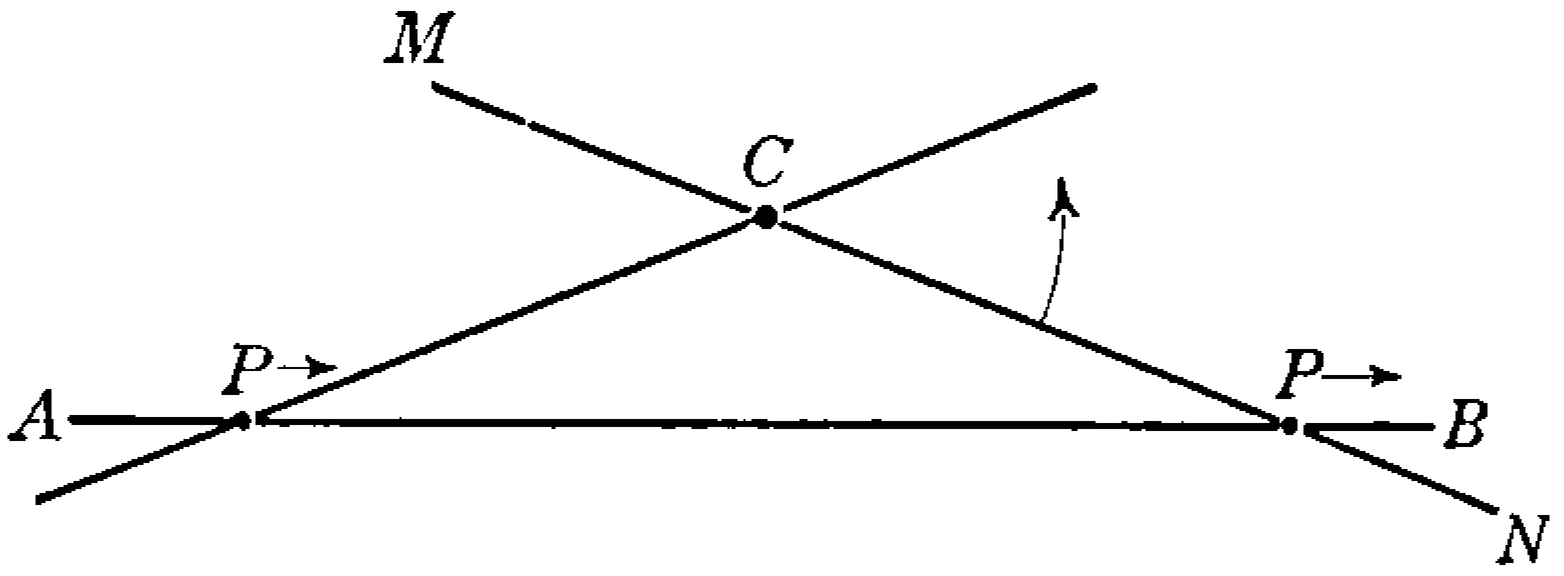
8.



Given:
$\angle a$ a rt. $\angle$
Prove:
$\angle x + \angle y = 90°$

SUPERIOR WORK

S1. Imagine that straight line *MN* rotates about *C* and intersects *AB* at *P*. We see *P* move to the right indefinitely and presently reappear from the left.



(*a*) At the moment of its reappearance from the left, does *P* exist on *AB* twice? If so, would this contradict the two-point postulate (§ 11)?

(*b*) Is there a time when *P* does not exist at all? If so, do *MN* and *AB* intersect during that time?

(*c*) If *MN* and *AB* do not intersect, we call them *parallel*. If the length of time referred to in (*b*) is but an instant, how many lines through *C* can be drawn parallel to *AB*? How many, if the time is longer?

Whether the time of "nonintersection" is only an instant or longer cannot be proved from what we have already accepted. Hence, we must have a new assumption.

PARALLEL LINES

Meaning of Parallel Lines

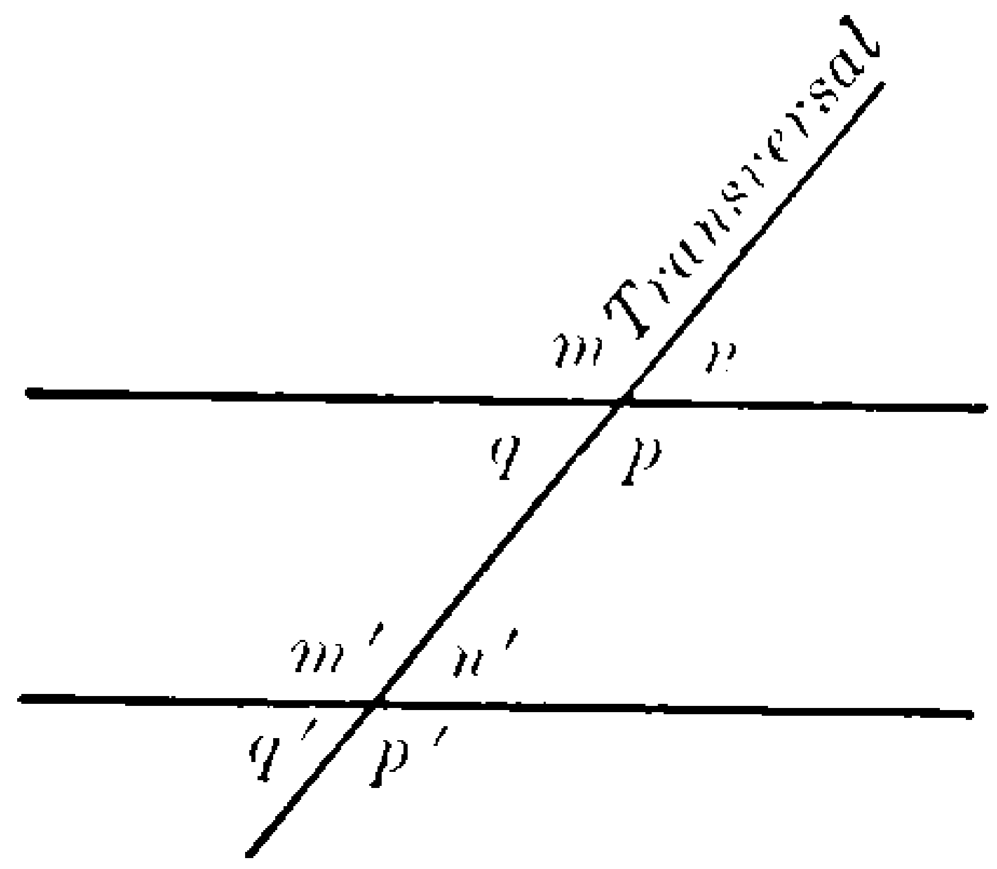
From the exercises that you have already done in this chapter, you may have seen that the sizes of certain angles have something to do with the direction of their sides. Since straight lines are infinitely long, they usually can meet and form angles. However, if two lines are in the same plane and do not meet no matter how far they are prolonged, they are called *parallel lines*. (Of course, all the figures in plane geometry are in the same plane.)

51. Definition. Parallel lines are straight lines in the same plane that do not meet no matter how far they are extended.

Angles Made by a Transversal

52. Since two parallel lines do not meet, they form no angles with each other. Thus, we cannot study the lines by comparing their angles directly. We therefore draw a third line across the two lines that we wish to study and use the angles that this third line makes with them. This third line is called a *transversal* (trans.). In order to compare the angles made by the transversal with the other lines, they are paired off as follows:

Corresponding Angles. Angles m and m' are called corresponding angles because they are in corresponding positions



—each is in the upper left-hand corner of the vertex. In the same way, angles n and n' are corresponding angles. Name the other two pairs of corresponding angles.

Alternate interior angles. Angles q and n' are called alternate interior angles because they are on alternate sides of the transversal and "inside" the figure. Name the other pair of alternate interior angles.

Interior angles on the same side of the transversal. Angles p and n' are merely described as a pair of interior angles on the same side of the transversal. Name another pair.

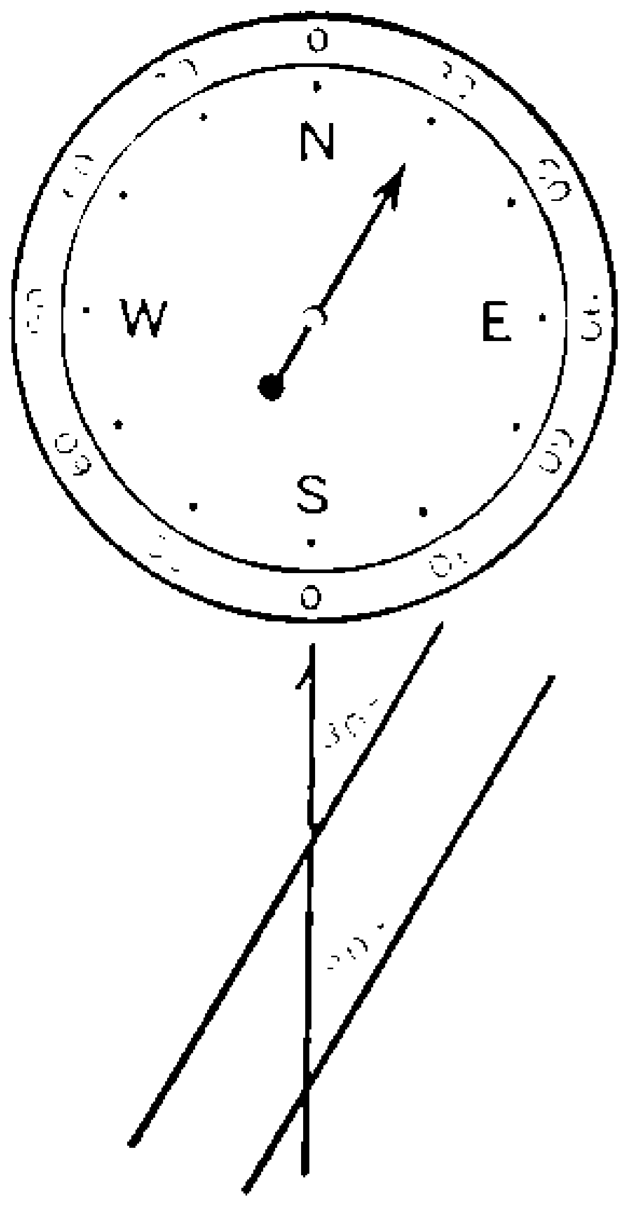
Direction of Lines

53. We often speak of lines having the same direction. How can we measure the direction of each line? If we are traveling a strange road, what will help us to find our direction?

To measure the direction of a line, we must refer to some other line—a standard line of reference. The direction of *our* line is found by measuring the angle between our line and the reference line.

In practice the north-and-south line is usually taken as fixed, and all other directions are given as angular measurements from this line. Thus the figure shows a direction of N. 30° E., from the north 30 degrees to the east.

Now suppose that two lines are both in the direction N. 30° E. It seems reasonable to say these lines are parallel. In other words, we assume that if two lines are in the same direction, they are parallel. In our figure we recognize the two 30° angles as the corresponding angles made by a transversal. We assume that if two lines are known to be parallel, then their directions are the same—no matter what reference line (transversal) is used to measure the directions.



In order to study parallel lines it is necessary to assume certain things without proof. We assume that two lines in the same direction are parallel and that the direction can be measured by any transversal. We shall state this as two postulates as follows:

54. The Parallel Postulates.

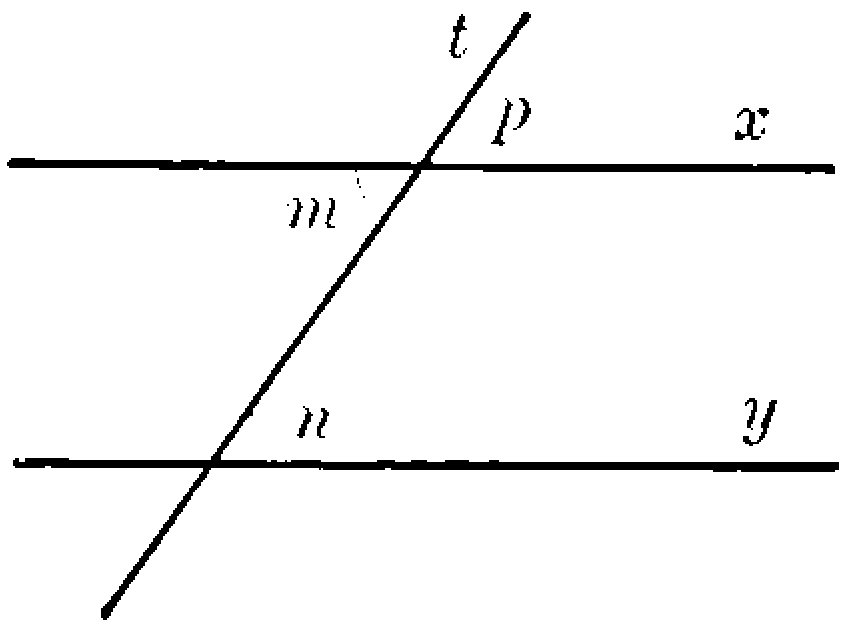
1. If two lines are cut by a transversal and a pair of corresponding angles are equal, the lines are parallel.

2. If two parallel lines are cut by a transversal, the corresponding angles are equal.

55. Corollary 1. If two lines are perpendicular to the same line, they are parallel.

56. Corollary 2. If a line is perpendicular to one of two parallel lines, then it is perpendicular to the other.

57. THEOREM. If two lines are cut by a transversal and a pair of alternate interior angles are equal, the lines are parallel.



Given: Lines x and y cut by the trans. t, making $\angle m = \angle n$

Prove: $x \parallel y$ (x is parallel to y)

PROOF

Statements	*Reasons*
1. $\angle p = \angle m$	1. Vert. ∠s are =.
2. $\angle m = \angle n$	2. Given.
3. $\angle p = \overset{\angle n}{(\angle m)}$	3. Subs. Ax.
4. $x \parallel y$	4. Corr. ∠s are =, ∥s. (See parallel postulate § 54, 1.)

Explain orally the reason for Statement 4 as follows: "If two lines" (point out lines x and y) "are cut by a transversal" (trace the line t) "and the corresponding angles" (sweep the ∠s p and n) "are equal, then the lines" (again point to x and y) "are parallel."

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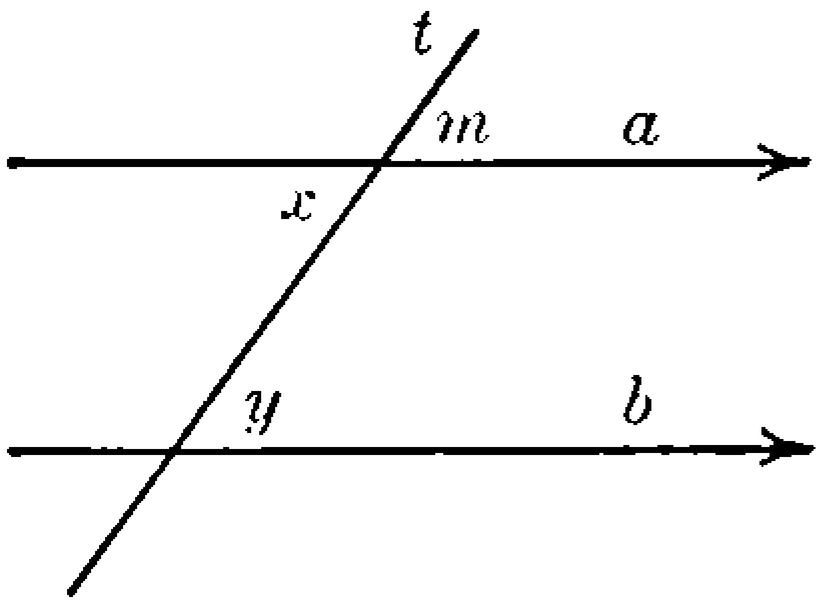
58. CONVERSE THEOREM. If two parallel lines are cut by a transversal, the alternate interior angles are equal.



Given: $a \parallel b$

Trans. t

Prove: $\angle x = \angle y$

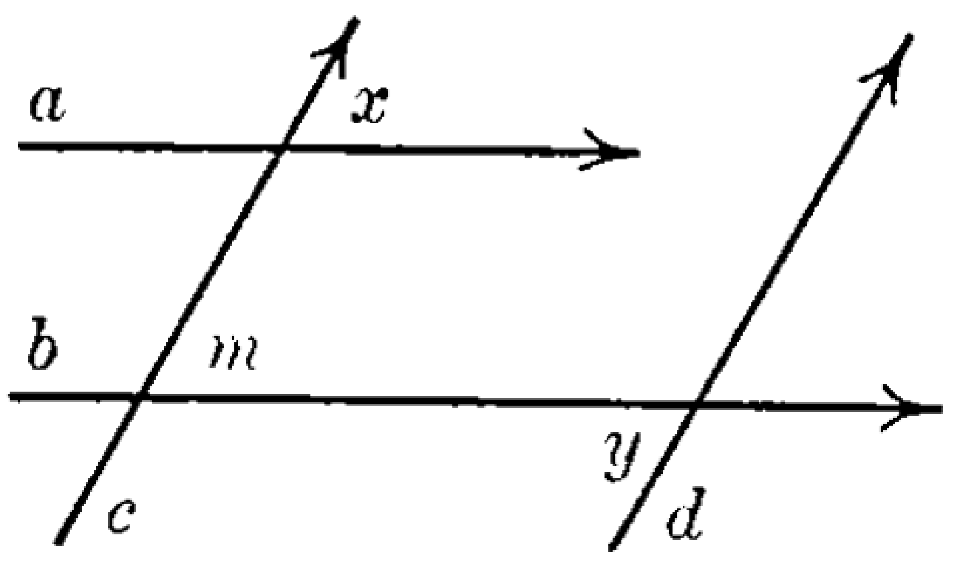
PROOF

Statements	*Reasons*
1. $a \parallel b$	1. Given.
2. $\angle m = \angle y$	2. ∥s, corr. ∠s are =. (See parallel postulate § 54, 2.)
3. $\angle m = \angle x$	3. Vert. ∠s are =.
4. $\overset{\angle x}{(\angle m)} = \angle y$	4. Subs. Ax.

Explain orally the reason for Statement 2 as follows: "If two parallel lines" (trace a and b) "are cut by a transversal" (trace t) "then the corresponding angles" (sweep ∠s m and y) "are equal."

EXERCISES

Illustrative Example



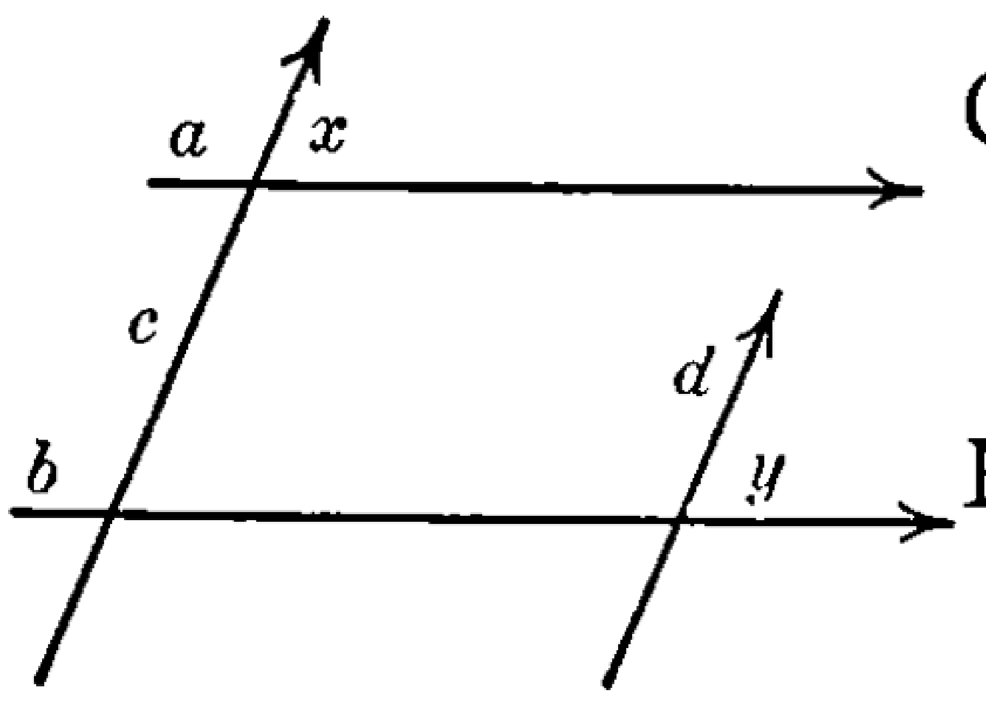
Given:
$a \parallel b$
$c \parallel d$
Prove:
$\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $a \parallel b$	1. Given.
2. $\angle x = \angle m$	2. $\parallel_s$, corr. $\angle$s are =.
3. $c \parallel d$	3. Given.
4. $\angle m = \angle y$	4. $\parallel_s$, alt. int. $\angle$s are =.
5. $\angle x = \overset{\angle y}{(\angle m)}$	5. Subs. Ax.

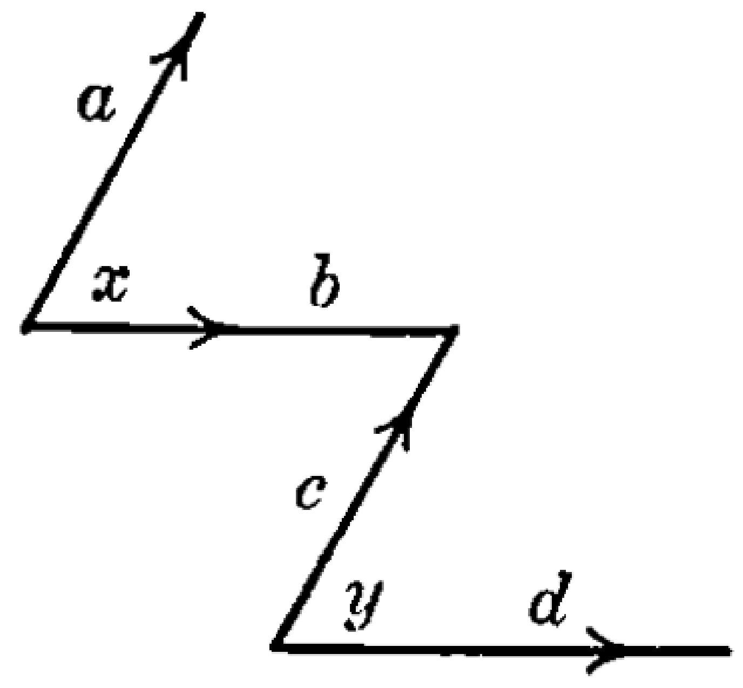
Point out how the reasons for Statements 2 and 4 apply, as explained above.

1.



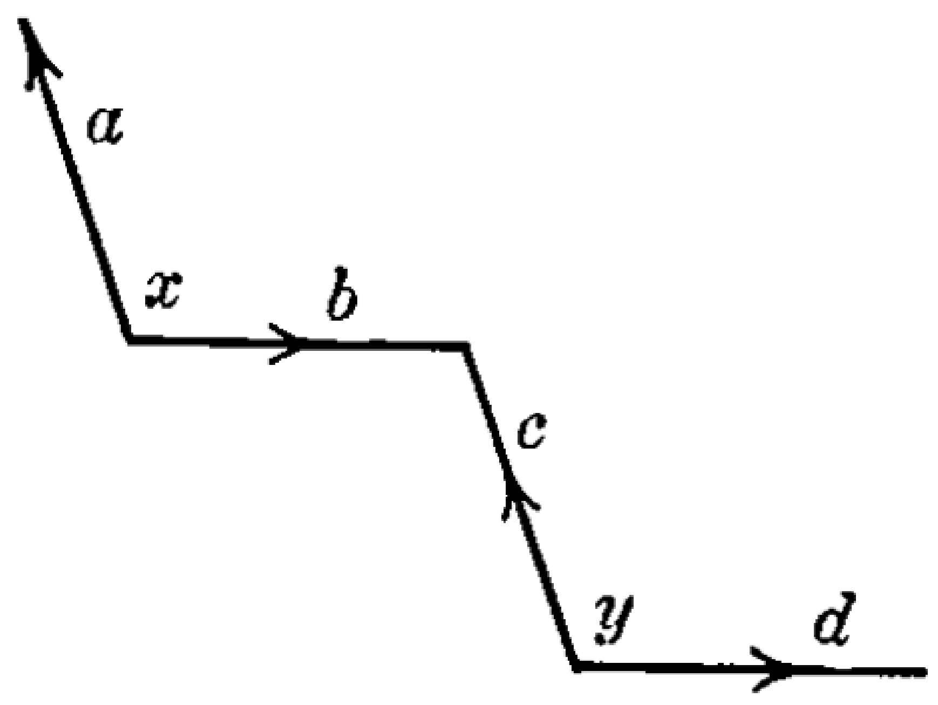
Given:
$a \parallel b$
$c \parallel d$
Prove:
$\angle x = \angle y$

2.



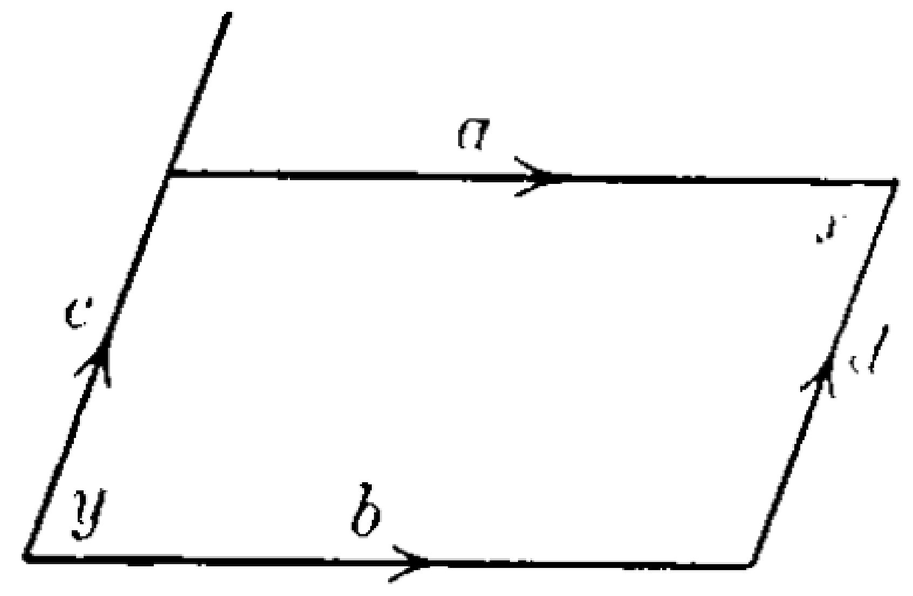
Given:
$a \parallel c$
$b \parallel d$
Prove:
$\angle x = \angle y$

3.



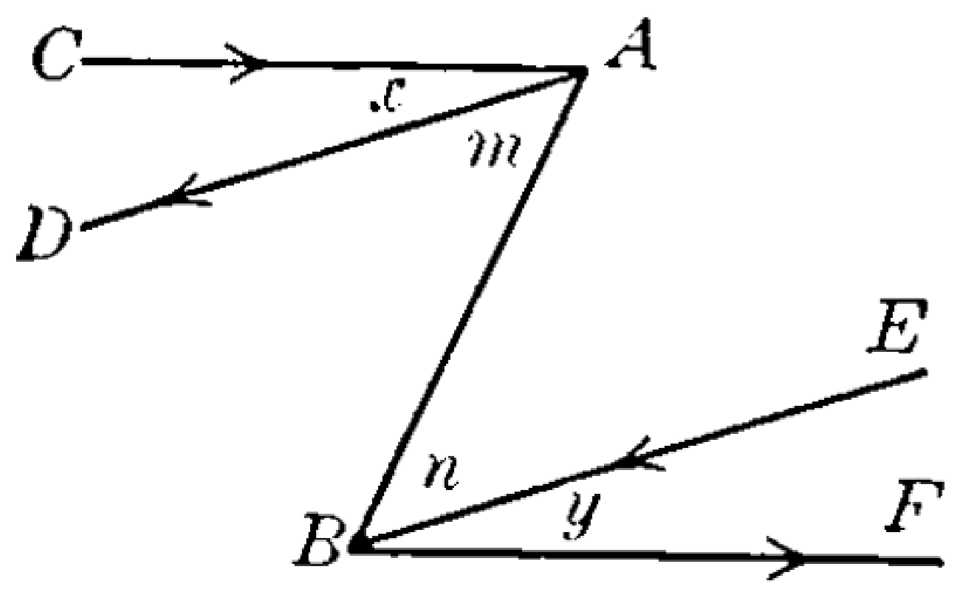
Given:
$a \parallel c$
$b \parallel d$
Prove:
$\angle x = \angle y$

4.



Given:
$a \parallel b$
$c \parallel d$
Prove:
$\angle x = \angle y$

Illustrative Example

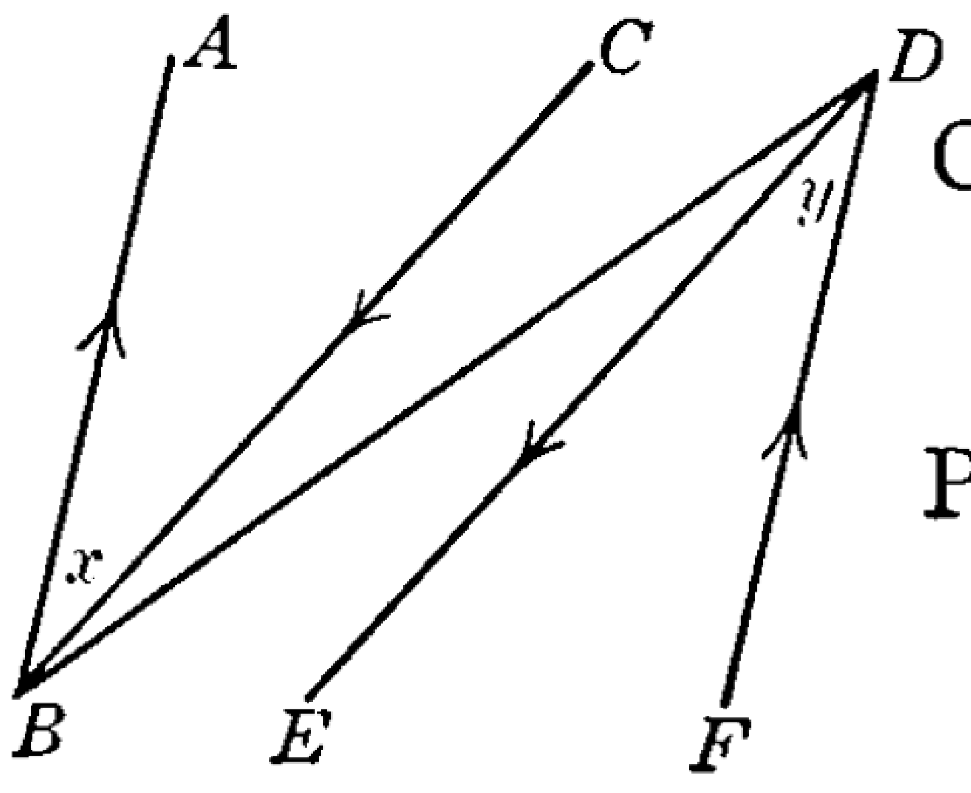


Given:
$CA \parallel BF$
$DA \parallel BE$
Prove:
$\angle x = \angle y$

PROOF

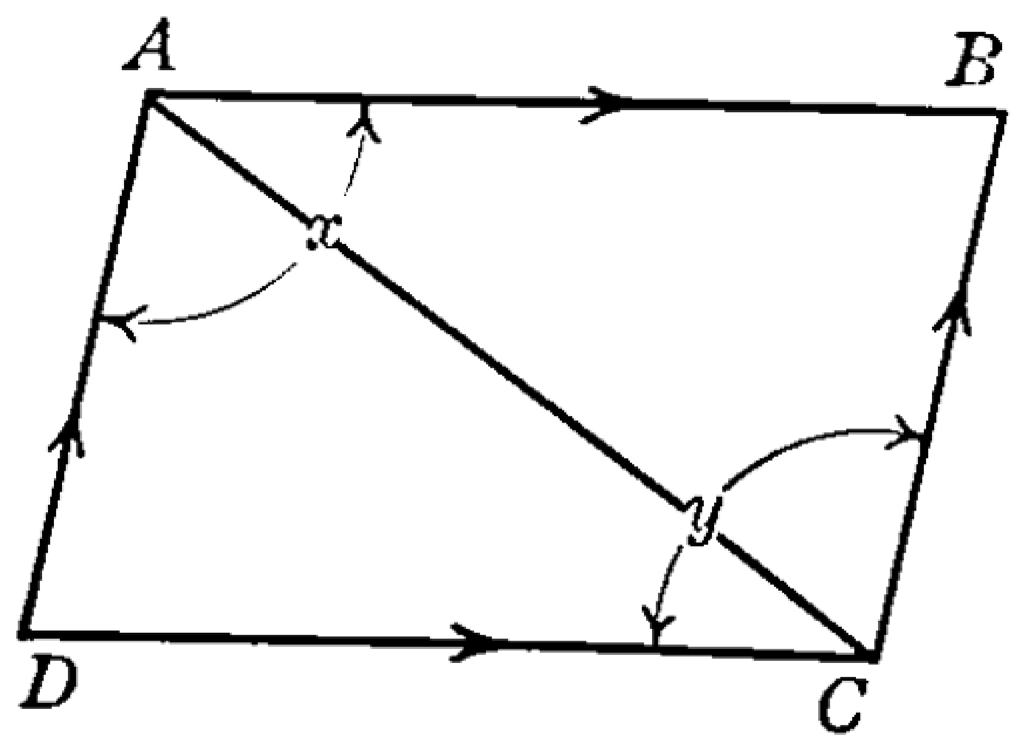
Statements	*Reasons*
1. $CA \parallel BF$	1. Given.
2. $\angle A = \angle B$	2. $\parallel_s$, alt. int. $\angle$s are =.
3. $DA \parallel BE$	3. Given.
4. $\angle m = \angle n$	4. Same as 2.
5. $\angle x = \angle y$	5. Subt. Ax.

5.



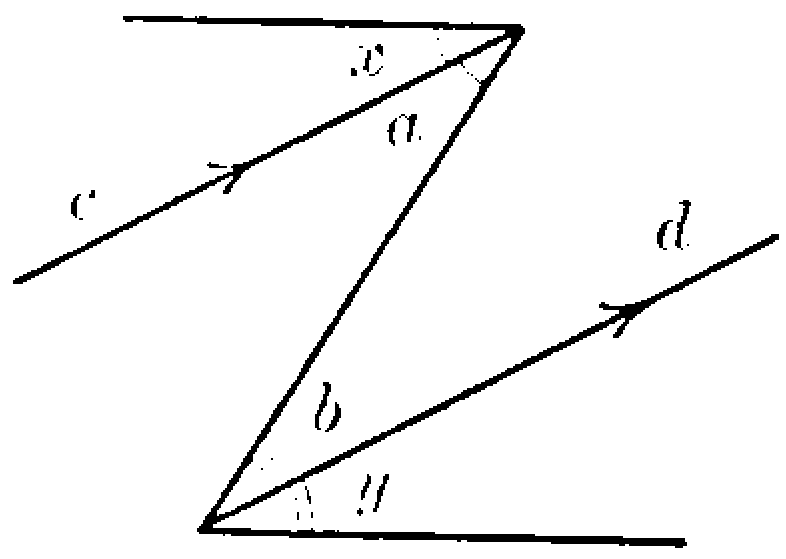
Given:
$AB \parallel DF$
$BC \parallel ED$
Prove:
$\angle x = \angle y$

6.



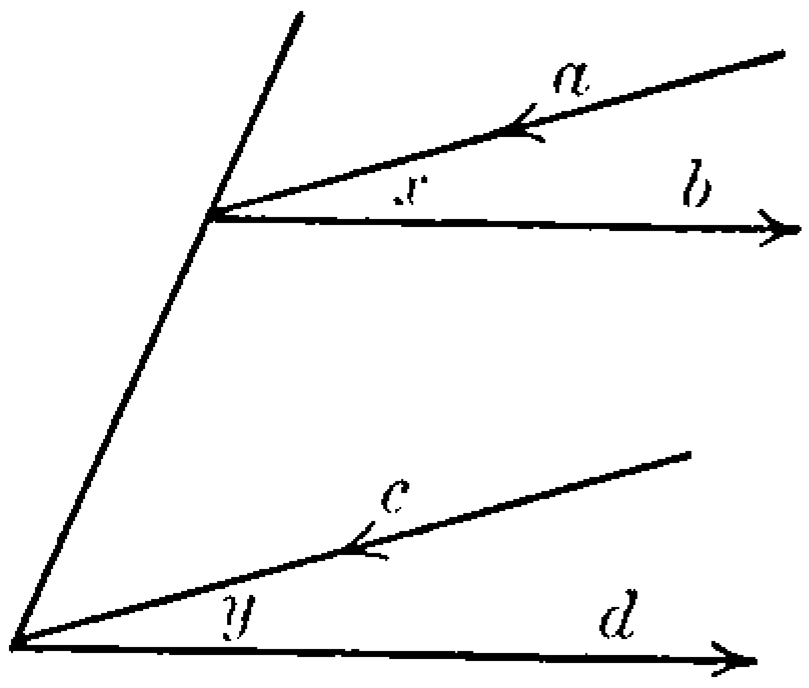
Given:
$AB \parallel DC$
$AD \parallel BC$
Prove:
$\angle x = \angle y$

7.



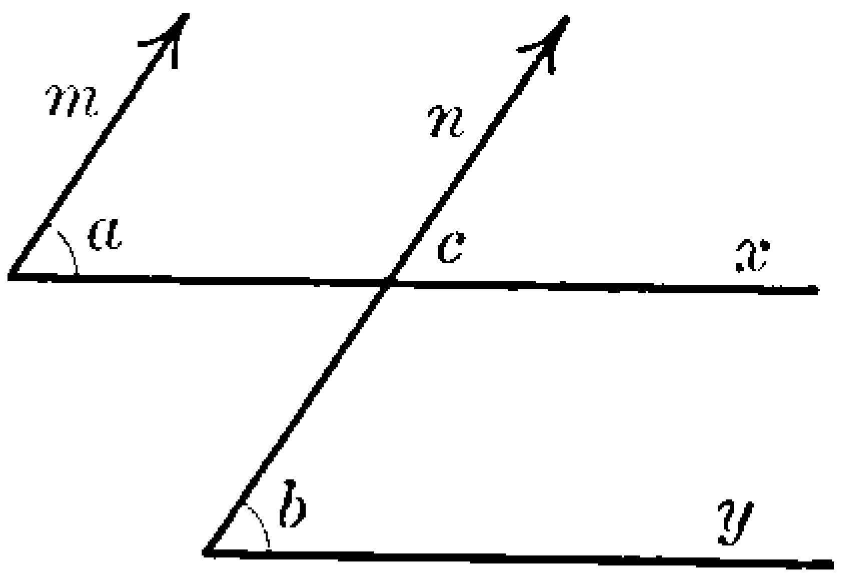
Given:
$c \parallel d$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

8.



Given:
$a \parallel c$
$b \parallel d$
Prove:
$\angle x = \angle y$

Illustrative Example

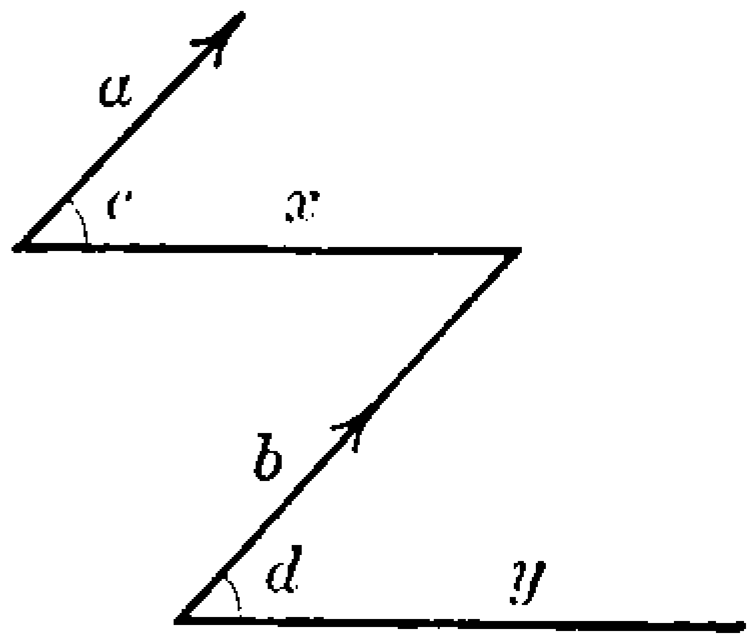


Given:
$m \parallel n$
$\angle a = \angle b$
Prove:
$x \parallel y$

PROOF

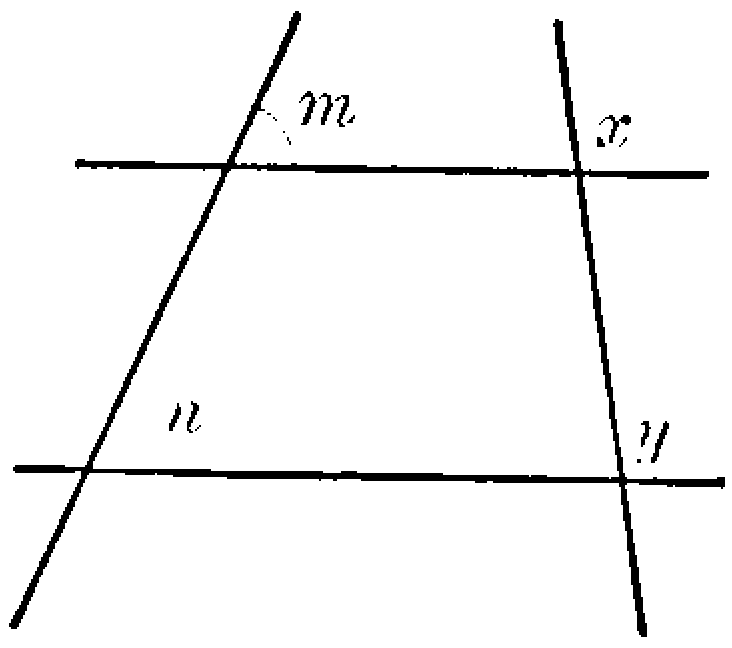
Statements	*Reasons*
1. $m \parallel n$	1. Given.
2. $\angle a = \angle c$	2. $\|_s$, corr. ∠s are =.
3. $\underset{\angle c}{\angle a} = \angle b$	3. Given.
4. $(\angle a) = \angle b$	4. Subs. Ax.
5. $x \parallel y$	5. Corr. ∠s are =, $\|_s$.

9.



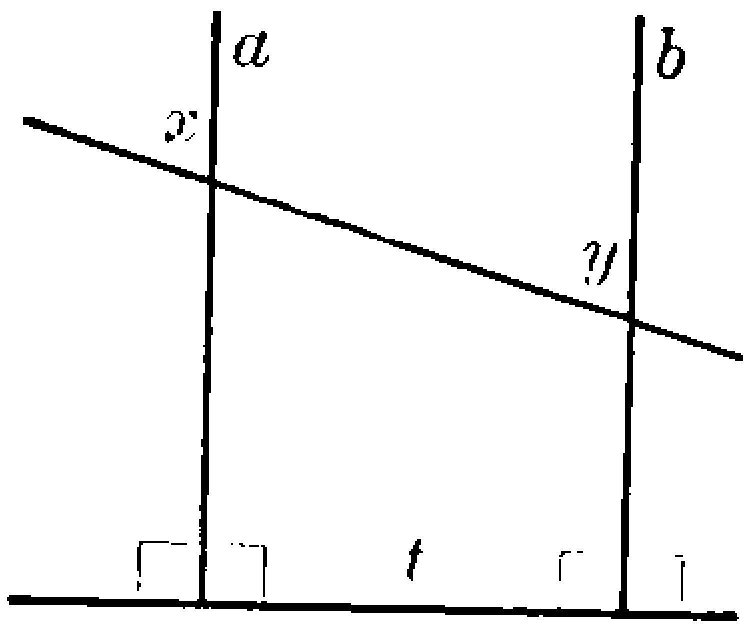
Given:
$a \parallel b$
$\angle c = \angle d$
Prove:
$x \parallel y$

10.



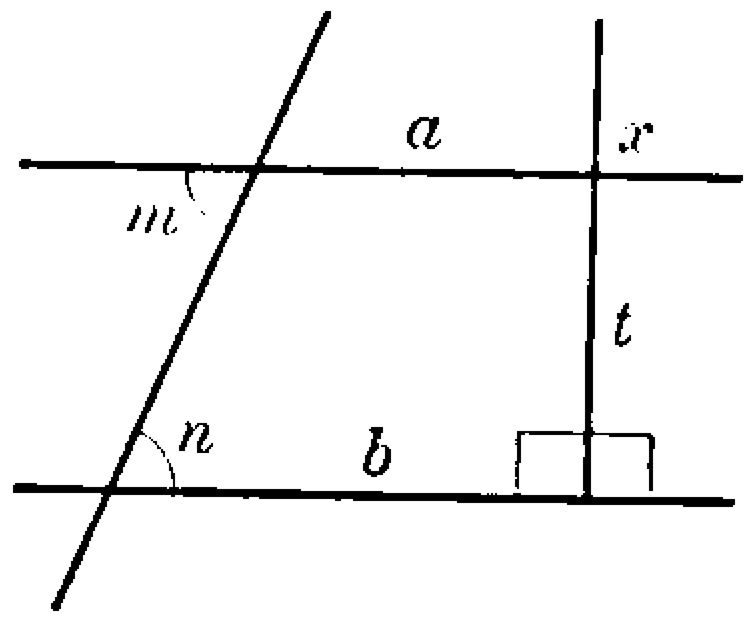
Given:
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

11.



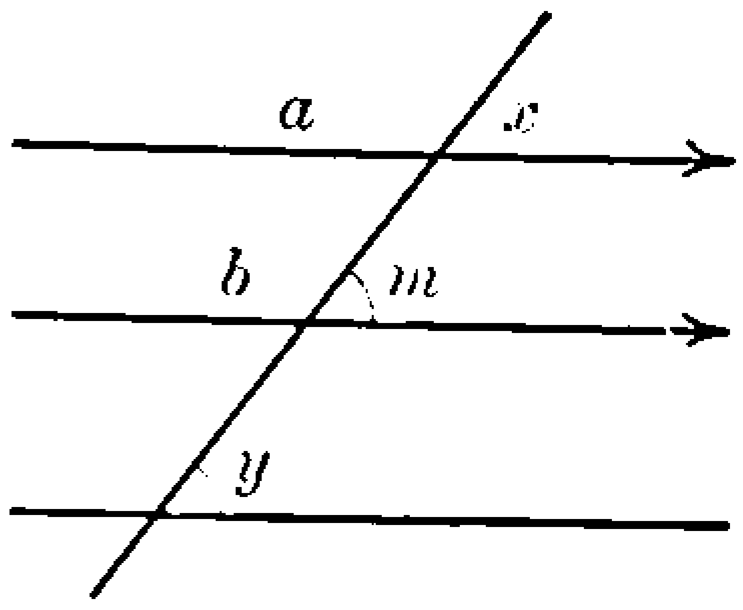
Given:
$a \perp t$
$b \perp t$
Prove:
$\angle x = \angle y$

12.



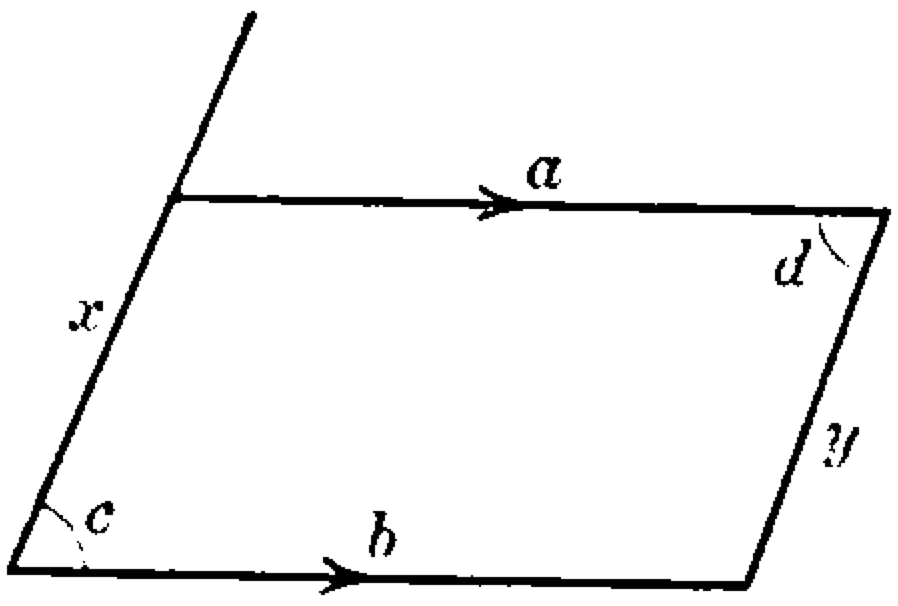
Given:
$\angle m = \angle n$
$t \perp b$
Prove:
$t \perp a$

13.



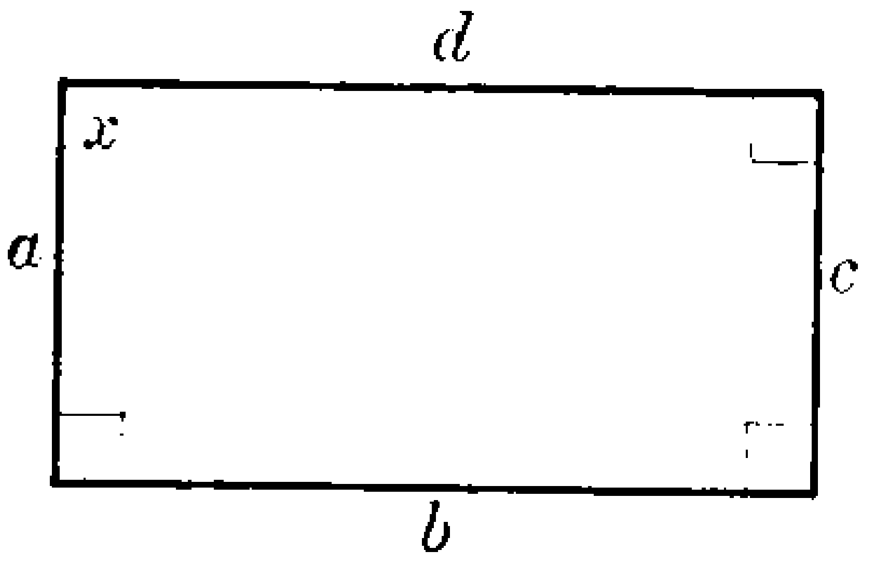
Given:
$a \parallel b$
$\angle m = \angle y$
Prove:
$\angle x = \angle y$

14.



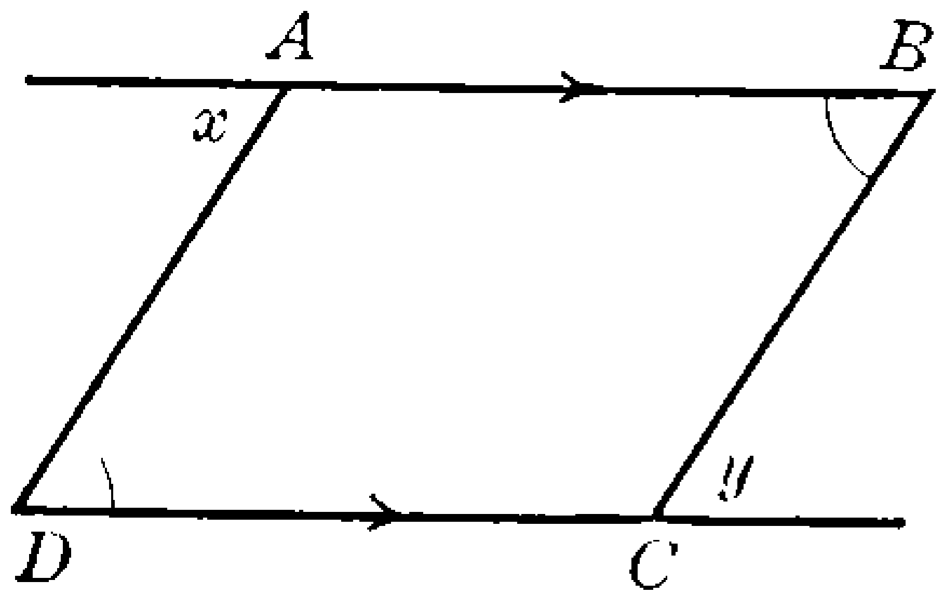
Given:
$a \parallel b$
$\angle c = \angle d$
Prove:
$x \parallel y$

15.



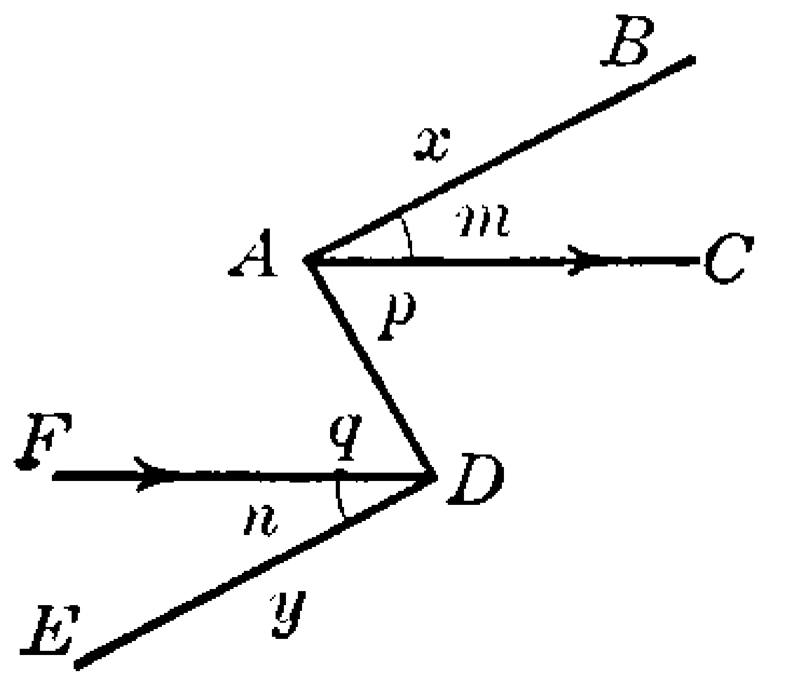
Given:
$a \perp b$
$b \perp c$
$c \perp d$
Prove:
$d \perp a$

16.



Given:
$AB \parallel DC$
$\angle B = \angle D$
Prove:
$\angle x = \angle y$

Illustrative Example

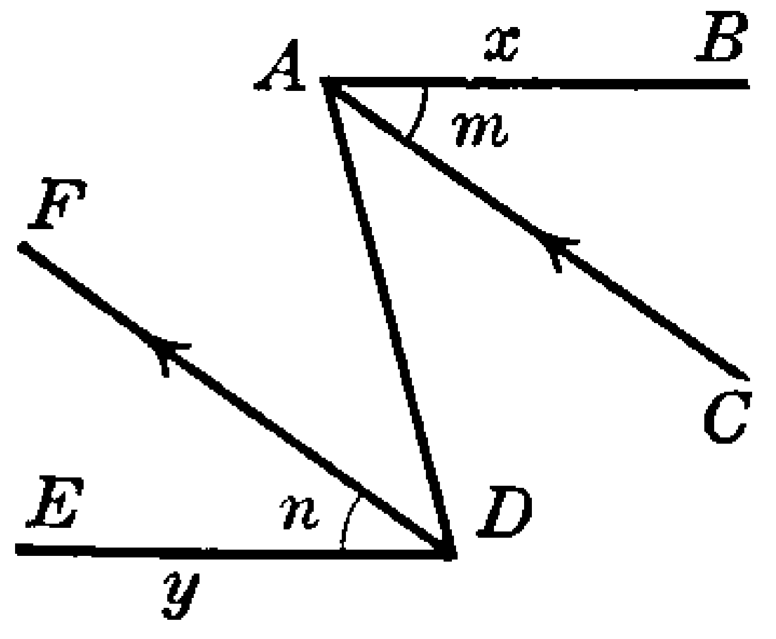


Given:
$AC \parallel FD$
$\angle m = \angle n$
Prove:
$AB \parallel ED$

PROOF

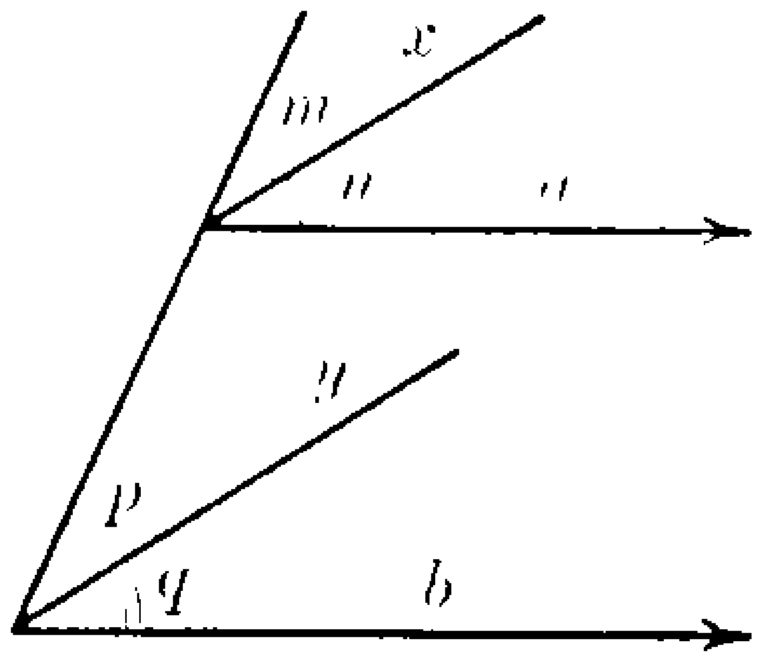
Statements	*Reasons*
1. $AC \parallel FD$	1. Given.
2. $\angle p = \angle q$	2. $\parallel$s, alt. int. $\angle$s are =.
3. $\angle m = \angle n$	3. Given.
4. $\angle A = \angle D$	4. Add. Ax.
5. $AB \parallel ED$	5. Alt. int. $\angle$s are =, $\parallel$s .

17.



Given:
$AC \parallel FD$
$\angle m = \angle n$
Prove:
$AB \parallel ED$

18.



Given:
$a \parallel b$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x \parallel y$

SUPERIOR WORK

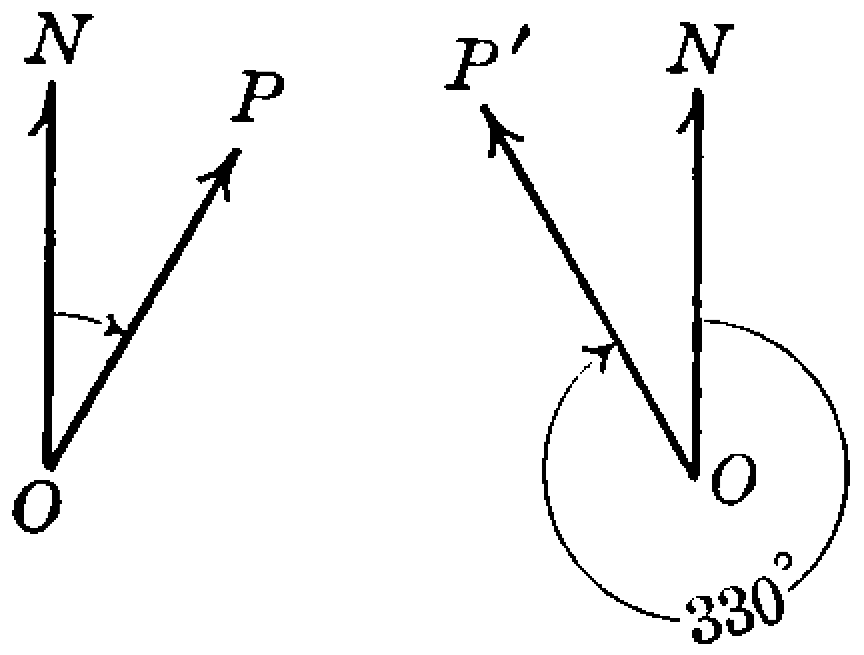
S1. Prove: If two lines are cut by a transversal and a pair of corresponding angles are equal, then the interior angles on the same side of the transversal are supplementary.

S2. Prove the converse of S1.

S3. Prove: If two lines are cut by a transversal and a pair of interior angles on the same side of the transversal are supplementary, then the exterior angles on the same side of the transversal are also supplementary.

S4. Prove: If two lines are cut by a transversal and a pair of alternate exterior angles are equal, the lines cannot meet. (From the definition of alternate interior angles, make up a definition for alternate exterior angles.)

S5. In aviation the direction of a ray (instead of a line) is measured from the north and *always* clockwise. In other words, our frame of reference is a ray *ON*. There is only one way in which the angle *O* is measured: clockwise, as indicated by the arrows. This method makes certain things simpler, but the number of degrees used to measure these angles may be as high as 360. Thus, in the figure, *OP′* is in a direction of 330°.



When direction is measured in this way it is called *azimuth.*

(*a*) Using your protractor, draw *OP* in the direction of 30° and *OP′* in the direction of 120°. What is the relationship between *OP* and *OP′*?

(*b*) Draw OP in the direction of 50° and OP' in the direction of 320°. What is the relationship between OP and OP'?

(*c*) Draw OP in the direction of 160° and OP' in the direction of 340°. What is the relationship between OP and OP'?

(*d*) A wind is blowing in the direction of 210°. *From* what direction is it blowing?

(*e*) A pilot who is heading in the direction of 240° observes that the wind is from his right. From what direction is the wind blowing?

(*f*) A plane is flying in the direction of 40° and the wind is from 220°. Compare the direction of the plane with the direction of the wind.

(*g*) A plane is flying in the direction of 300°. From what direction is a tail wind? a head wind?

(*h*) A plane is headed in the direction of 60° but, because of the wind, it is really flying in the direction of 45°. What is the *drift angle?* Is the plane drifting to the right or to the left?

Interior Angles on the Same Side of the Transversal

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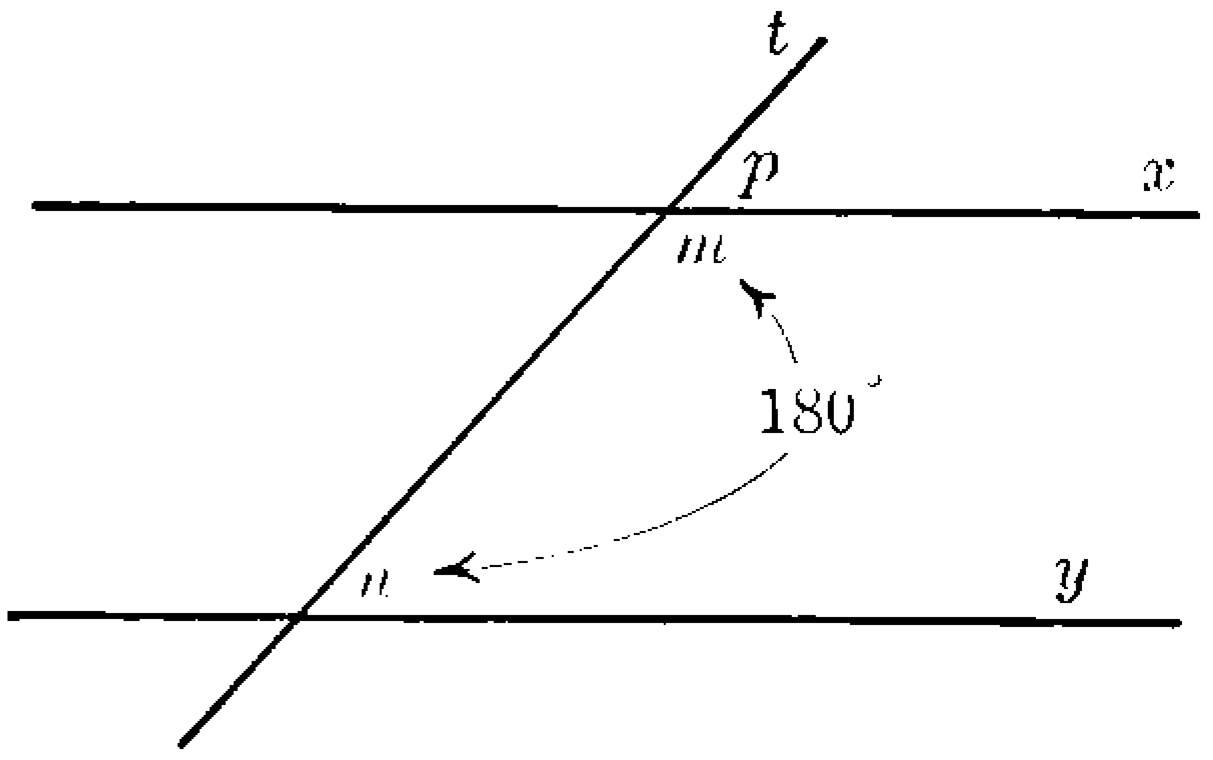
59. THEOREM. If two lines are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel.



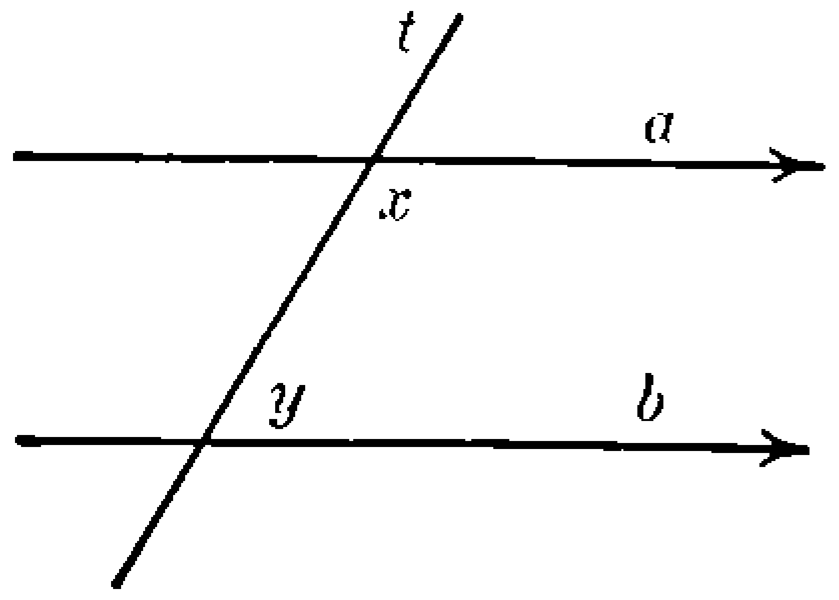
Given: Lines x and y cut by the trans. t
$\angle m + \angle n = 180°$
Prove: $x \parallel y$

PROOF

Statements	*Reasons*
1. $\angle m + \angle n = 180°$	1. Given.
2. $\angle m + \angle p = 180°$	2. Def. st. $\angle$.
3. $\angle n = \angle p$	3. Sups. of the same $\angle$ are $=$.
4. $x \parallel y$	4. Corr. $\angle$s are $=$, $\parallel$s.

---

60. CONVERSE THEOREM. If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.



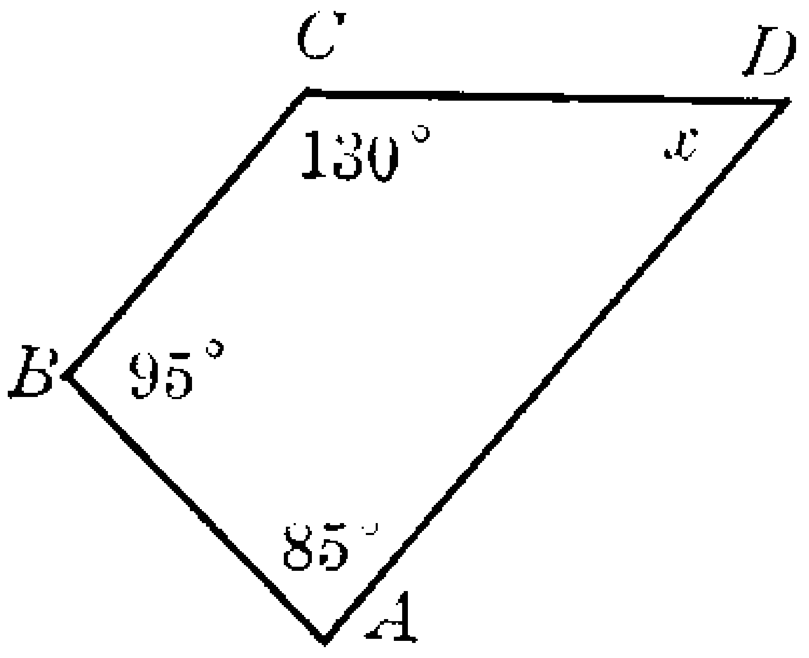
Given: $a \parallel b$
Trans. t
Prove: $\angle x + \angle y = 180°$

Write out the proof.

EXERCISES

Write out the proof for each exercise. Use the form shown in the illustrative example.

Illustrative Example



Given:
∠A = 85°
∠B = 95°
∠C = 130°
Prove:
∠D = 50°

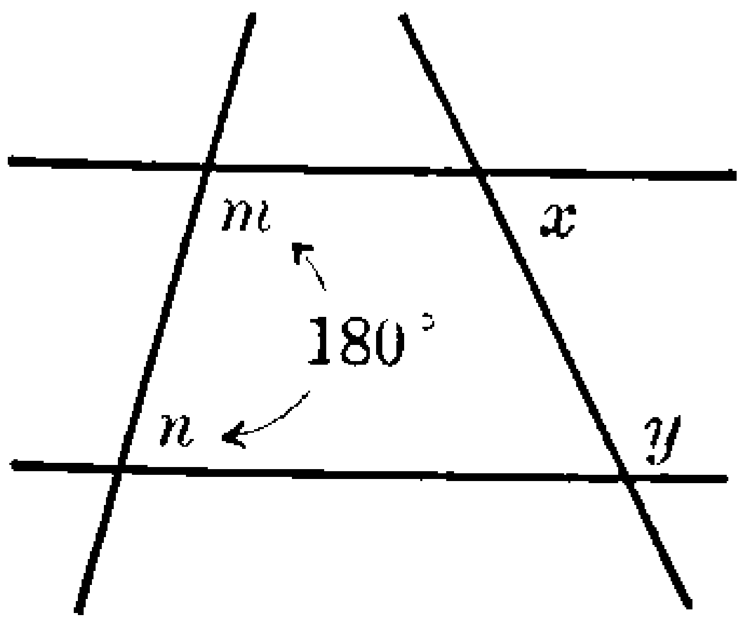
PROOF

Statements	*Reasons*
1. ∠A = 85° ∠B = 95°	1. Given.
2. ∠A + ∠B = 180°	2. Add. Ax.
3. BC ∥ AD	3. Int. ∡ sup., ∥s (§ 59).
4. ∠C + ∠D = 180°	4. ∥s, int. ∡ sup. (§ 60).
5. ∠C = 130°	5. Given.
6. ∠D = 50°	6. Subt. Ax.

Explain Statement 3 as follows: "If two lines" (trace BC and AD) "are cut by a transversal" (trace BA) "and the interior angles on the same side of the transversal" (sweep ∡ A and B) "are supplementary, then the lines" (point to BC and AD) "are parallel."

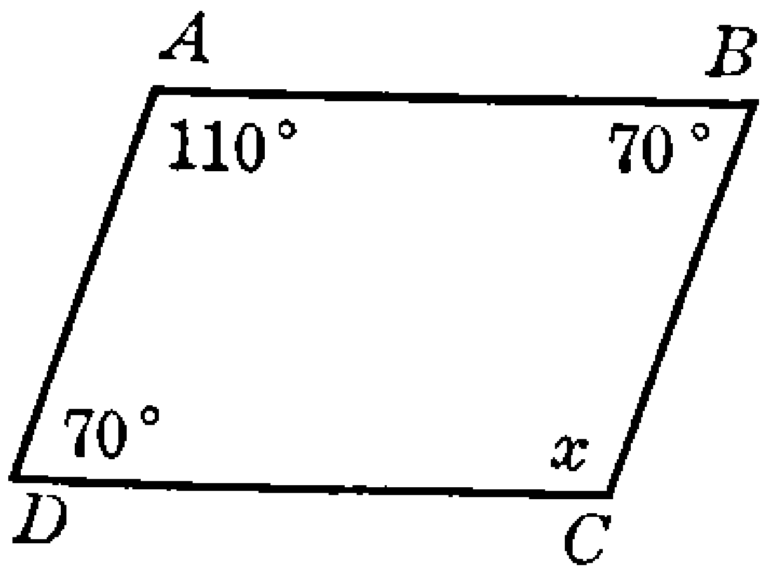
In like manner, explain Statement 4.

1.



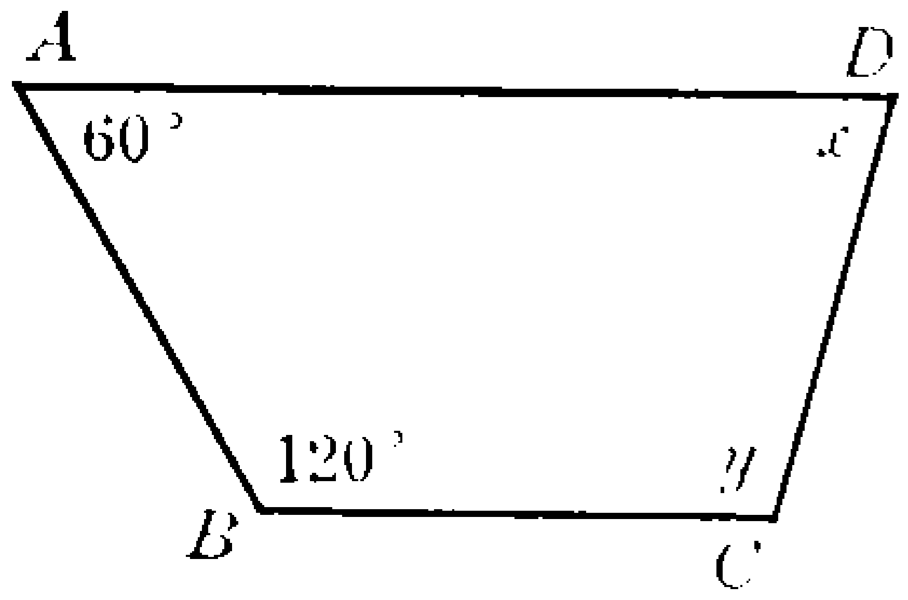
Given:
∠m + ∠n = 180°
Prove:
∠x + ∠y = 180°

2.



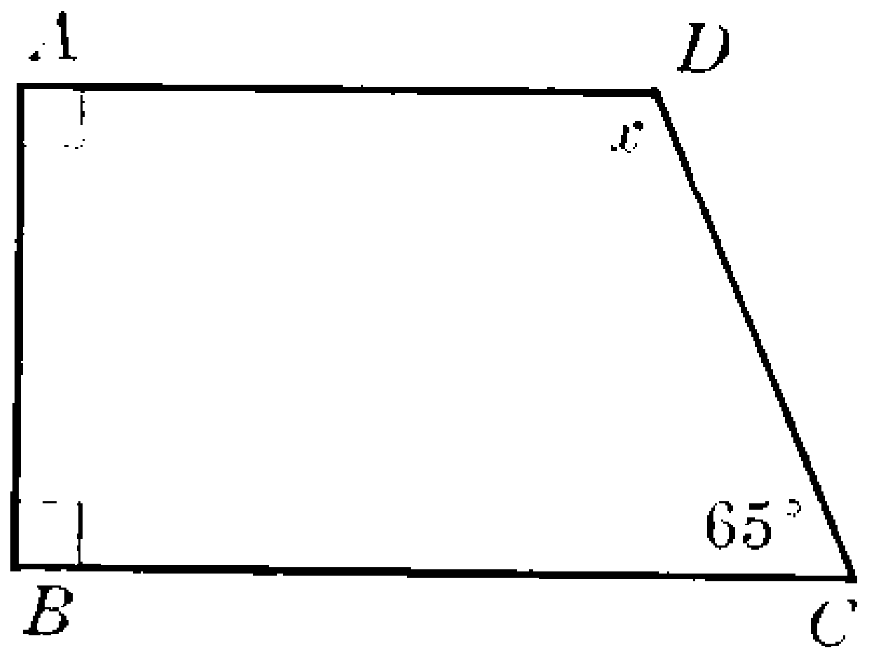
Given:
∠A = 110°
∠B = 70°
∠D = 70°
Prove:
∠C = 110°

3.



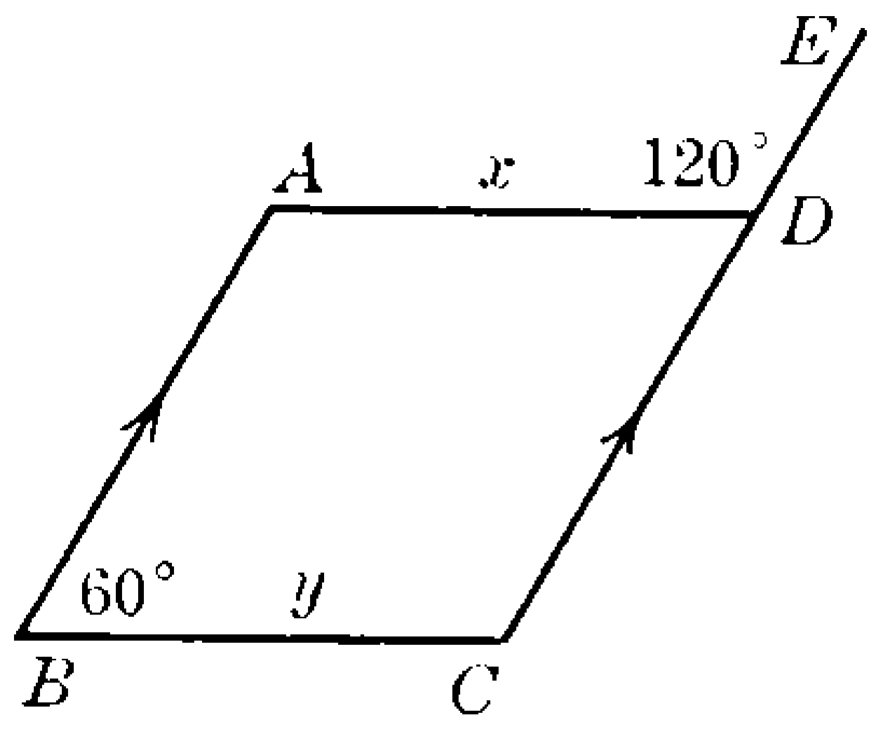
Given:
∠A = 60°
∠B = 120°
Prove:
∠x + ∠y = 180°

4.



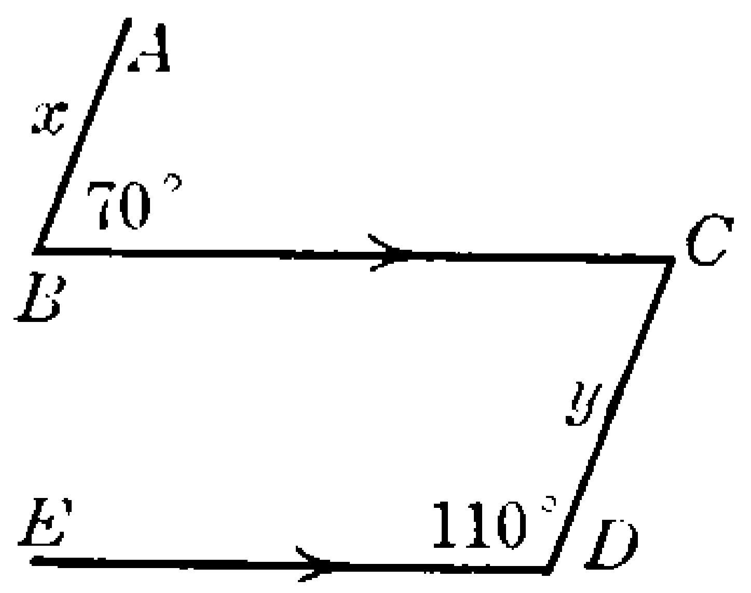
Given:
AD ⊥ AB
BC ⊥ AB
∠C = 65°
Prove:
∠x = 115°

5.



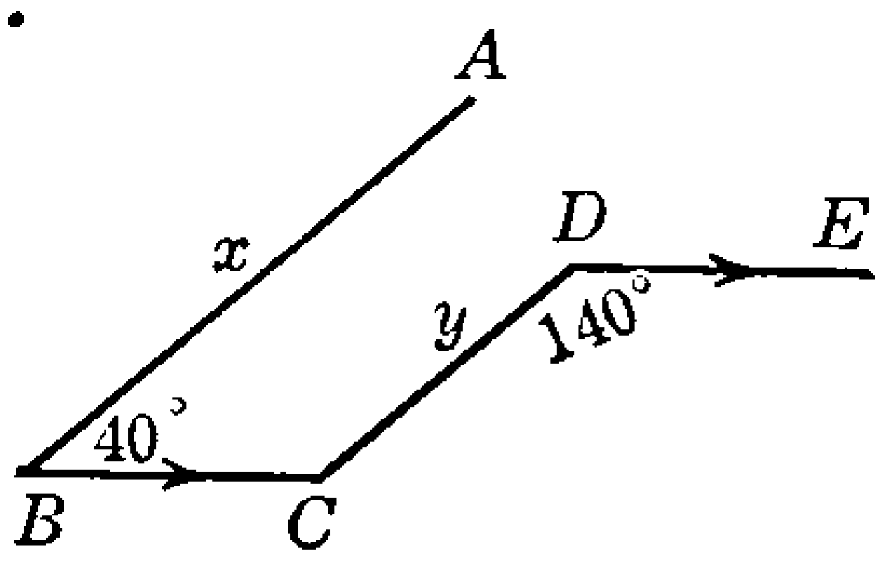
Given:
AB ∥ EC
∠B = 60°
∠ADE = 120°
Prove:
AD ∥ BC

6.



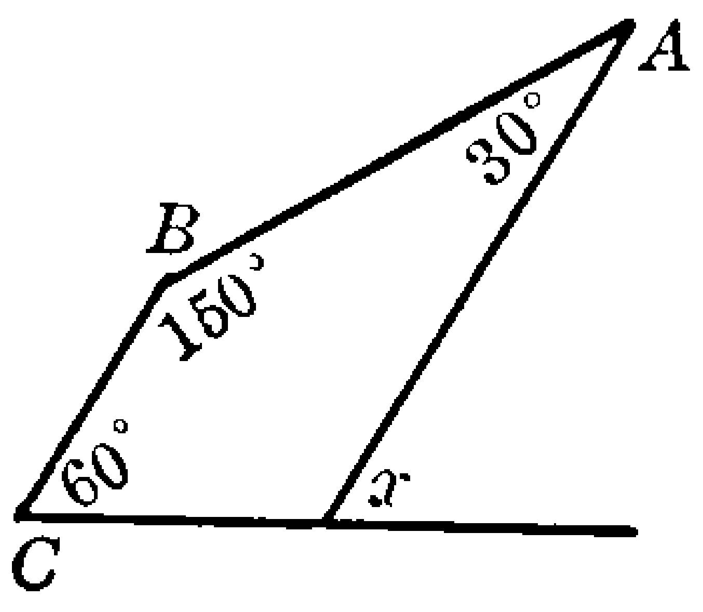
Given:
BC ∥ ED
∠B = 70°
∠D = 110°
Prove:
AB ∥ CD

7.



Given:
DE ∥ BC
∠B = 40°
∠D = 140°
Prove:
AB ∥ DC

8.



Given:
∠A = 30°
∠B = 150°
∠C = 60°
Prove:
∠x = 60°

SUPERIOR WORK

S1. If two parallel lines are cut by a transversal and

(*a*) one of the interior angles on the same side of the transversal is twice the other, then these angles are —?—° and —?—°.

(*b*) one of the interior angles on the same side of the transversal exceeds the other by 10 degrees, then these angles are —?—° and —?—°.

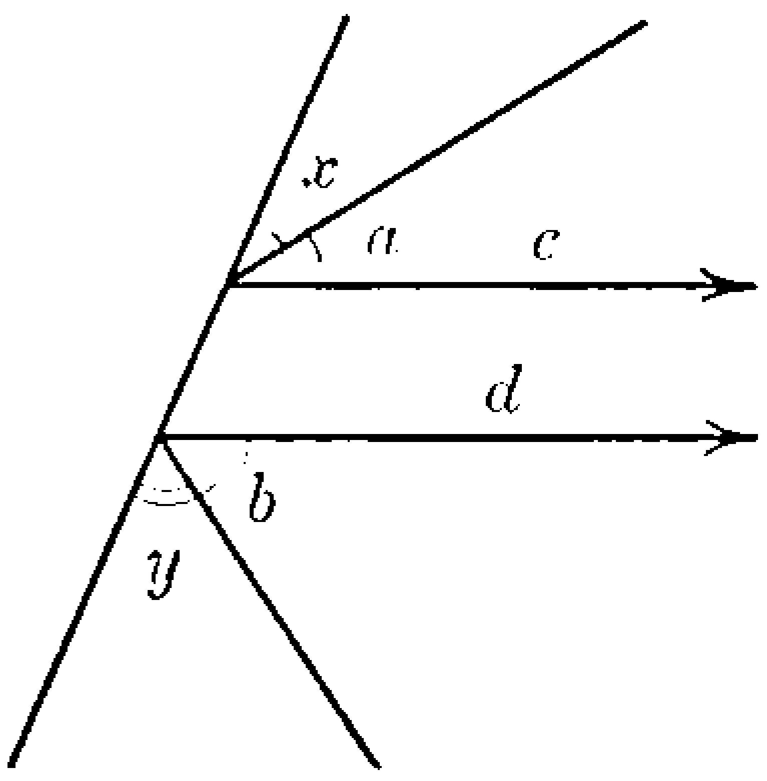
S2. Complete the following statements:

(*a*) If a is perpendicular to b, and b is perpendicular to c, then a is —?— to c.

(*b*) If a is perpendicular to b, and b is parallel to c, then a is —?— to c.

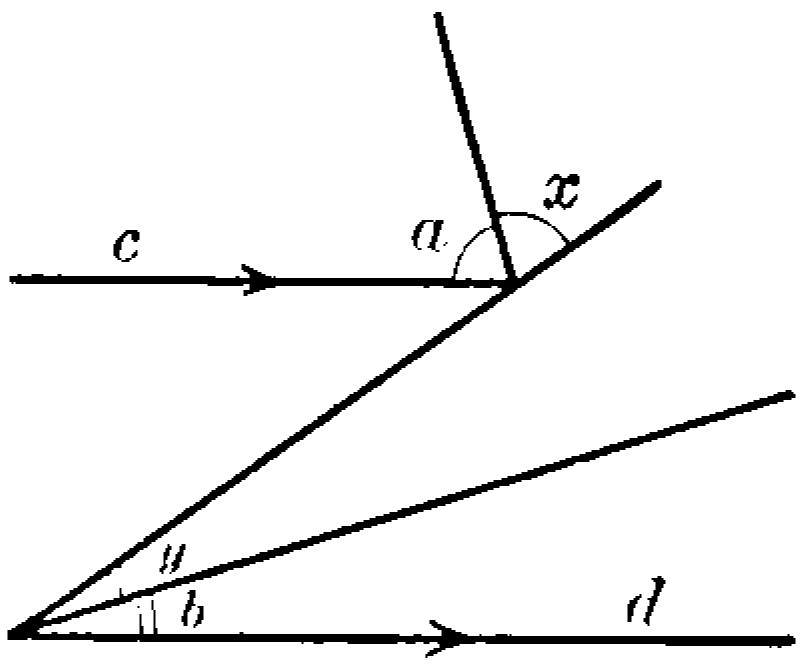
(*c*) If two angles are not —?—, they cannot be the interior angles on the same side of a transversal of two parallel lines.

S3.



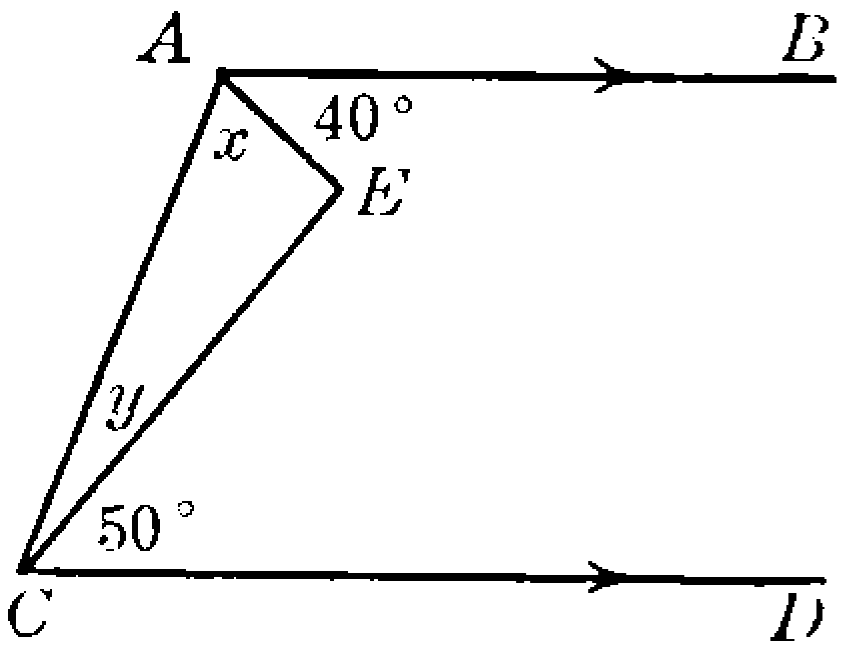
Given:
$c \parallel d$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x + \angle y = 90°$

S4.



Given:
$c \parallel d$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x + \angle y = 90°$

S5.

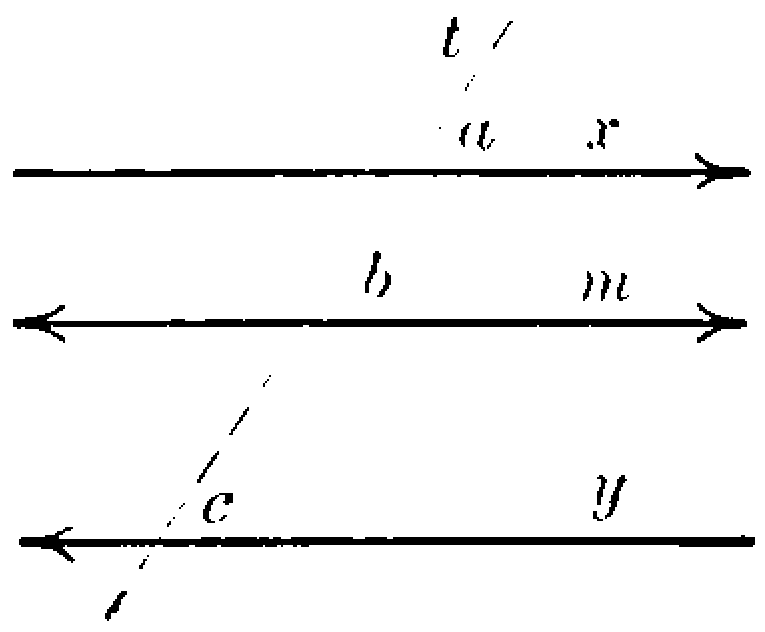


Given:
$AB \parallel CD$
$\angle BAE = 40°$
$\angle ECD = 50°$
Prove:
$\angle x + \angle y = 90°$

Using Parallels

In problems where two lines are parallel to the same line, the following theorem is used. An example of an exercise in which the theorem is used as part of the proof follows the theorem.

61. THEOREM. If two lines are parallel to the same line, then they are parallel to each other.

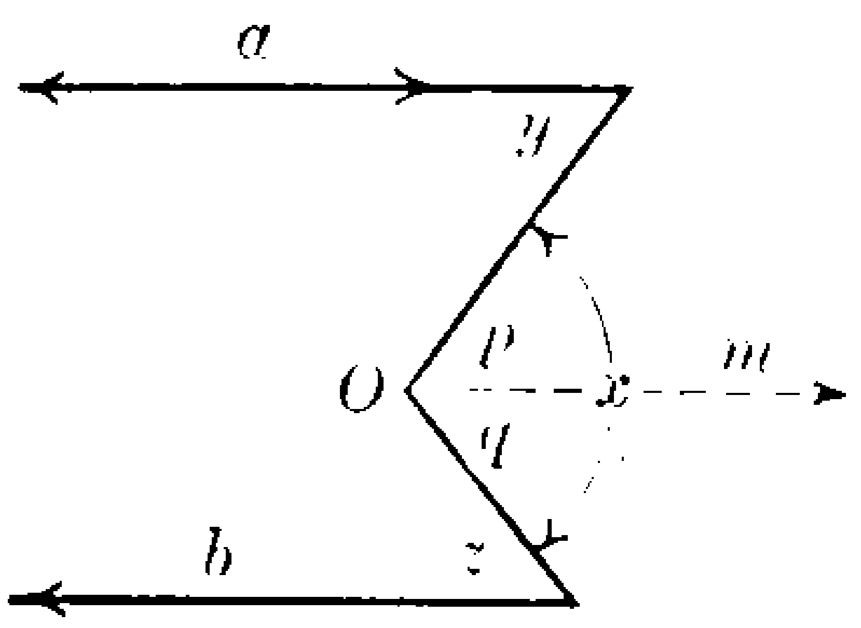


Given: $x \parallel m$
$m \parallel y$
Prove: $x \parallel y$
Cons.: Draw any trans. t.

PROOF

Statements	*Reasons*
1. $x \parallel m$	1. Given.
2. $\angle a = \angle b$	2. $\|_s$, corr. $\measuredangle$ are =.
3. $m \parallel y$	3. Given.
4. $\angle b = \angle c$	4. Same as 2.
5. $\overset{\angle a}{(\angle b)} = \angle c$	5. Subs. Ax.
6. $x \parallel y$	6. Corr. $\measuredangle$ are =, $\|_s$.

Illustrative Example



Given:
$a \parallel b$
Prove:
$\angle x = \angle y + \angle z$

Cons.: Through O, draw $m \parallel a$.

PROOF

Statements	*Reasons*
1. $m \parallel a$	1. Cons.
2. $\angle p = \angle y$	2. $\parallel$s, alt. int. $\angle$s are =.
3. $a \parallel b$	3. Given.
4. $m \parallel b$	4. Lines $\parallel$ to same line (line a) are $\parallel$ to each other (§ 61).
5. $\angle q = \angle z$	5. Same as 2.
6. $\angle x = \angle y + \angle z$	6. Add. Ax.

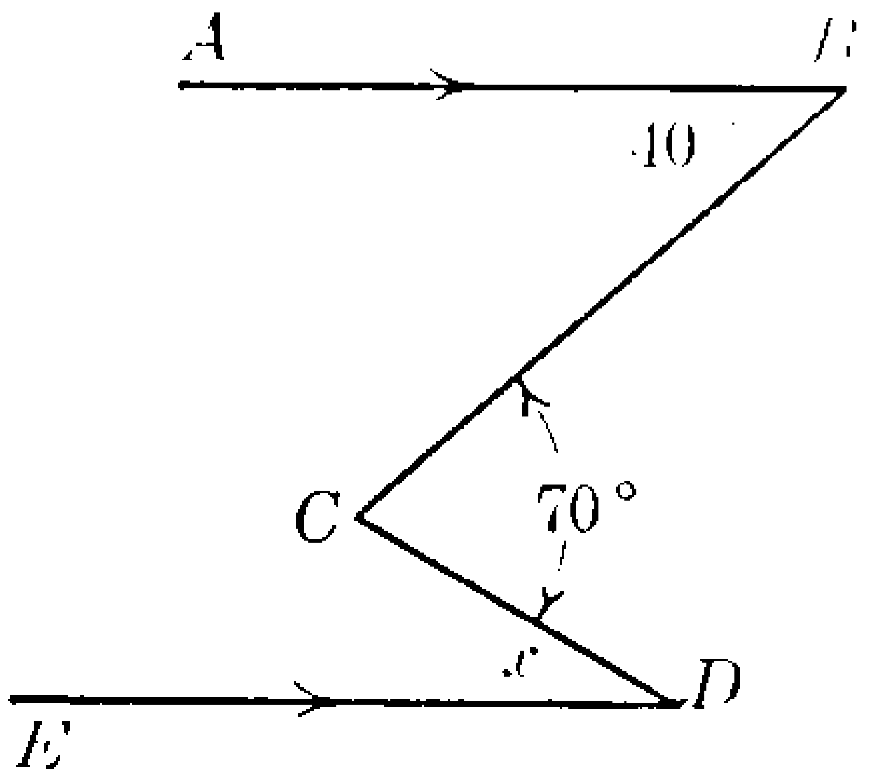
Determining a Line. The two-point postulate (§ 11) tells us that two points determine a straight line. A line through one point is not determined because its direction is not known. We may think of the second point in the two-point postulate as determining the direction of the line. A line is determined when we know one point on it and the direction of the line.

In making constructions, we may draw a line through two given points: or we may draw a line through one given point having a given direction, such as parallel or perpendicular to a given line. In our construction above, we drew m: (1) through point O and (2) parallel to line a.

A common mistake is to draw a line that passes through three points, or a line that passes through two points and is parallel to another line. A line may be drawn to do only two of these three things. If a line is suspected of meeting more than two conditions, then that fact must be proved and *not constructed.*

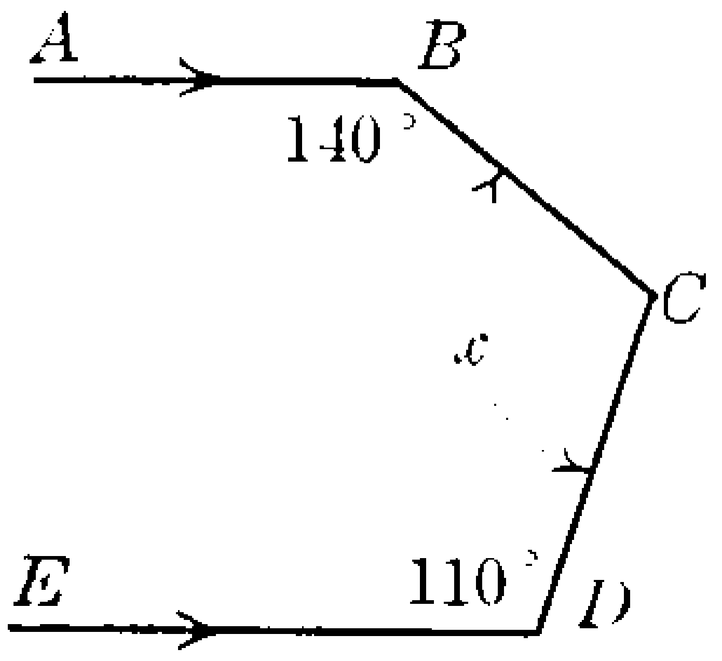
EXERCISES

1.



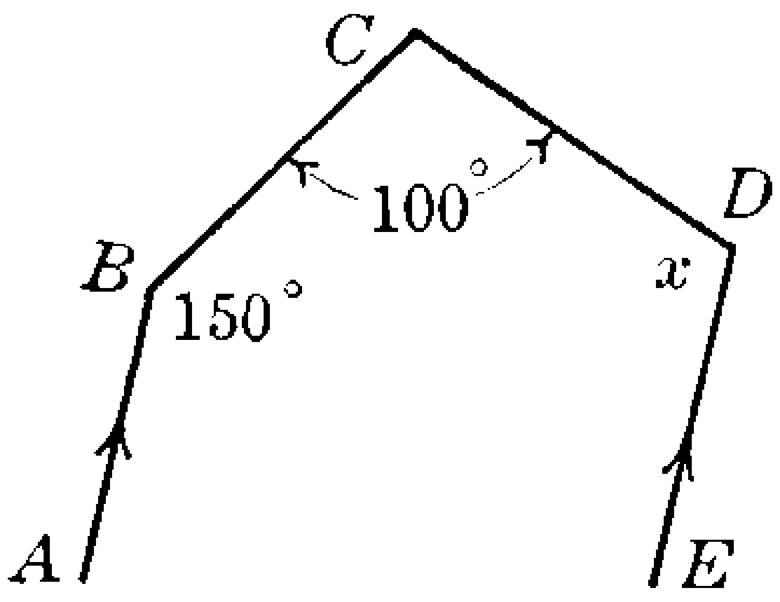
Given:
$AB \parallel ED$
$\angle B = 40°$
$\angle C = 70°$
Prove:
$\angle D = 30°$

2.



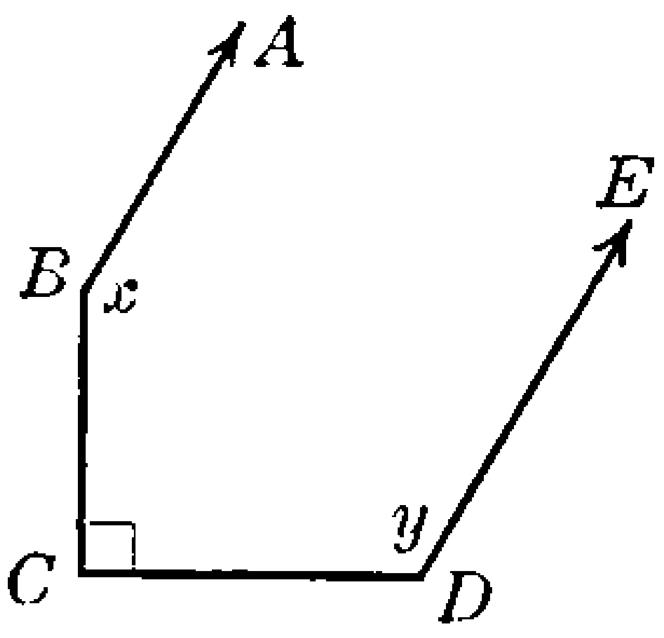
Given:
$AB \parallel ED$
$\angle B = 140°$
$\angle D = 110°$
Prove:
$\angle C = 110°$

3.



Given:
$AB \parallel ED$
$\angle B = 150°$
$\angle C = 100°$
Prove:
$\angle D = 110°$

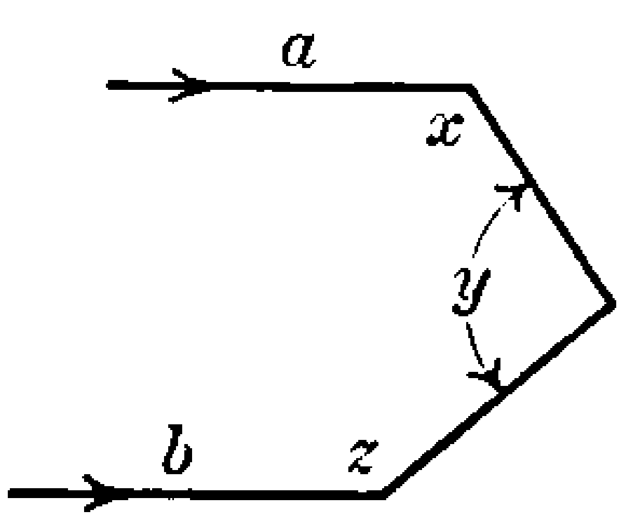
4.



Given:
$AB \parallel ED$
$\angle C = 90°$
Prove:
$\angle x + \angle y = 270°$

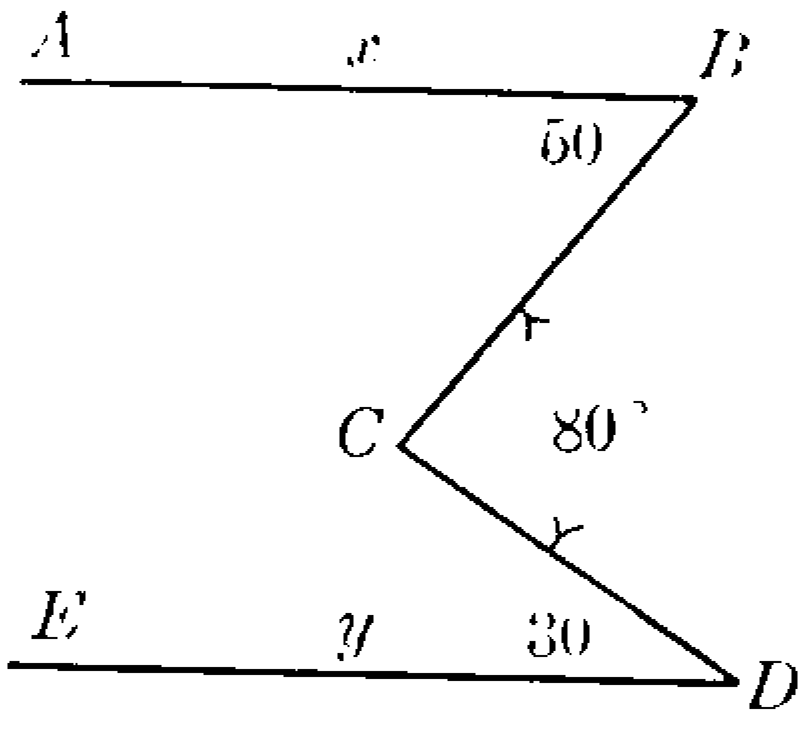
SUPERIOR WORK

S1.



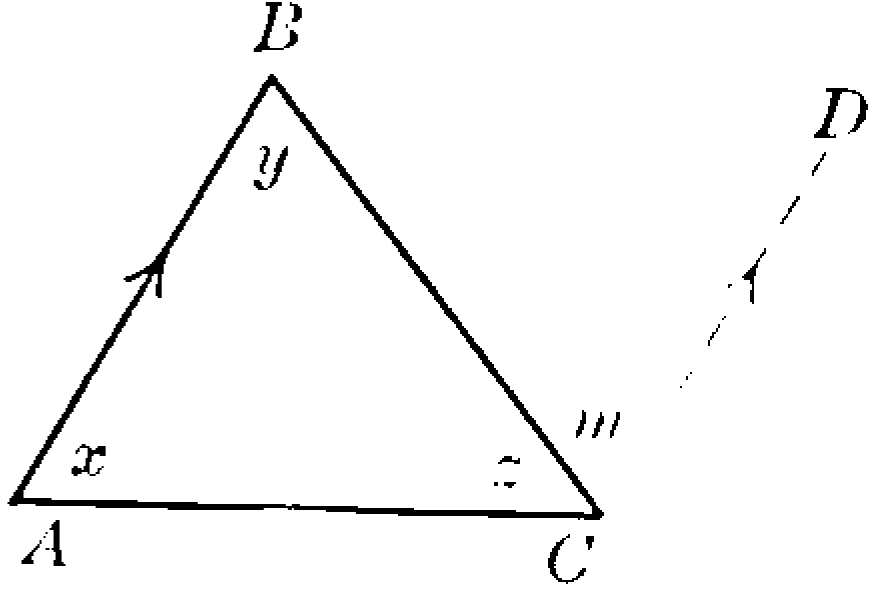
Given:
$a \parallel b$
Prove:
$\angle x + \angle y +$
$\angle z = 360°$

S2.



Given:
$\angle B = 50^\circ$
$\angle C = 80^\circ$
$\angle D = 30^\circ$
Prove:
$AB \parallel ED$

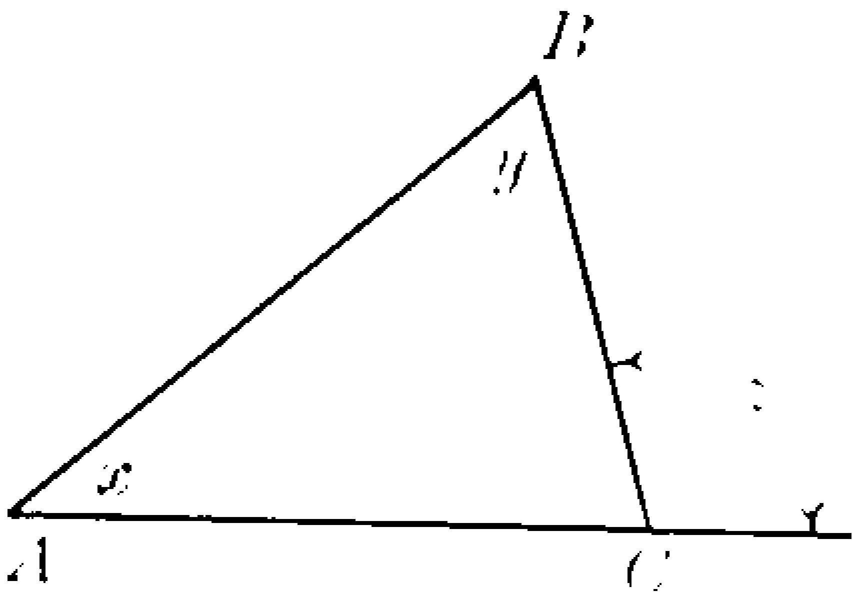
S3.



Given:
$\triangle ABC$
Prove:
$\angle x + \angle y +$
$\angle z = 180^\circ$

Cons.: Through C, draw $CD \parallel AB$.

S4.



Given:
$\triangle ABC$ with
AC extended
Prove:
$\angle x + \angle y = \angle z$

Cons.: Through C, draw $CD \parallel AB$.

The Angles of a Triangle

From our knowledge of parallels we can now prove the following fundamental theorem about the angles of a triangle.

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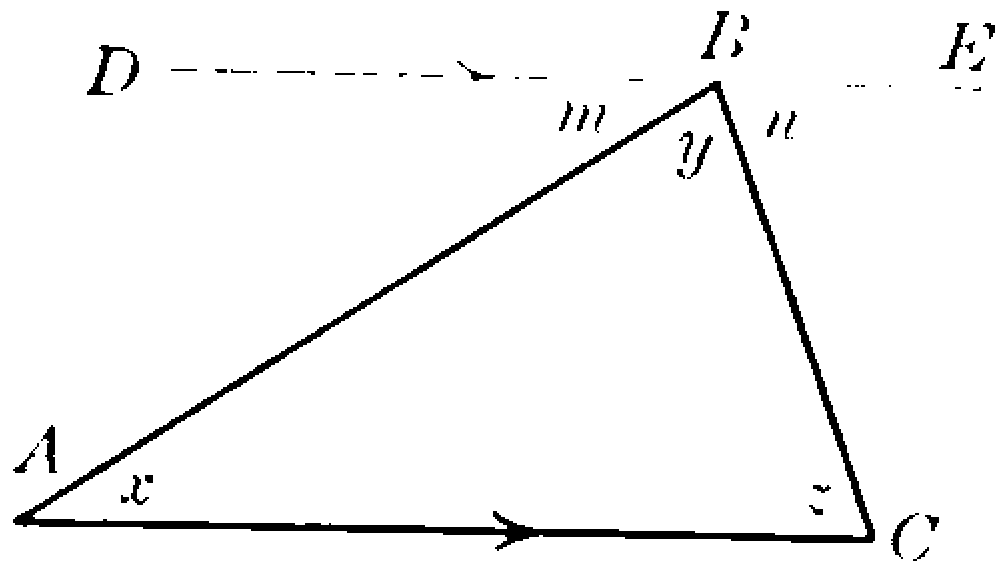
62. THEOREM. The sum of the angles of a triangle is 180°.



Given: $\triangle ABC$
Prove: $\angle x + \angle y + \angle z = 180^\circ$
Cons.: Through B, draw $DE \parallel AC$.

PROOF

Statements	*Reasons*
1. $\angle m + \angle y + \angle n = 180^\circ$	1. Def. st. ∠.
2. $\angle m = \angle x$ $\angle n = \angle z$	2. ∥s, alt. int. ∠s are =.
3. $(\overset{\angle x}{\angle m}) + \angle y + (\overset{\angle z}{\angle n}) = 180^\circ$	3. Subs. Ax.

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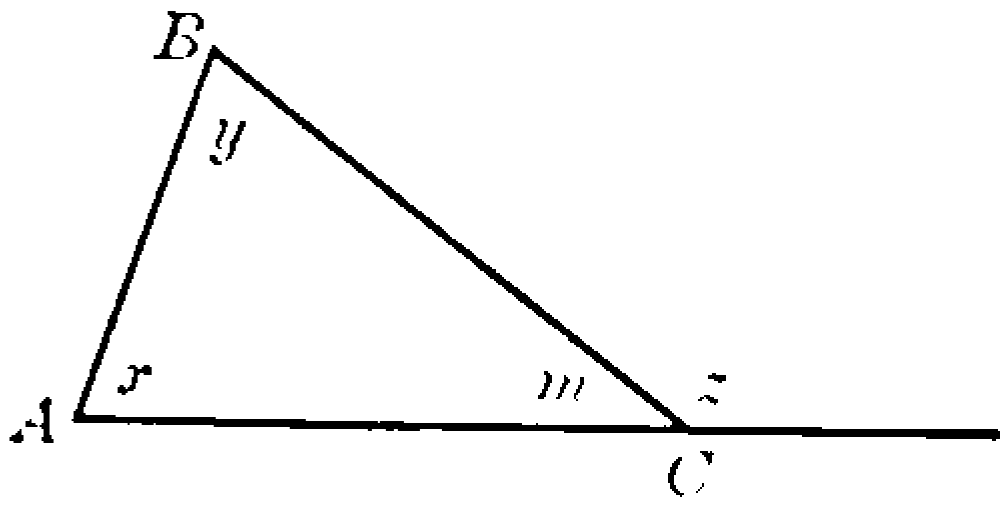
63. Definition. An exterior angle of a triangle is an angle that is formed by one side of the triangle and the prolongation of another side.

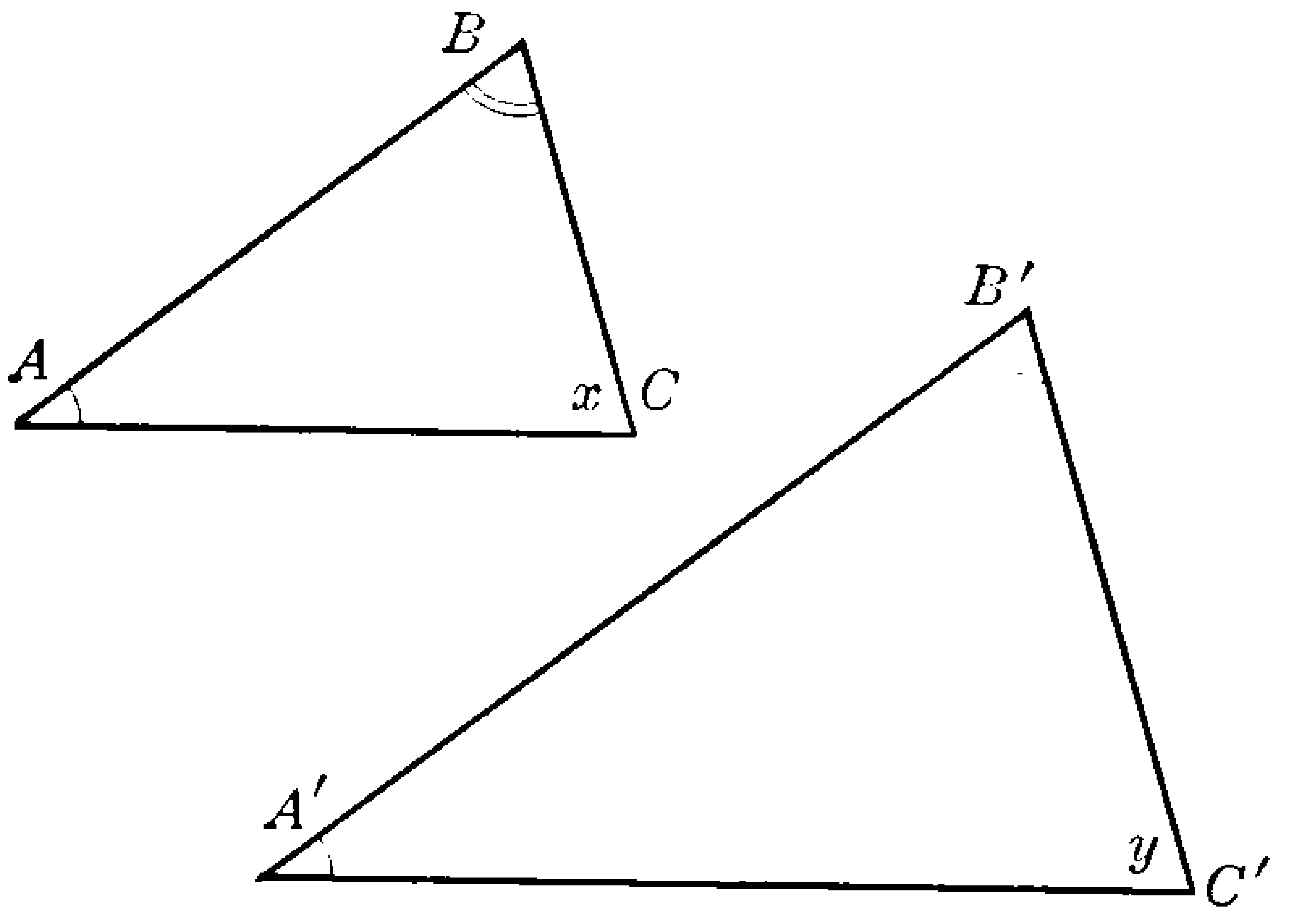


In the figure, $\angle z$ is an exterior angle of $\triangle ABC$ because it is formed by the side BC and the prolongation of the side AC.

From the theorem in § 62, the sum of the angles x and y, that is, $\angle x + \angle y$, may be thought of as forming the supplement of $\angle m$. The exterior $\angle z$ is also the supplement of $\angle m$. Therefore, $\angle x + \angle y = \angle z$.

64. Corollary. An exterior angle of a triangle equals the sum of the two nonadjacent interior angles.

In the figure the angles A and B are *respectively* equal to the angles A' and B', that is, $\angle A = \angle A'$ and $\angle B = \angle B'$.



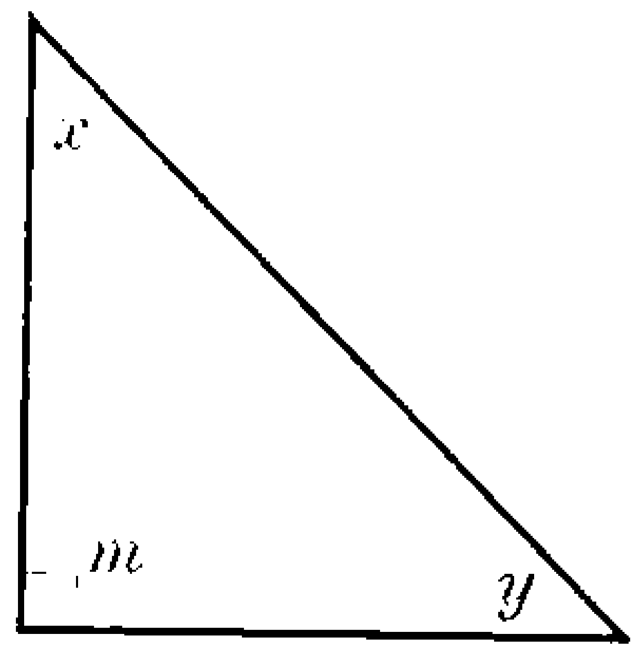
From the theorem that the sum of the angles of a triangle is 180°, $\angle x$ may be thought of as the supplement of $(\angle A + \angle B)$. Also, $\angle y$ may be thought of as the supplement of $(\angle A' + \angle B')$. Therefore, $\angle x$ is equal to $\angle y$, because supplements of equal angles are equal.

65. Corollary. If two angles of one triangle are equal respectively to two angles of another triangle, then the remaining angles are equal.

As the corollary is stated, it refers to *two* complete triangles at rest. If we were thinking of a changing figure, we would consider only one triangle that changes. But the triangle changes in such a way that two of its angles always remain constant. The conclusion would be that the third angle also remains constant. To fit a geometry of motion, the corollary could be stated as follows: Two angles of a triangle determine the third angle.

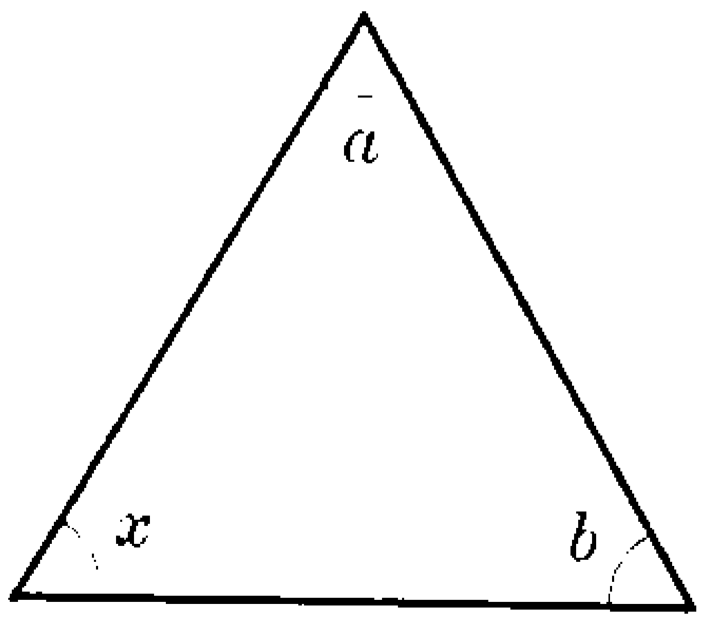
EXERCISES

1.



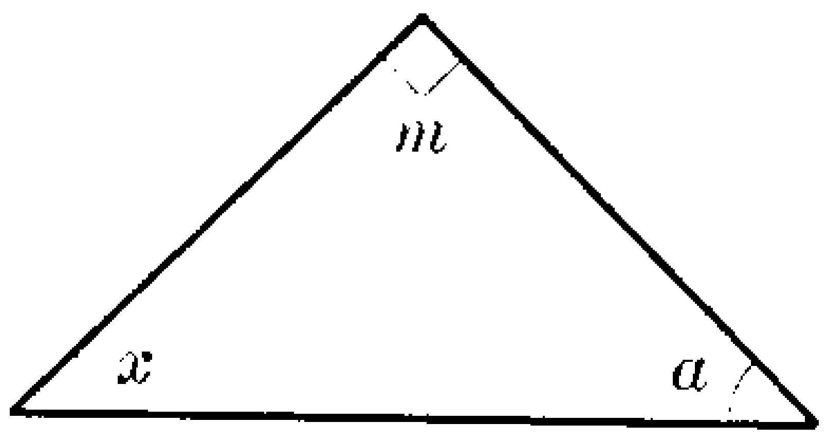
Given:
$\angle m = 90°$
Prove:
$\angle x + \angle y = 90°$

2.



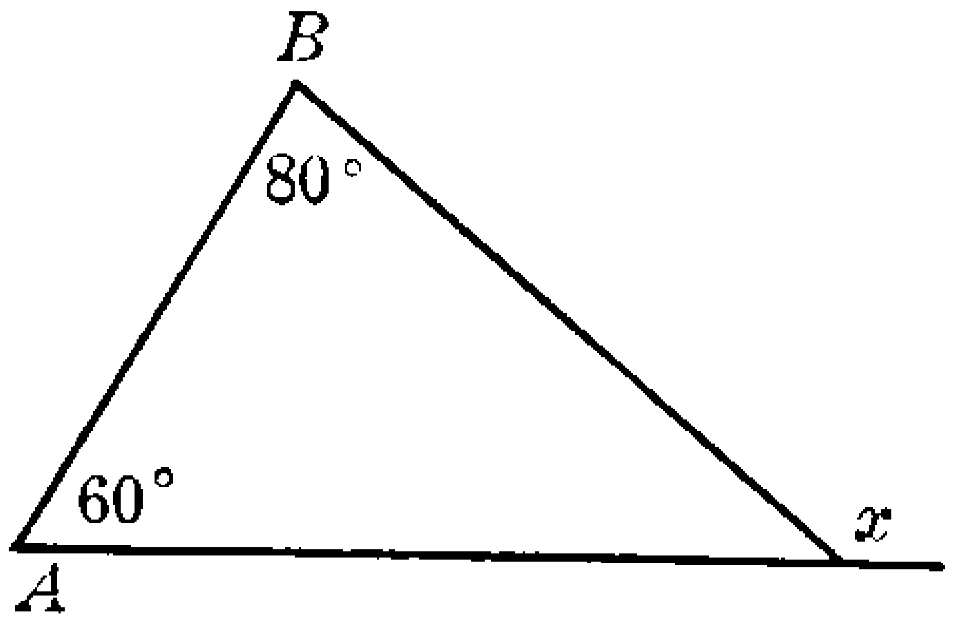
Given:
$\angle x = \angle a = \angle b$
Prove:
$\angle x = 60°$

3.



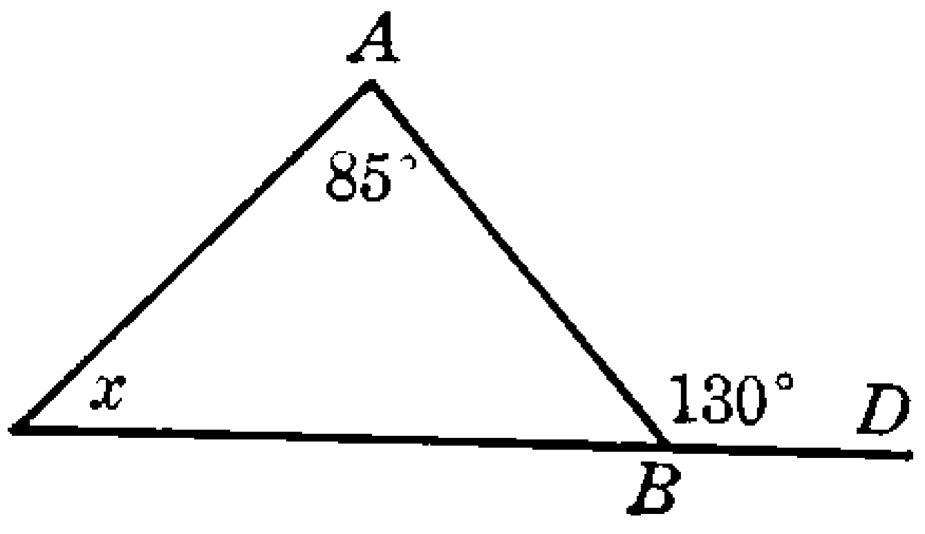
Given:
$\angle m = 90°$
$\angle x = \angle a$
Prove:
$\angle x = 45°$

4.



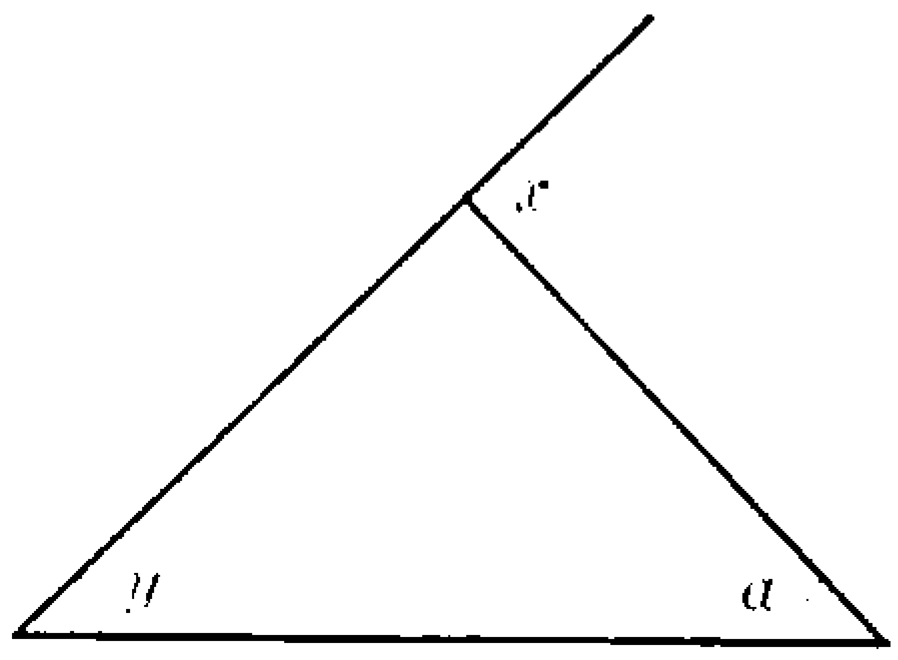
Given:
$\angle A = 60°$
$\angle B = 80°$
Prove:
$\angle x = 140°$

5.



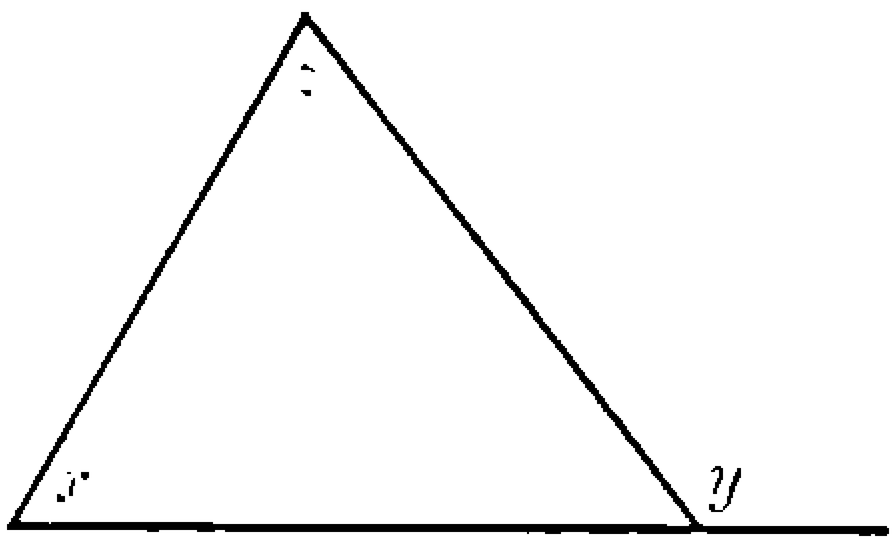
Given:
$\angle ABD = 130°$
$\angle A = 85°$
Prove:
$\angle x = 45°$

6.



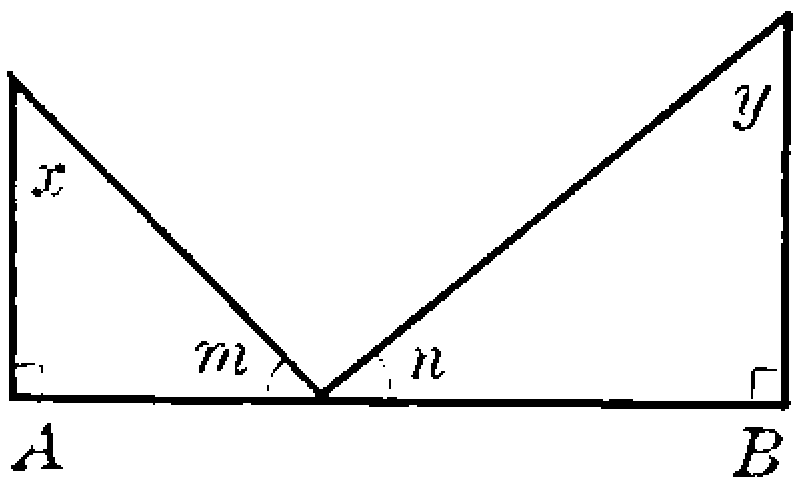
Given:
$\angle y = \angle a$
Prove:
$\angle x = 2 \cdot \angle y$

7.



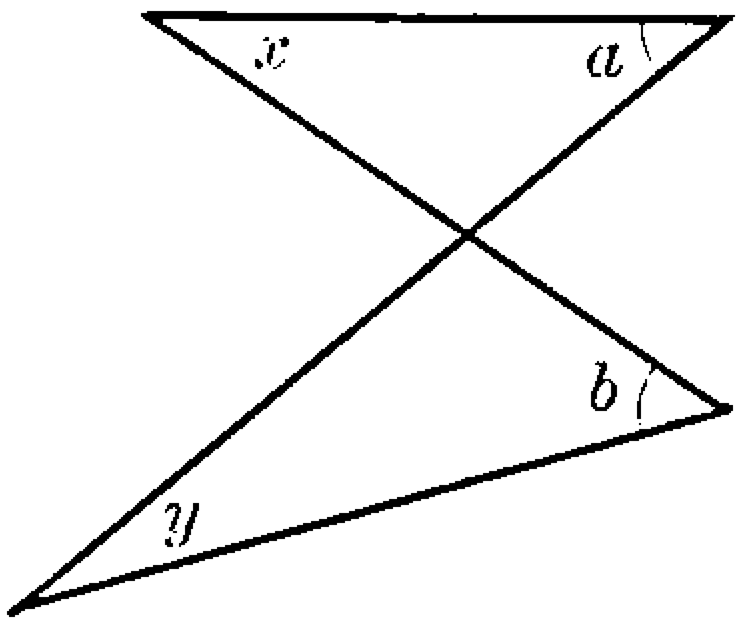
Given:
$\angle y$ ext.
$\angle$ of $\triangle$
Prove:
$\angle x = \angle y - \angle z$

8.



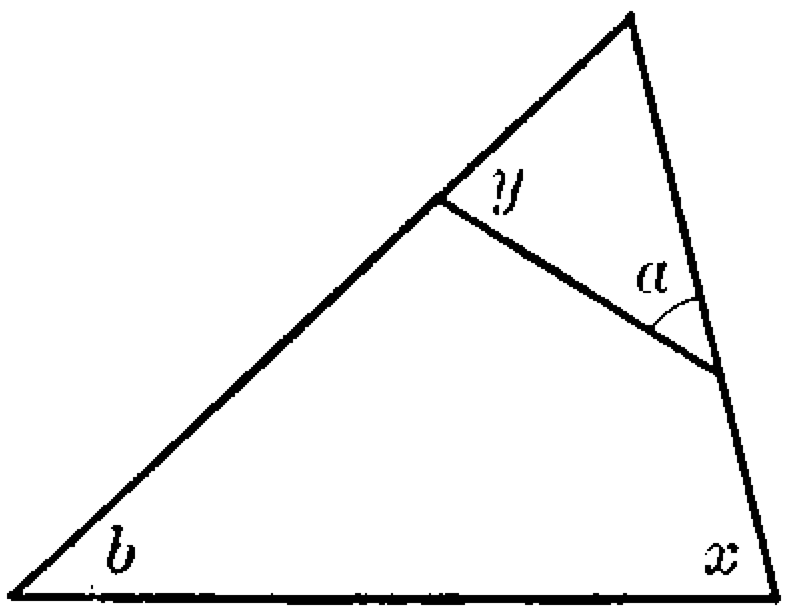
Given:
$\angle A$ a rt. $\angle$
$\angle B$ a rt. $\angle$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

9.



Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

10.

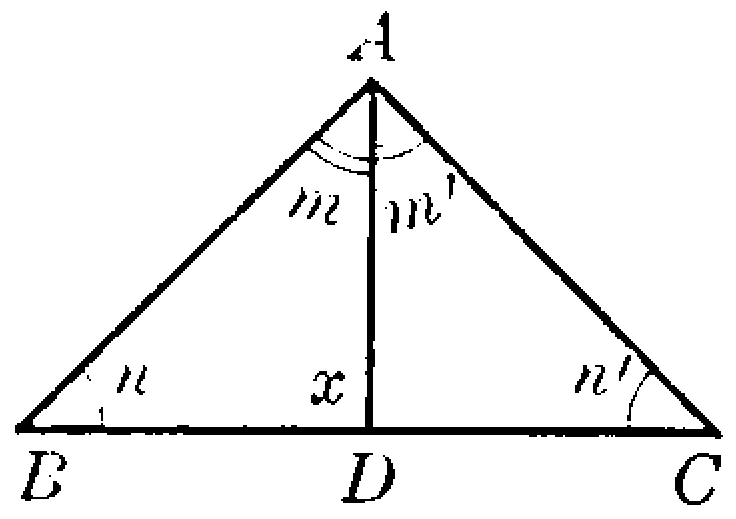


Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

Help: Compare the ∡ in the two overlapping ◬.

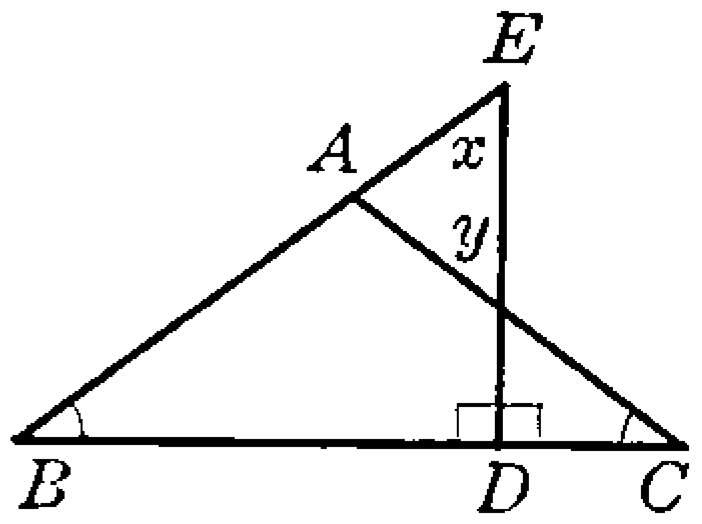
SUPERIOR WORK

S1.



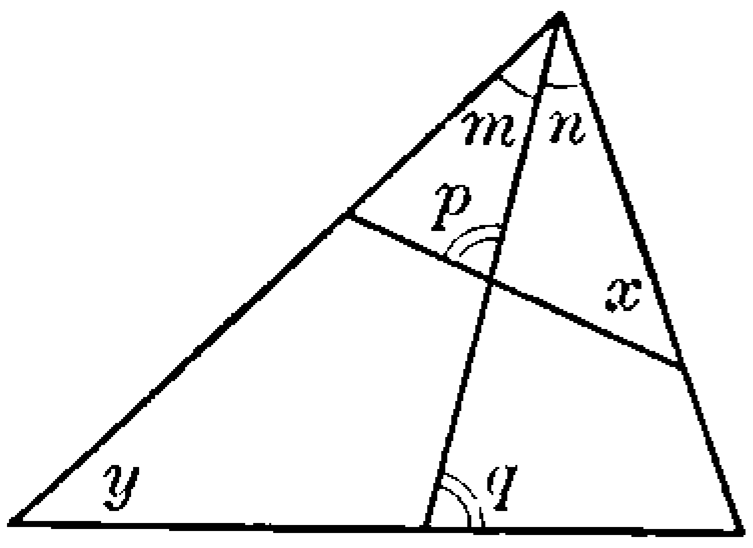
Given:
In $\triangle ABC$
$\angle m = \angle m'$
$\angle n = \angle n'$
Prove:
$AD \perp BC$

S2.



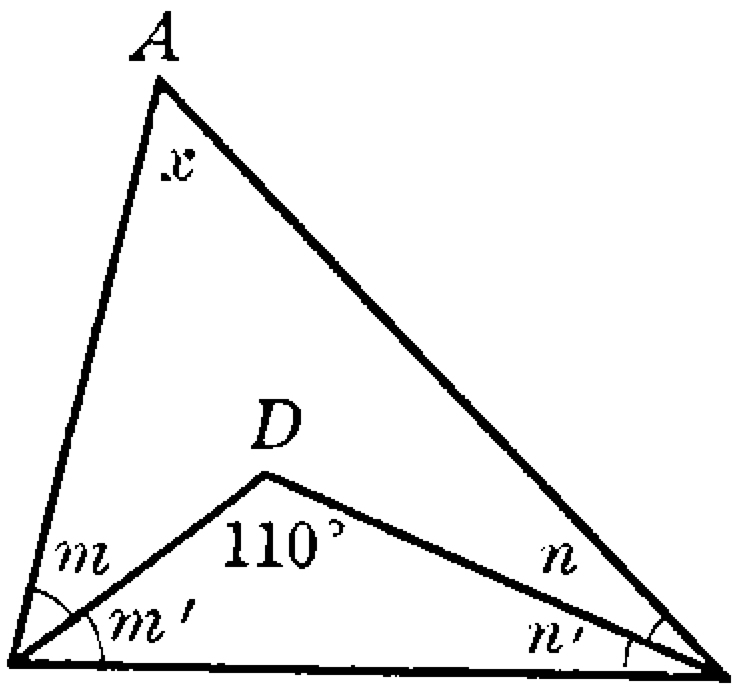
Given:
$\angle B = \angle C$
$ED \perp BC$
Prove:
$\angle x = \angle y$

S3.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$\angle x = \angle y$

S4.



Given:
$\angle m = \angle m'$
$\angle n = \angle n'$
$\angle D = 110°$
Prove:
$\angle x = 40°$

SUMMARY

66. In this chapter we have been concerned with the size of certain angles and the direction of their sides.

Starting from the assumption that all straight angles are equal, we have proved that all right angles are equal; that supplements and complements of the same or equal angles are equal; that vertical angles are equal.

We defined parallel lines as straight lines that never meet even though they are in the same plane. In actual practice, we think of two parallel lines as being "in the same direction." We saw that the directions of such lines are really the corresponding angles made by some transversal. Therefore we assumed that if a transversal of two lines makes the corresponding angles equal, the lines are parallel; and, conversely, if two parallel lines are cut by a transversal, the corresponding angles are equal. From these assumptions and from the propositions about the size of certain angles, we have the following:

Two lines are parallel if:

(*a*) The corresponding angles are equal.
(*b*) The alternate interior angles are equal.
(*c*) The interior angles on the same side of the transversal are supplementary.
(*d*) They are both perpendicular to the same line.
(*e*) They are both parallel to the same line.

If two lines are parallel, then:

(*a*) The corresponding angles are equal.
(*b*) The alternate interior angles are equal.
(*c*) The interior angles on the same side of the transversal are supplementary.
(*d*) A perpendicular to one of them is also perpendicular to the other.

Using the relationships between the angles whose sides are parallel, we proved the following theorems about the angles of a triangle:

The sum of the angles of a triangle equals a straight angle.

An exterior angle of a triangle equals the sum of the two remote interior angles.

If two angles of one triangle are equal respectively to two angles of a second triangle, then the remaining angles are equal.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

corollary § 45
angular degree (1°) § 41
parallel lines (|| lines) § 51
transversal (trans.) § 52
corresponding angles (corr. ∠s) § 52
alternate interior angles (alt. int. ∠s) § 52
interior angles on the same side of the transversal (int. ∠s) § 52
exterior angle of a triangle (ext. ∠ of a △) § 63

DEPENDENCE CHART

67. The chart on page 51 summarizes the twelve main theorems in this chapter and shows how they depend on three assumptions about space. By referring to this chart answer the following:

1. How many theorems depend directly or indirectly on Postulates 2 and 3? on Postulates 1 and 3?

2. State the theorems that depend *directly* on Postulate 1; on Theorem 6; on Corollary 7.

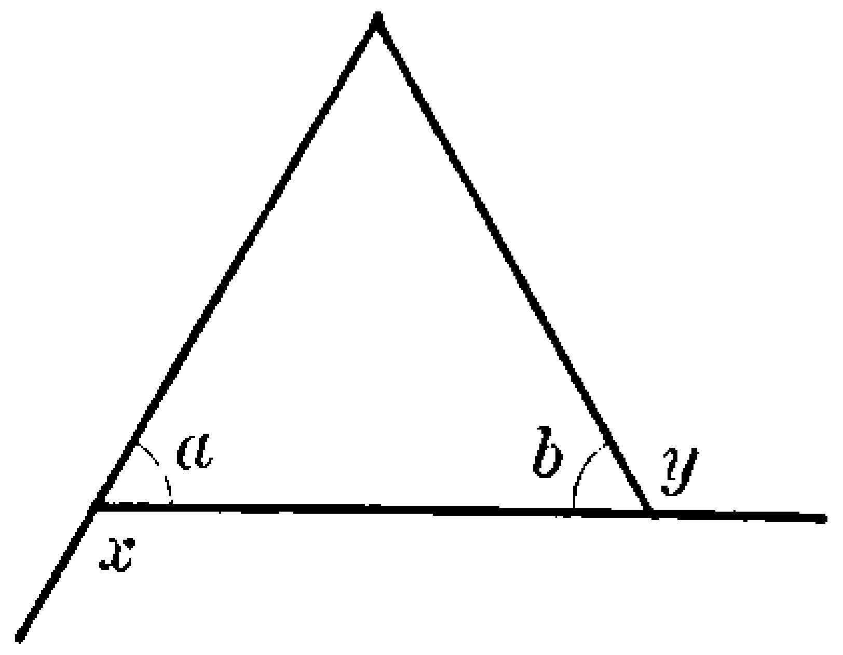
3. How many theorems would be false if we denied Postulate 1? Postulate 2? Postulate 3?

4. Would the theorem about the sum of the angles of a triangle be true if we denied Postulate 1? Postulate 2? Postulate 3?

5. What propositions about space are used directly in the proof of Theorem 9? Theorem 13? Corollary 15?

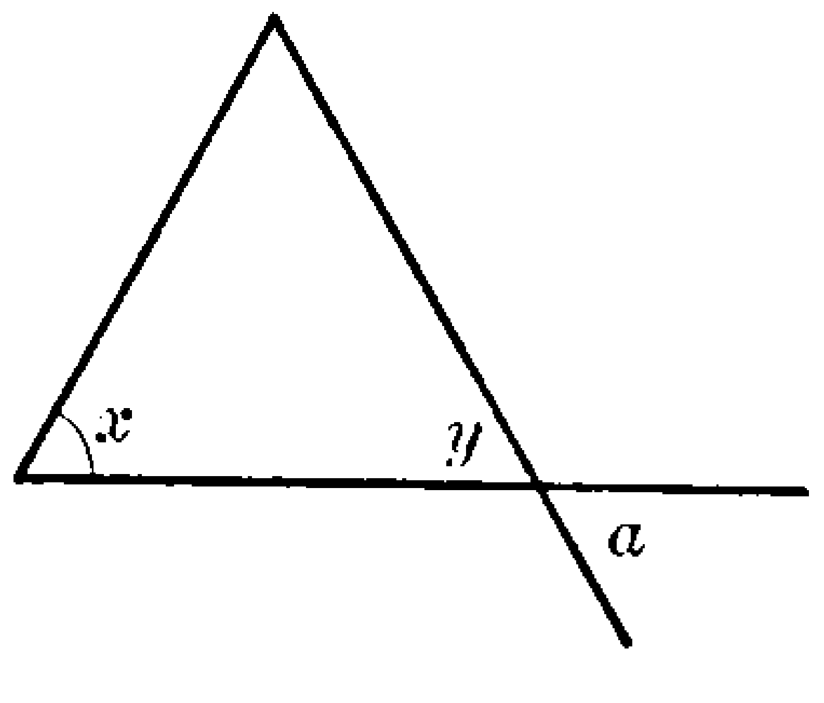
REVIEW EXERCISES

1.



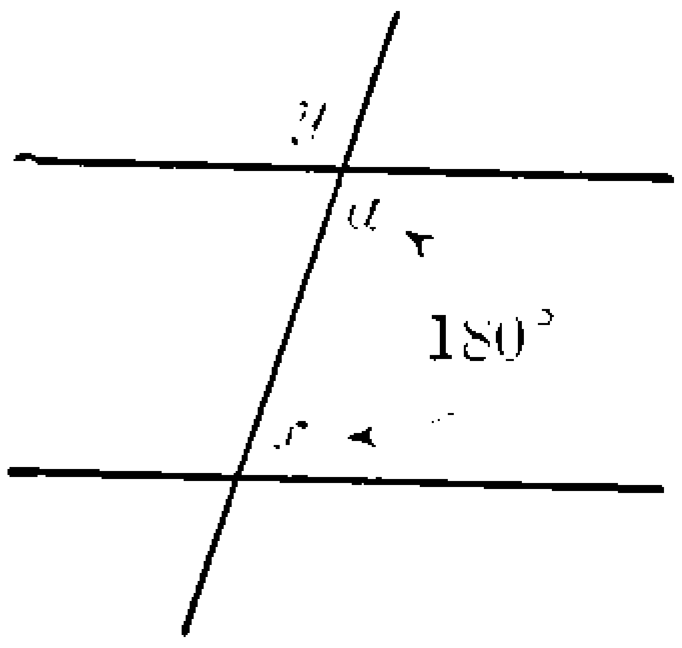
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

2.



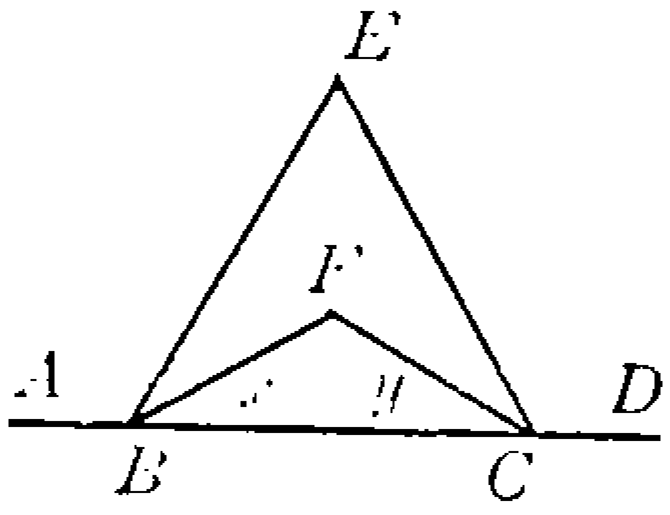
Given:
$\angle x = \angle a$
Prove:
$\angle x = \angle y$

3.



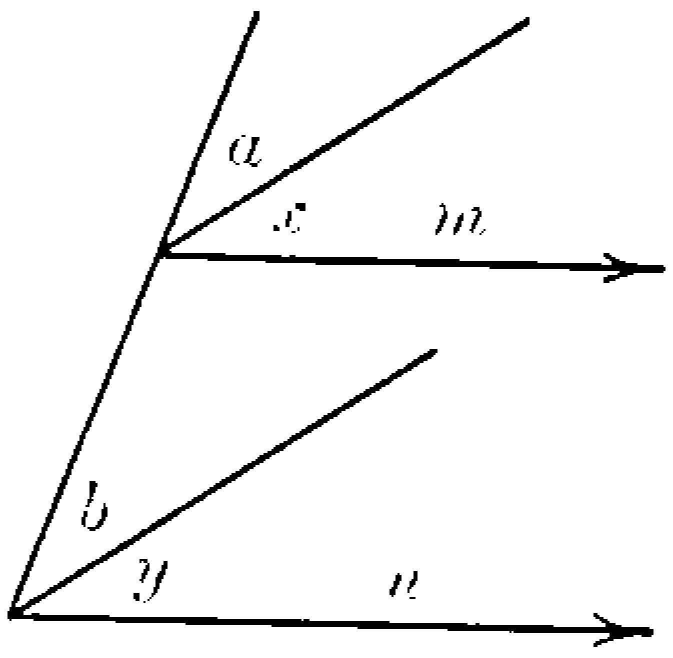
Given:
$\angle x + \angle a = 180^\circ$
Prove:
$\angle x + \angle y = 180^\circ$

4.



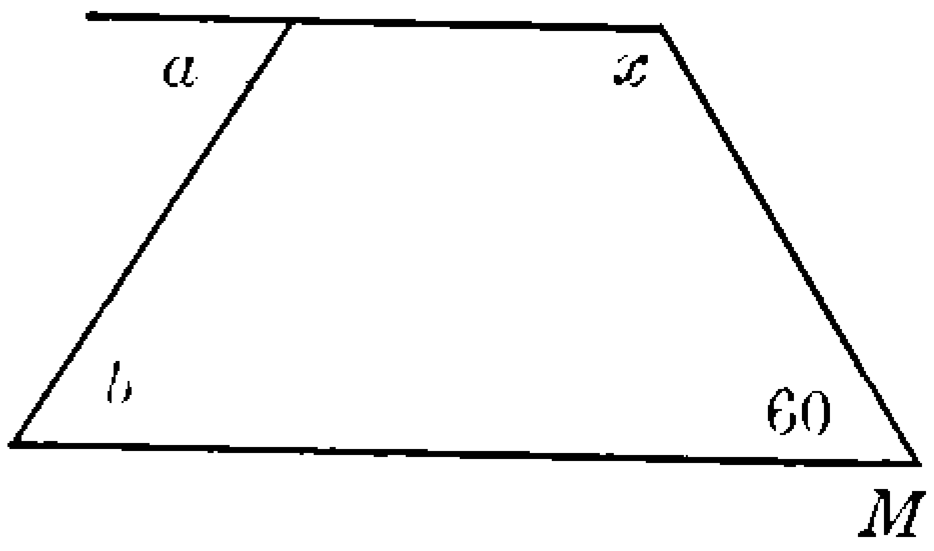
Given:
$\angle ABE = \angle DCE$
$\angle EBF = \angle FBC$
$\angle ECF = \angle FCB$
Prove:
$\angle FBC = \angle FCB$

5.



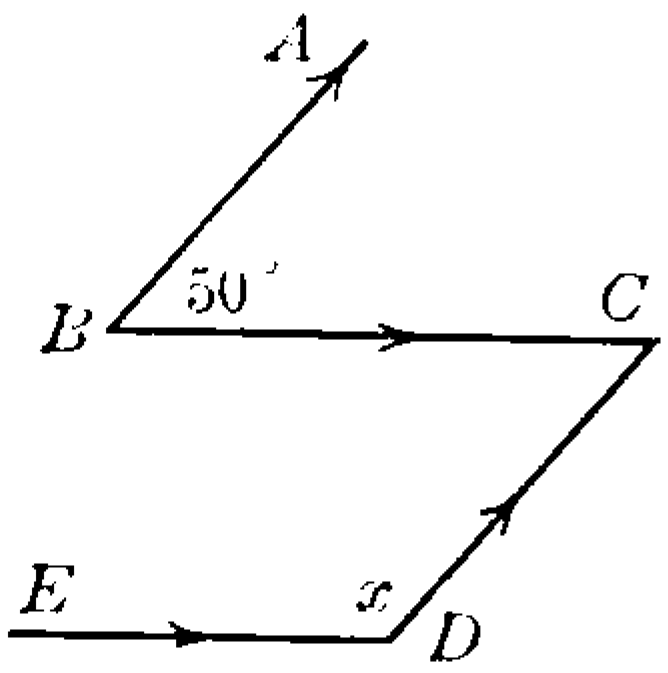
Given:
$m \parallel n$
$\angle x = \angle a$
$\angle y = \angle b$
Prove:
$\angle x = \angle y$

6.



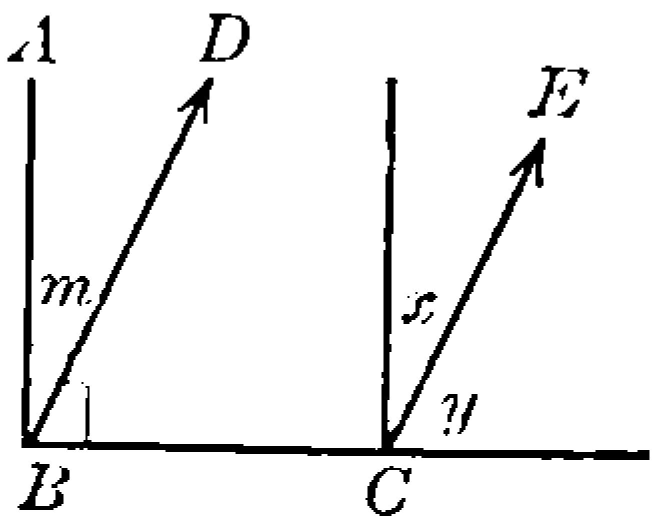
Given:
$\angle a = \angle b$
$\angle M = 60^\circ$
Prove:
$\angle x = 120^\circ$

7.



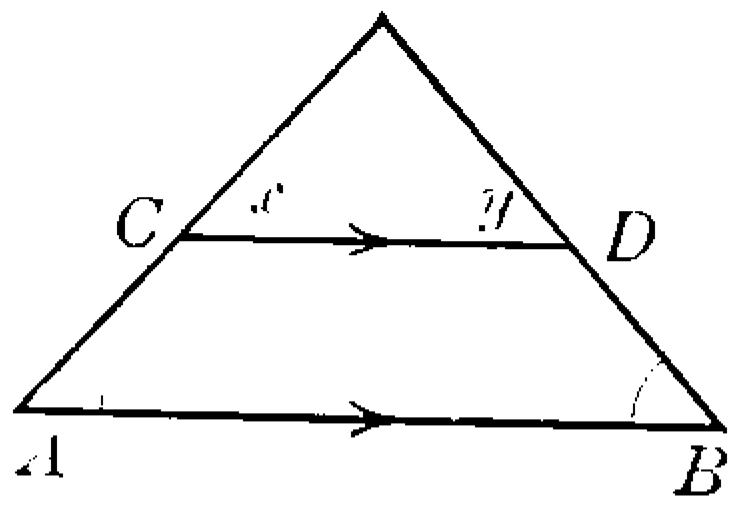
Given:
$AB \parallel CD$
$BC \parallel ED$
$\angle B = 50^\circ$
Prove:
$\angle D = 130^\circ$

8.



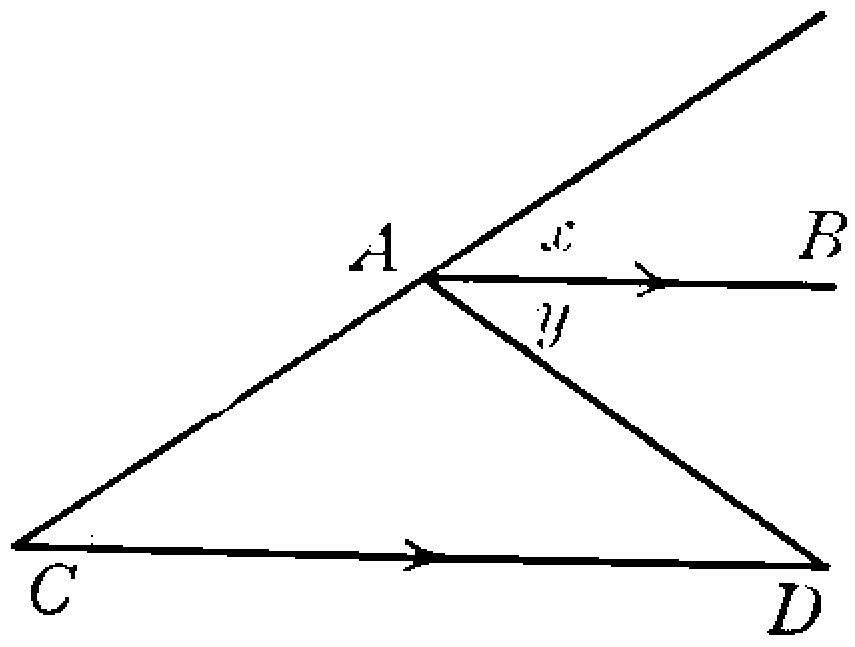
Given:
$AB \perp BC$
$BD \parallel CE$
$\angle m = \angle x$
Prove:
$\angle x + \angle y = 90^\circ$

9.



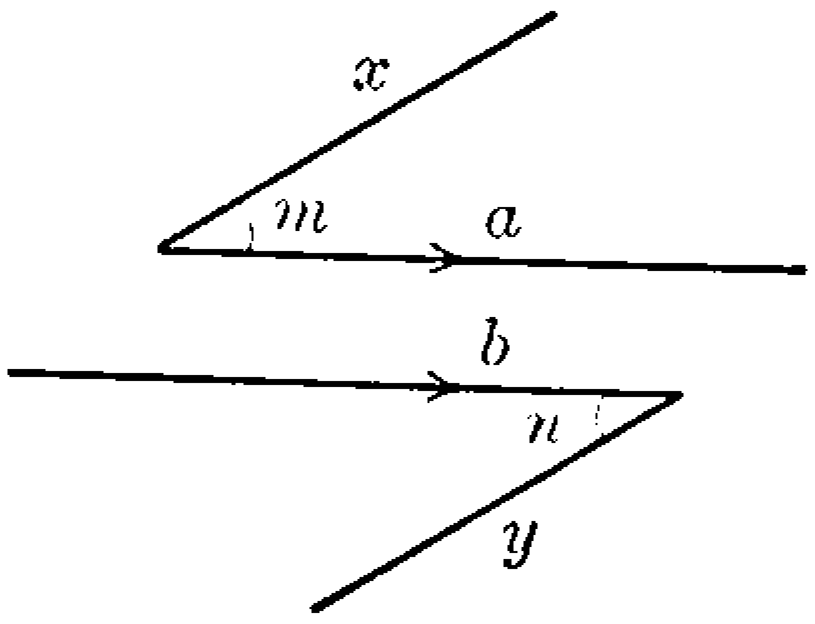
Given:
$CD \parallel AB$
$\angle A = \angle B$
Prove:
$\angle x = \angle y$

10.



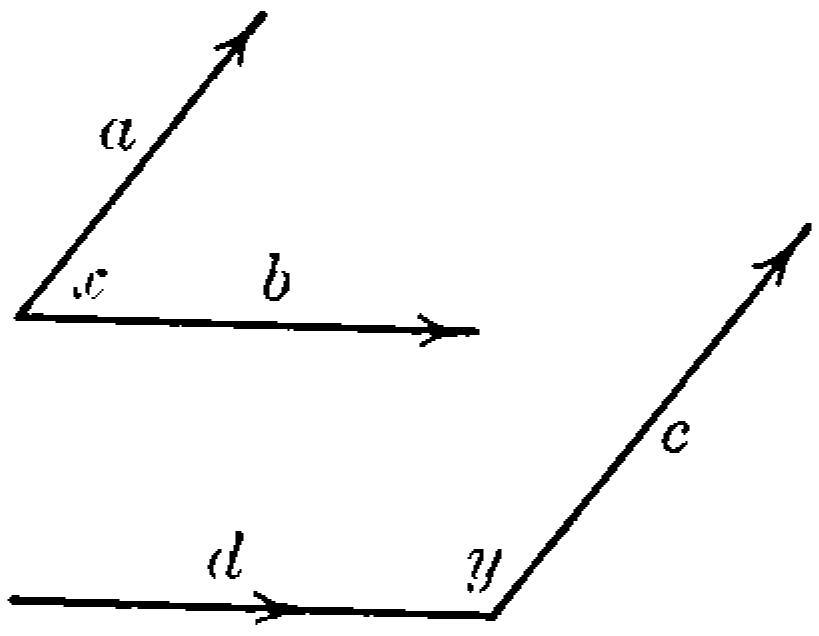
Given:
$AB \parallel CD$
$\angle C = \angle D$
Prove:
$\angle x = \angle y$

11.



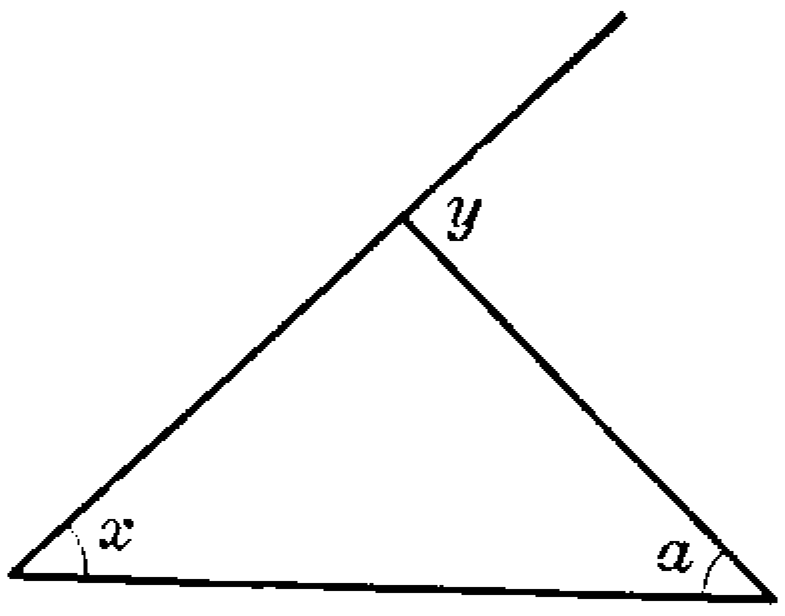
Given:
$a \parallel b$
$\angle m = \angle n$
Prove:
$x \parallel y$

12.



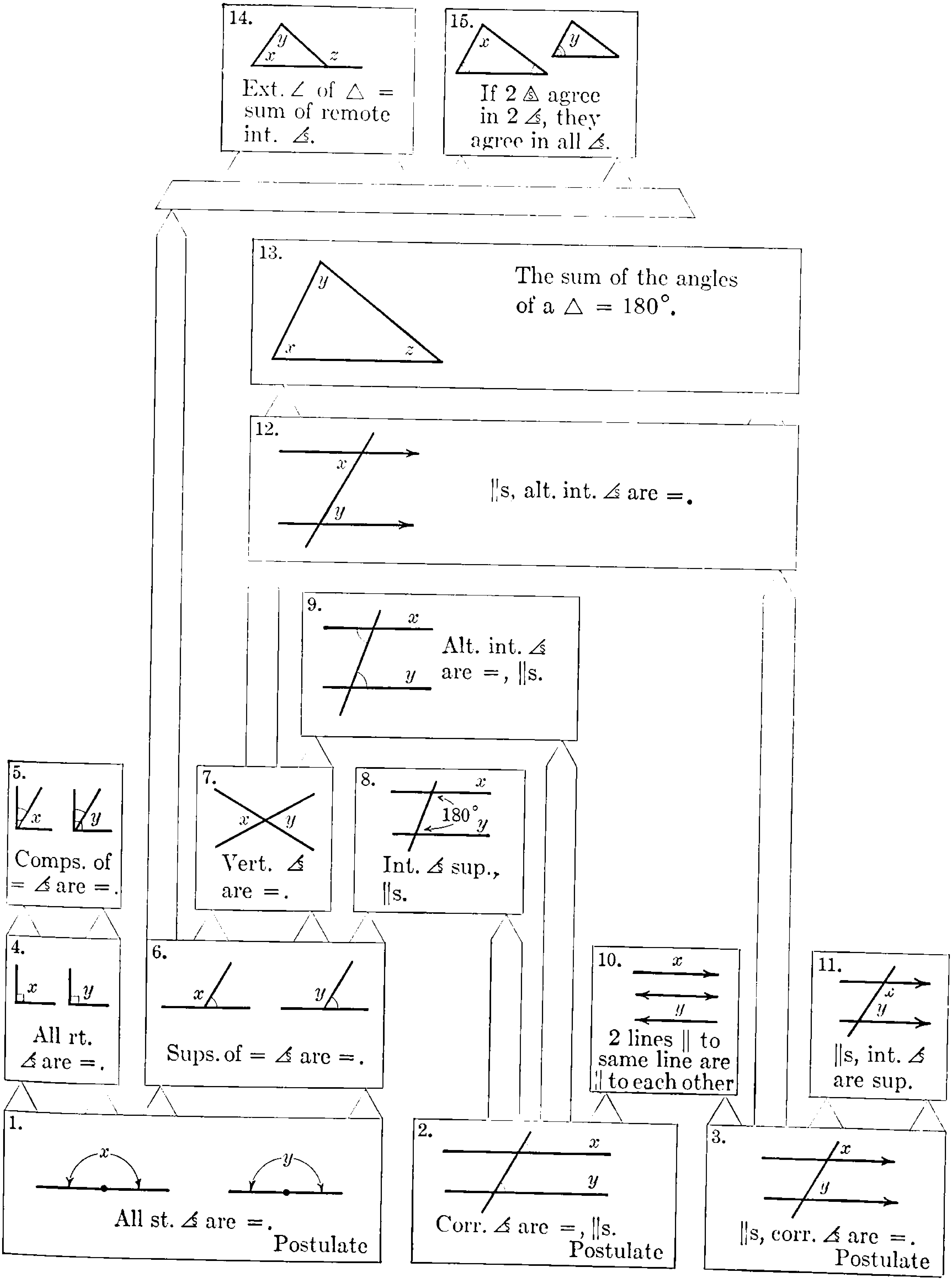
Given:
$a \parallel c$
$b \parallel d$
Prove:
$\angle x + \angle y = 180^\circ$

13.

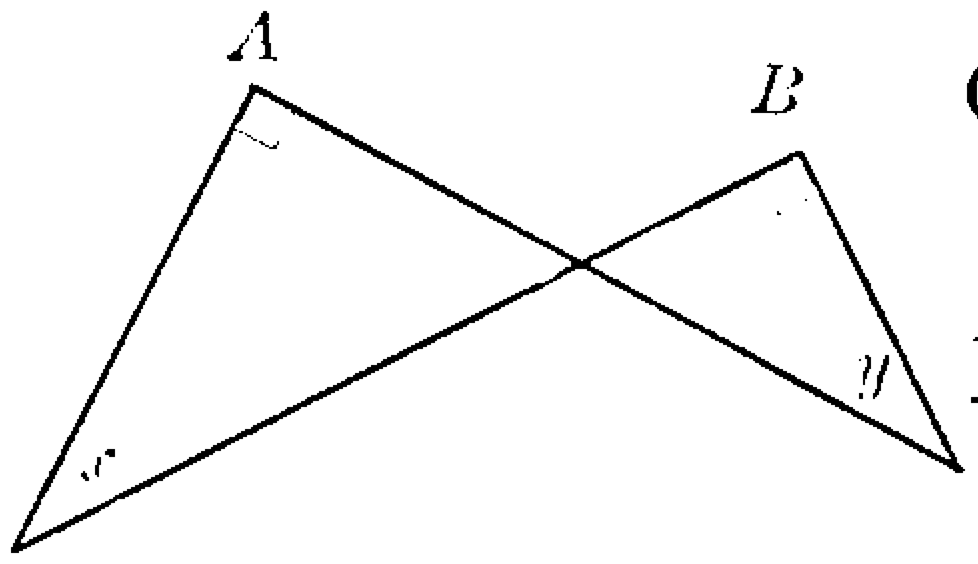


Given:
$\angle x = \angle a$
Prove:
$\angle x = \frac{1}{2}\angle y$

DEPENDENCE CHART

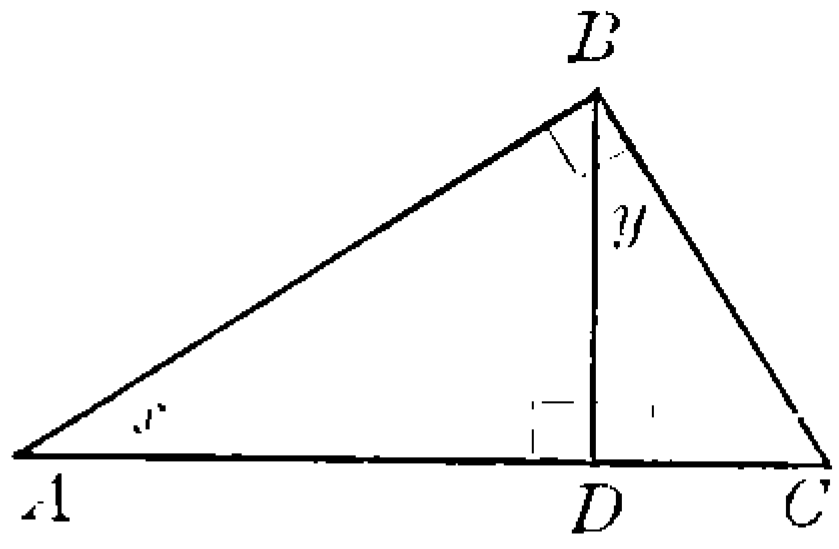


14.



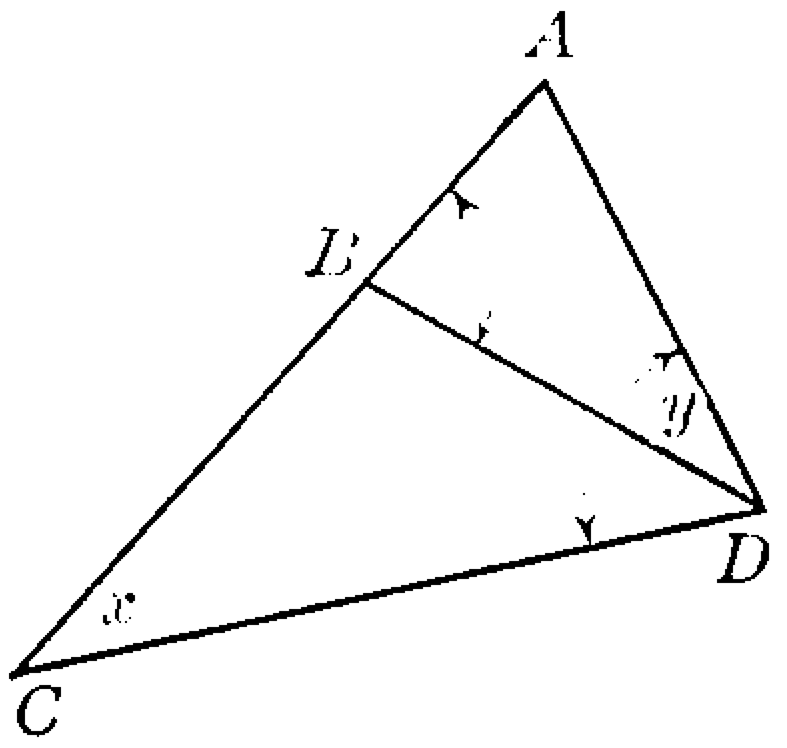
Given:
$\angle A$ and $\angle B$
rt. $\angle$s
Prove:
$\angle x = \angle y$

15.



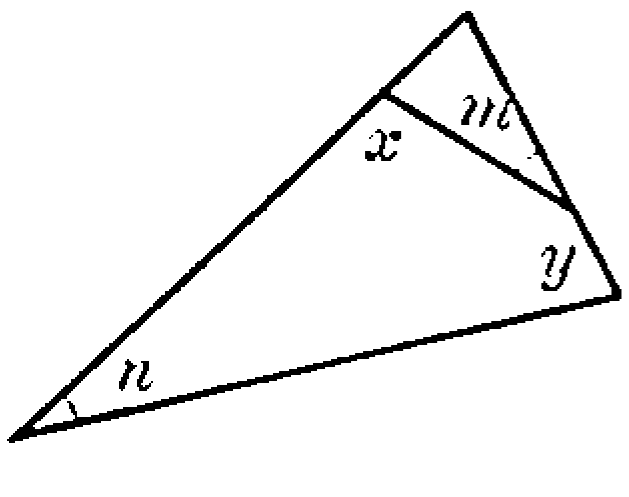
Given:
$AB \perp BC$
$BD \perp AC$
Prove:
$\angle x = \angle y$

16.



Given:
$\angle ABD = \angle D$
Prove:
$\angle x = \angle y$

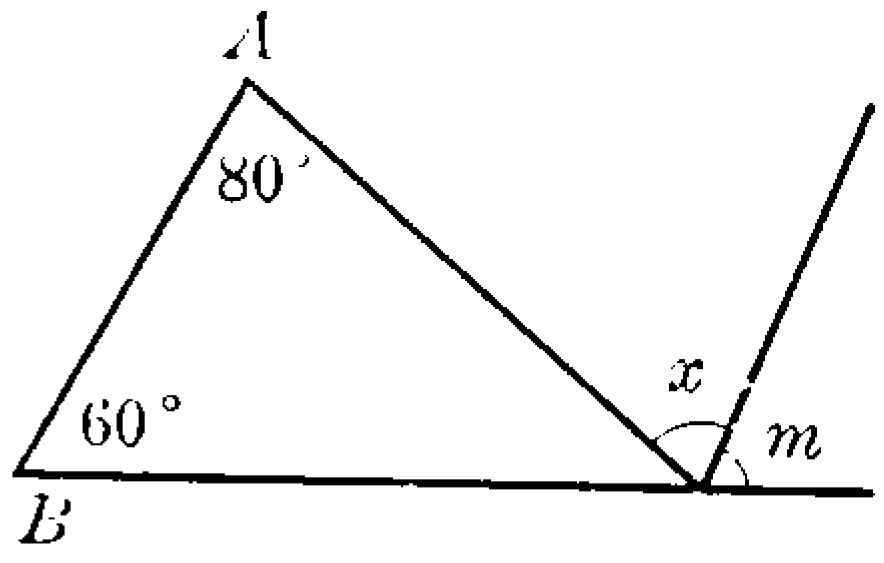
17. Solve the following problems by using algebra:
- (*a*) One angle of a triangle exceeds another angle by 20°, and the third angle equals the sum of the other two. Find the number of degrees in each of the three angles.
- (*b*) One angle of a triangle is three times another angle, and the third angle is half the sum of the other two. Find the number of degrees in each of the three angles.
- (*c*) An exterior angle of a triangle is 140°, and the difference between the nonadjacent interior angles is 20°. Find the number of degrees in each of the three angles of the triangle.
- (*d*) In the figure, $\angle m = \angle n$, and $\angle x$ is twice $\angle y$. Find the number of degrees in $\angle x$ and $\angle y$.



SUPERIOR WORK

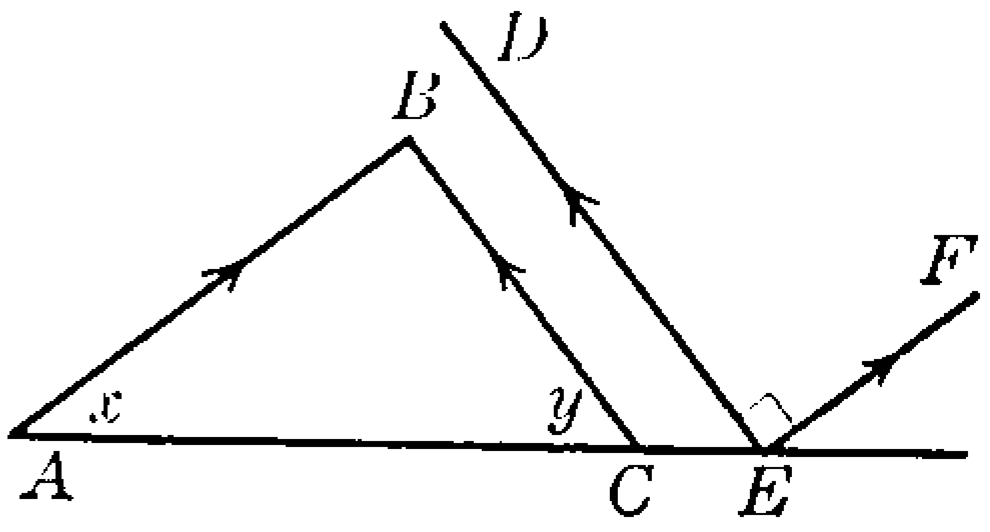
S1. Prove: The bisectors of two adjacent supplementary angles are perpendicular to each other. (To *bisect* means to divide into two equal parts.)

S2.



Given:
$\angle A = 80°$
$\angle B = 60°$
$\angle x = \angle m$
Prove:
$\angle x = 70°$

S3.



Given:
$AB \parallel EF$
$BC \parallel DE$
$\angle DEF = 90°$
Prove:
$\angle x + \angle y = 90°$

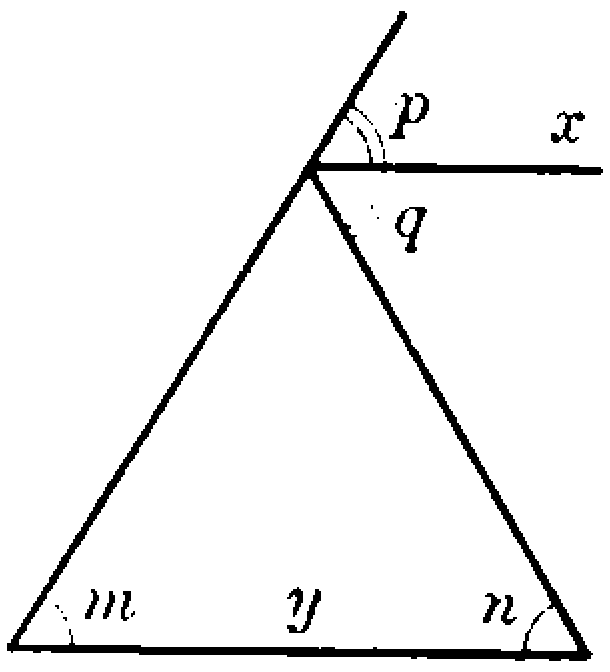
S4. (*a*) Prove: If two angles have their sides respectively parallel, they are either equal or supplementary.

(*b*) Prove: If two angles have their sides respectively perpendicular, they are either equal or supplementary.

S5. Prove: If two perpendicular lines are cut by a transversal, then the sum of the interior angles on the same side of the transversal is either 90° or 270°.

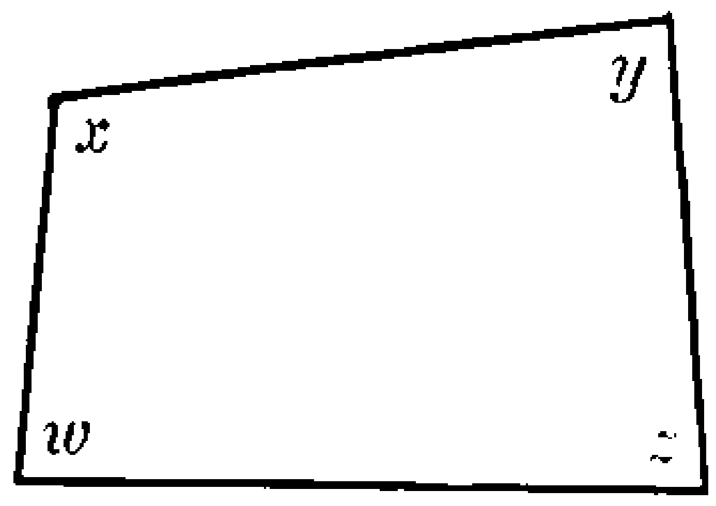
S6. Prove: If two parallel lines are cut by a transversal, then the bisectors of a pair of interior angles on the same side of the transversal are perpendicular to each other.

S7. Prove: If two angles of a triangle are equal, then the bisector of the nonadjacent exterior angle is parallel to the opposite side.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x \parallel y$

S8. Prove: The sum of the angles of a quadrilateral (a closed four-sided figure) is 360°.



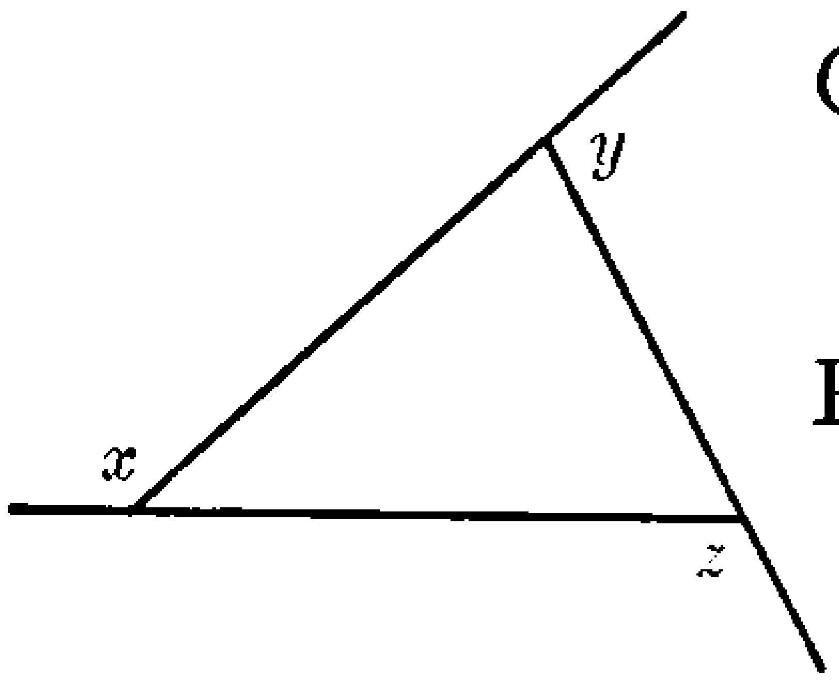
Given:

Quad. with $\angle$s $w, x, y,$ and z

Prove:

$\angle w + \angle x + \angle y + \angle z = 360°$

S9. Prove: The sum of the three exterior angles of a triangle is 360°.



Given:

$\angle$s $x, y,$ and z ext. $\angle$s of $\triangle$

Prove:

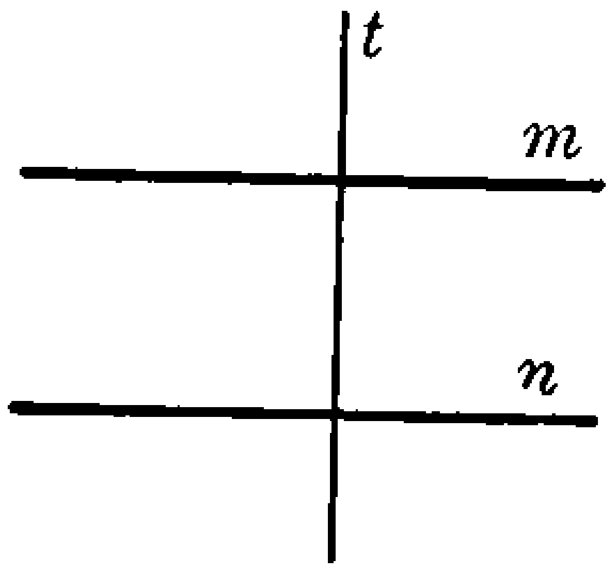
$\angle x + \angle y + \angle z = 360°$

S10. (*a*) Two lines that are not in the same plane are called *skew* lines. Complete the statement: If two lines never meet and yet are not parallel, they are —?— lines.

(*b*) Write a definition of parallel planes.

(*c*) Define a line that is parallel to a plane.

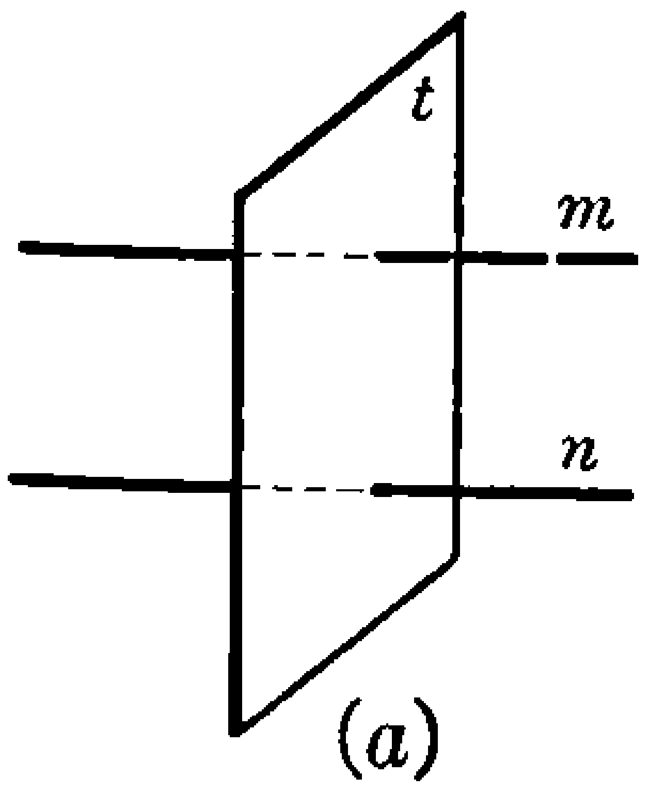
S11. Consider this proposition in plane geometry: If two lines m and n are each perpendicular to a line t, then m and n are parallel.



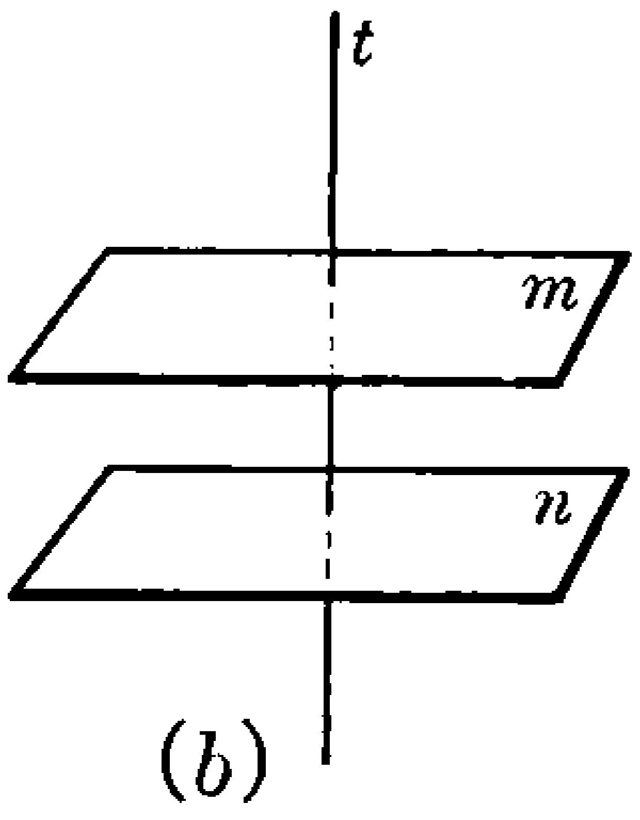
Is this proposition valid if:

(*a*) Line t is changed to plane t?

(*b*) Lines m and n are changed to planes m and n?



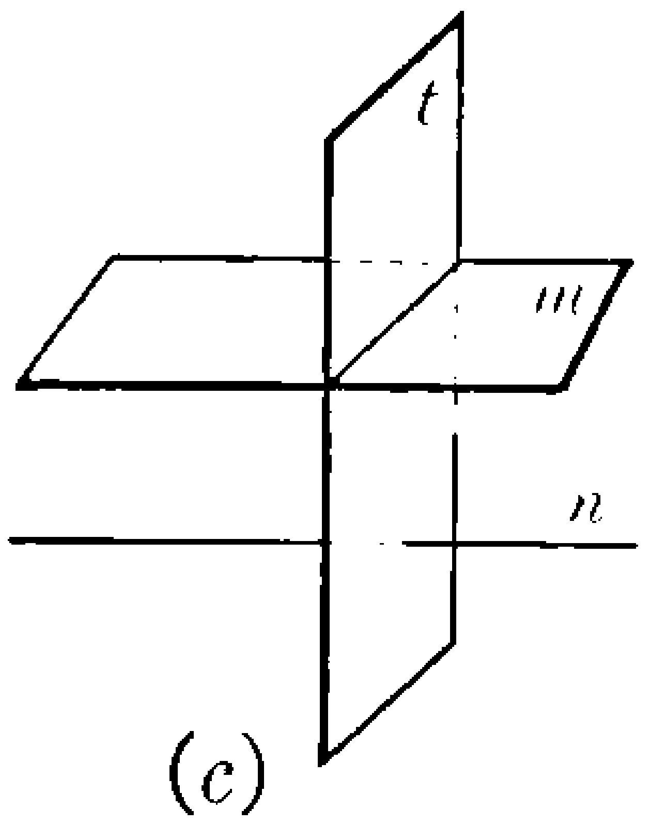
(*a*)



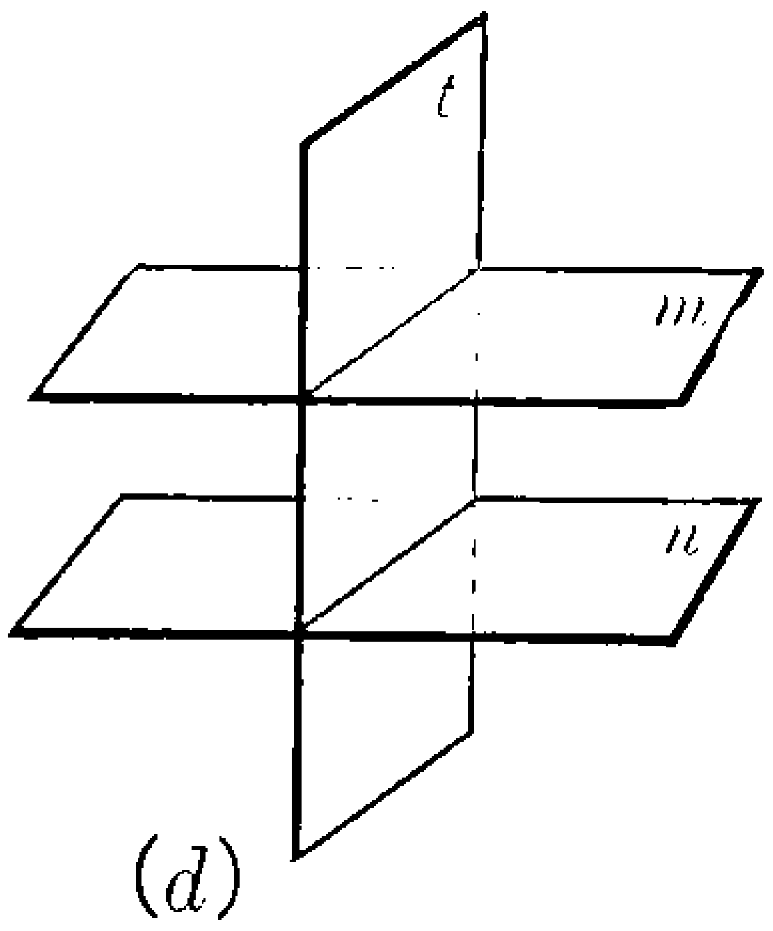
(*b*)

(*c*) Lines m and t are changed to planes m and t?

(*d*) All the lines are changed to planes?



(*c*)

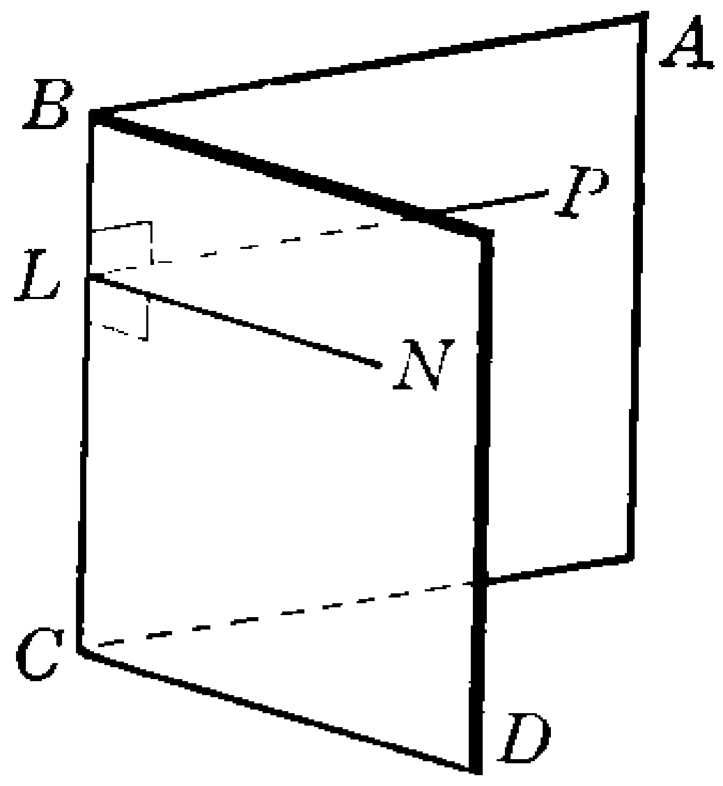


(*d*)

S12. Consider this proposition in plane geometry: If a line t is perpendicular to one of two parallel lines m and n, then it is perpendicular to the other.

Answer the same four questions asked in S11.

S13. In the figure, A-BC-D is a dihedral angle. PL lies in plane AC and is perpendicular to BC at L. NL lies in plane BD and is perpendicular to BC at L. $\angle PLN$ is called a *plane angle* of the dihedral angle.



(*a*) Write a definition of a plane angle of a dihedral angle.

(*b*) How are the sides of two plane angles of the same dihedral angle related?

(*c*) Is the proposition proved in S4 for plane geometry true for general space?

(*d*) A dihedral angle is said to have the same measure (*i.e.*, number of degrees) as its plane angle. Recall the propositions about supplementary, complementary, and vertical plane angles. Do these hold true for dihedral angles? Explain.

814. A plane n contains a line l. What statement can you make about the plane n if:

(a) The line l is perpendicular to another plane m?

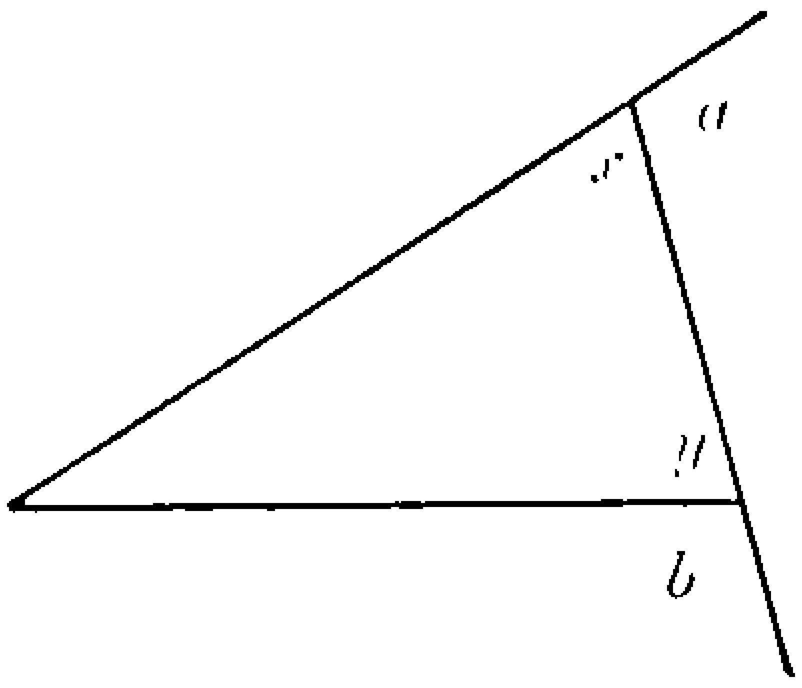
(b) The line l is parallel to another plane m?

(c) The line l is parallel to another line m?

Draw figures to illustrate your answers.

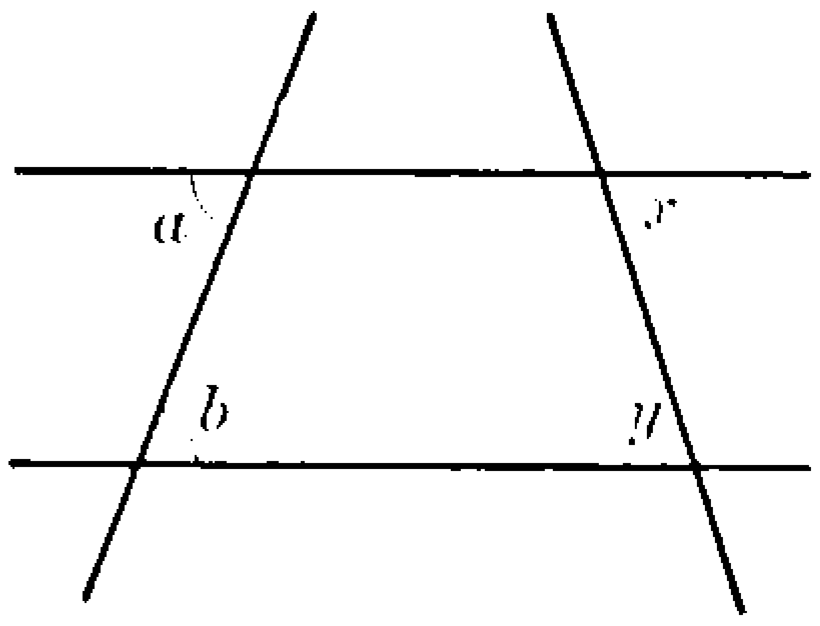
TEST

1.



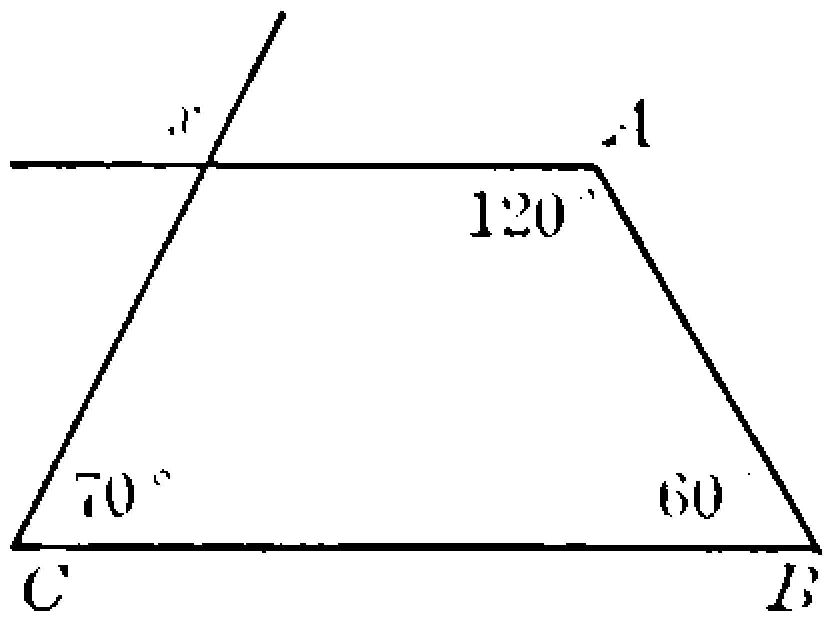
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

2.



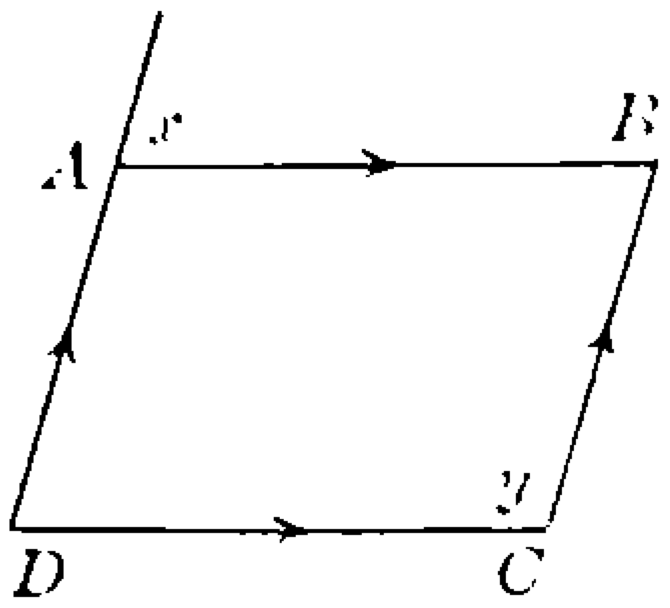
Given:
$\angle a = \angle b$
Prove:
$\angle x = \angle y$

3.



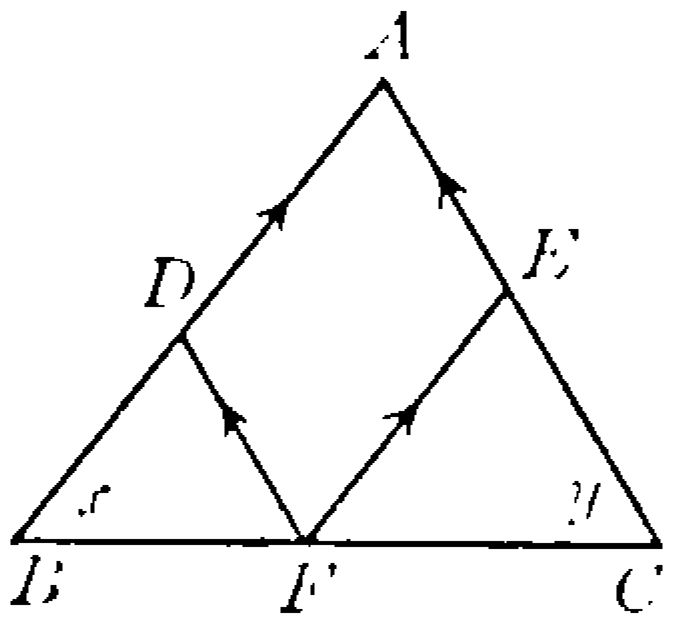
Given:
$\angle A = 120^\circ$
$\angle B = 60^\circ$
$\angle C = 70^\circ$
Prove:
$\angle x = 110^\circ$

4.



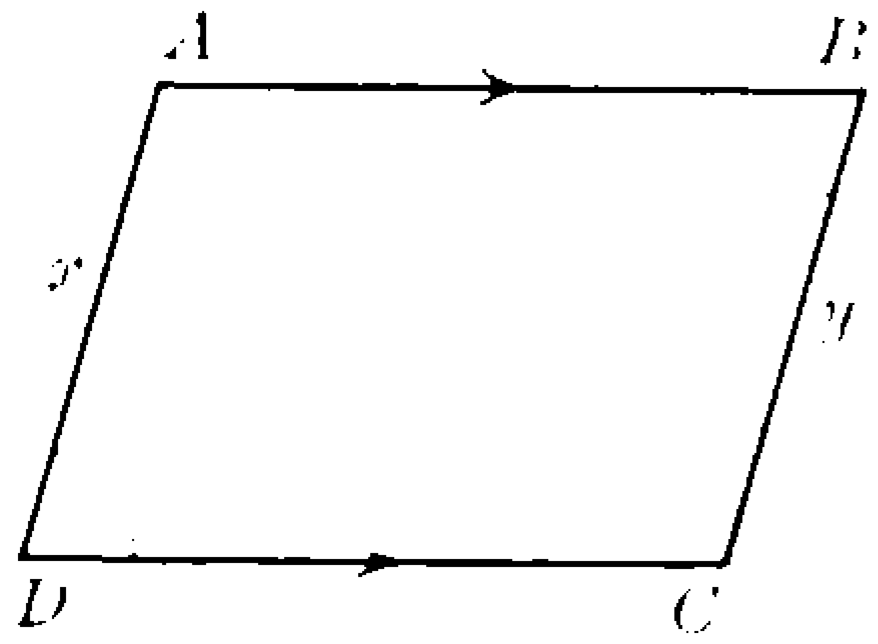
Given:
$AB \parallel DC$
$AD \parallel BC$
Prove:
$\angle x + \angle y = 180^\circ$

5.



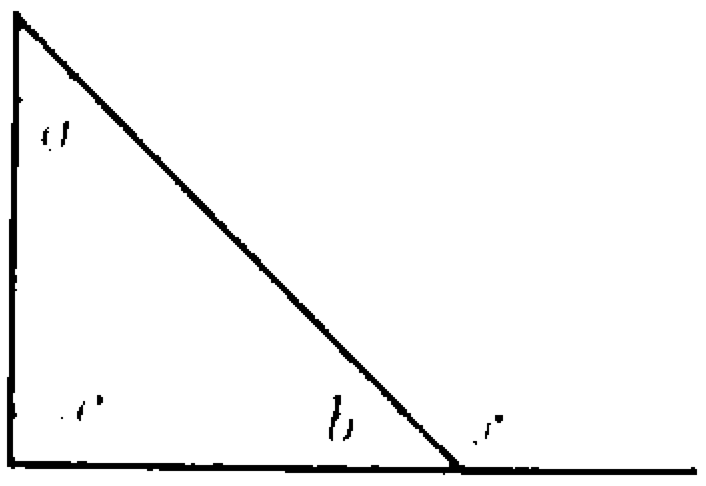
Given:
$AC \parallel DF$
$AB \parallel EF$
$\angle DFB = \angle EFC$
Prove:
$\angle x = \angle y$

6.



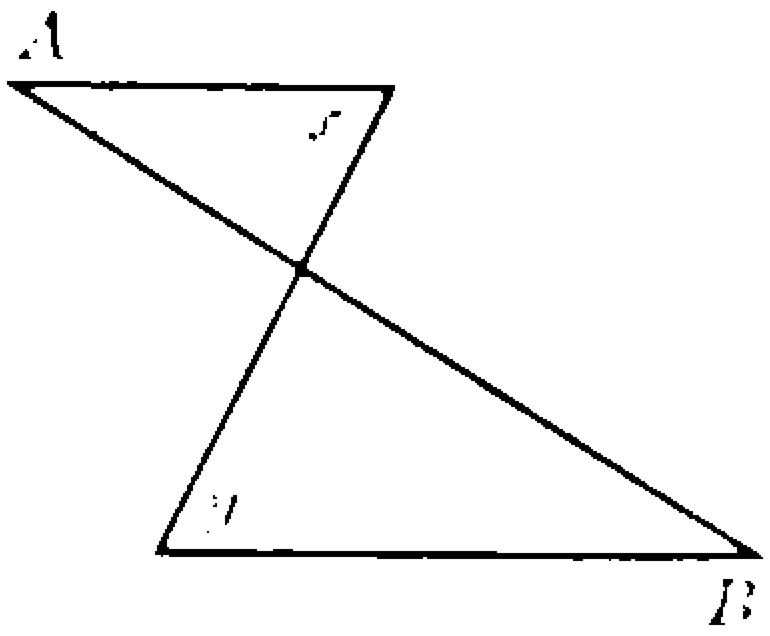
Given:
$AB \parallel DC$
$\angle B = \angle D$
Prove:
$x = y$

7.



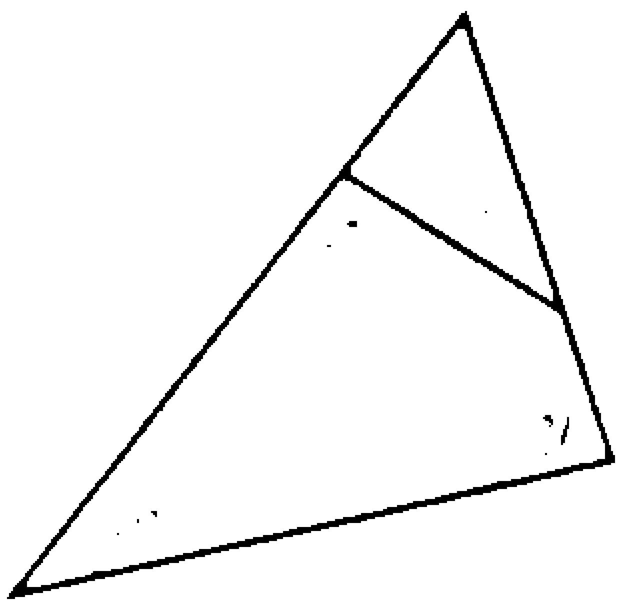
Given:
$\angle c = 90^\circ$
$\angle a = \angle b$
Prove:
$\angle x = 135^\circ$

8.



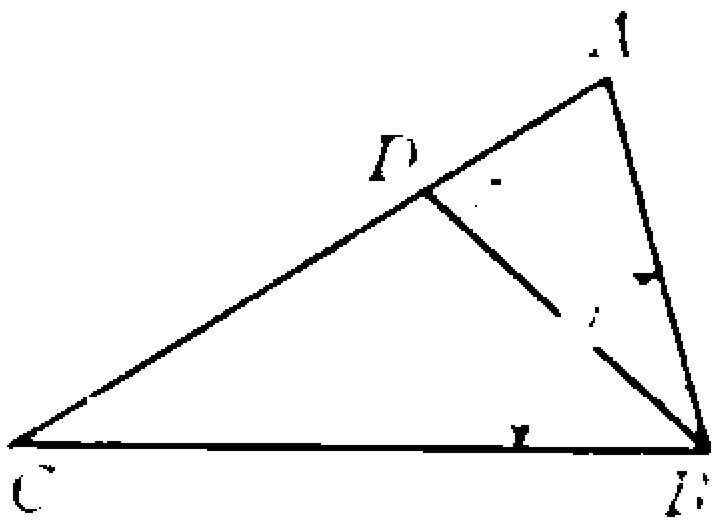
Given:
$\angle A = \angle B$
Prove:
$\angle x = \angle y$

9.



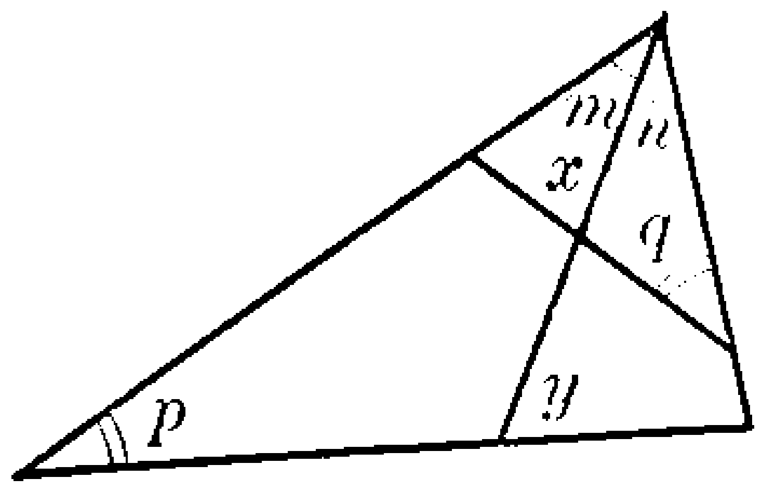
Given:
$\angle m = \angle r$
Prove:
$\angle x + \angle y = 180^\circ$

10.



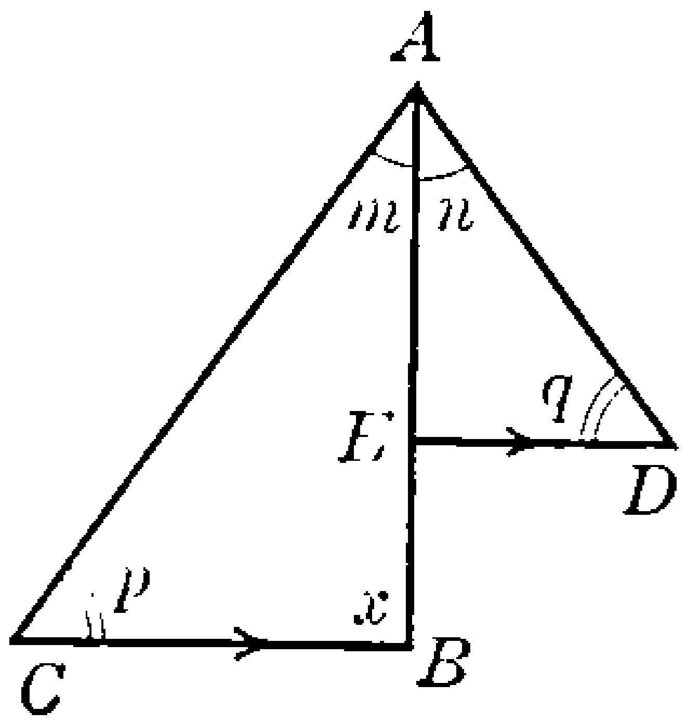
Given:
$\angle ABD = \angle C$
Prove:
$\angle ADB = \angle B$

11.



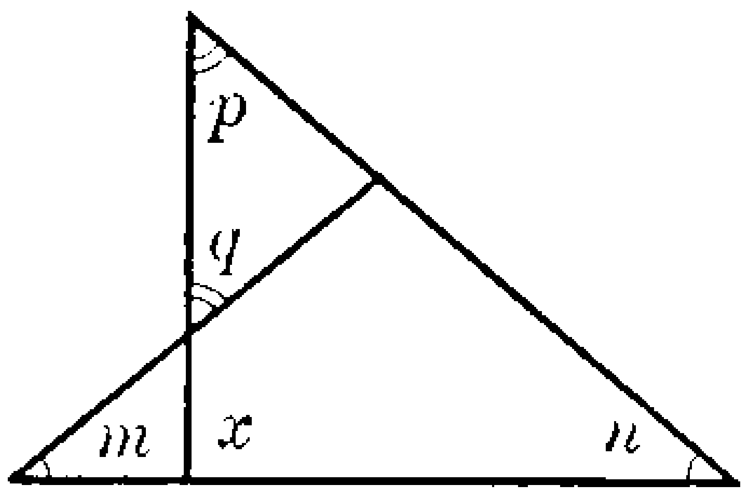
Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$\angle x = \angle y$

12.



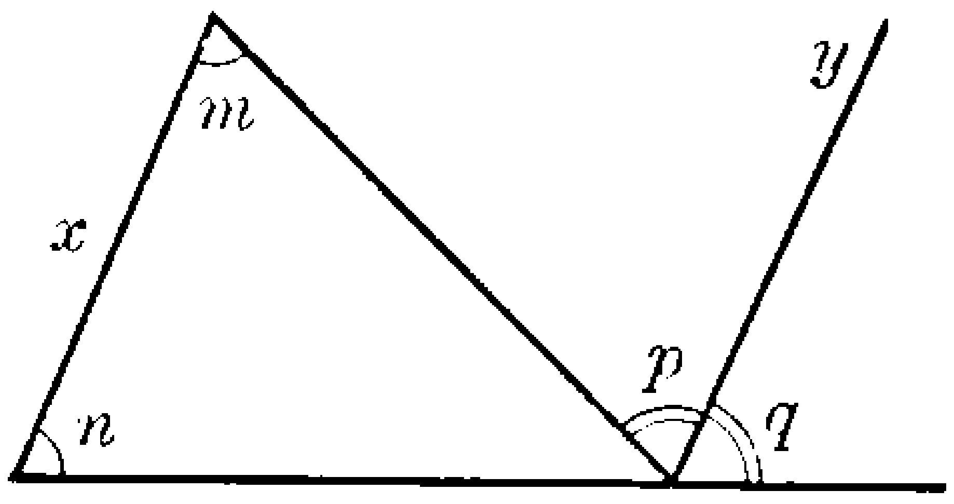
Given:
$ED \parallel CB$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$AB \perp BC$

13.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$\angle x = 90°$

14.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x \parallel y$

3

The Proposition

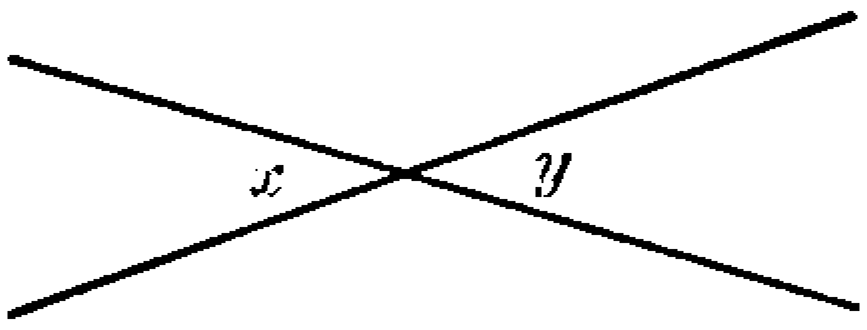
How Propositions are Related

ILLUSTRATING GENERAL PROPOSITIONS

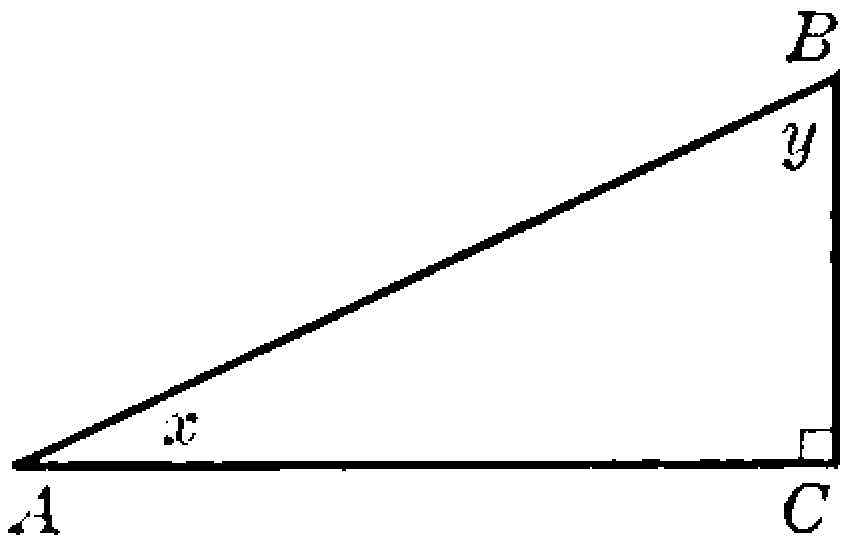
In this chapter we shall learn how to form general propositions, how to illustrate them, how to form new propositions from them, and how to prove a proposition indirectly by proving another proposition.

Why We Illustrate

68. A general proposition is a statement about many things of a certain kind. The statement that vertical angles are equal is an example of a general proposition because it has to do with all pairs of vertical angles. To prove this proposition, as we did in Chapter 2, we draw *one* typical pair of vertical angles x and y and prove them equal. The drawing is an *illustration* of the general proposition. You find the same idea outside the field of geometry where diagrams and pictures are used to illustrate some general description or explanation.

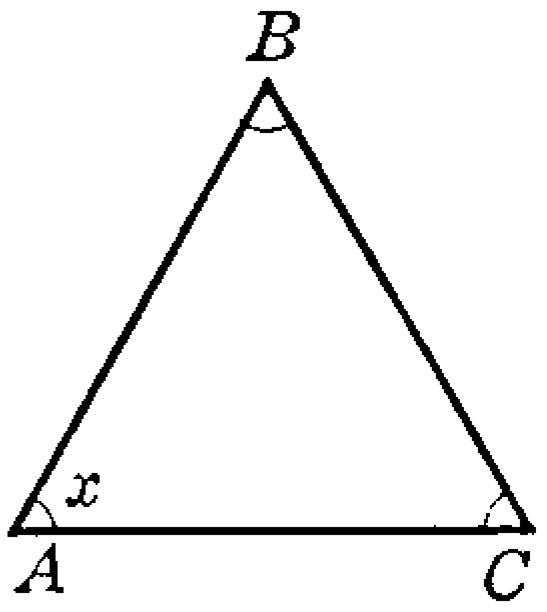


Illustrative Example. Illustrate the following general proposition: If one angle of a triangle is a right angle, the other two angles are complementary.



Given:
  In $\triangle ABC$
  $\angle C = 90°$
Prove:
  $\angle A + \angle B = 90°$

Illustrative Example. Illustrate the general proposition: If the three angles of a triangle are equal, each one is 60°.

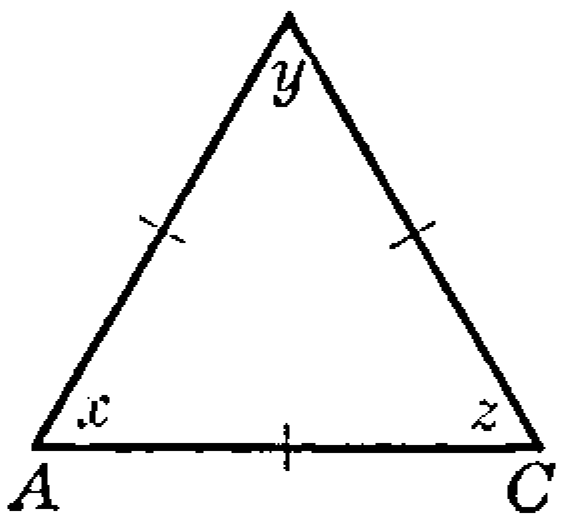


Given:
  In $\triangle ABC$
  $\angle A = \angle B = \angle C$
Prove:
  $\angle A = 60°$

Do you see that the illustration of a general proposition makes its meaning clear and enables us to prove it more easily by studying one typical case?

Forming a General Proposition

69. Suppose that a diagram is given and that you are asked to write the general proposition illustrated by it. The diagram chosen might be the one shown in which $AB = BC = CA$, and in which you are asked to prove that $\angle A = \angle B = \angle C$. The general proposition might be stated as follows: *If the three sides of a triangle are equal, then the three angles are equal.*



Propositions about Triangles

70. A general proposition need not always be stated in exactly the same terms. For example, after you have learned the names of certain triangles and know that a triangle with three equal sides is equilateral and a triangle with three equal angles is equiangular, the general proposition stated in § 69 might be worded as follows: *If a triangle is equilateral, then it is equiangular.*

In order to discuss triangles clearly and in as few words as possible, it is necessary to make use of the following definitions:

71. Definition. A polygon is a closed broken line (§ 22).

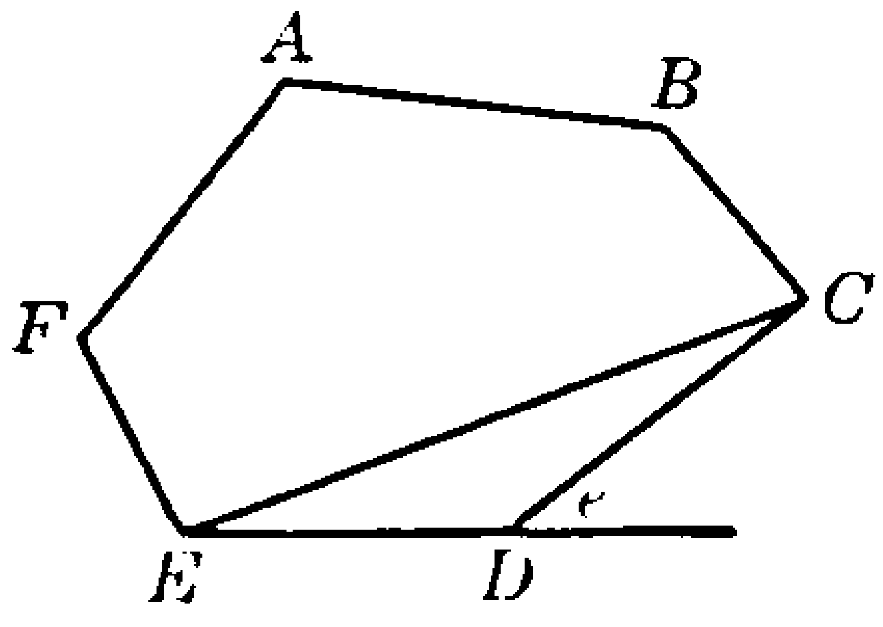
72. Definition. A vertex of a polygon is the vertex of an angle of the polygon.

73. Definition. A diagonal of a polygon is a line segment connecting two nonconsecutive vertices of the polygon.

A *convex* polygon is one in which all the diagonals fall inside the figure. It is the only kind we shall study.

74. Definition. An exterior angle of a polygon is an angle formed by one side of the polygon and the prolongation of an adjacent side.

Name a vertex, a diagonal, and an exterior angle of the polygon $ABCDEF$.



When we use the term *angles of a polygon* without either *interior* or *exterior*, we mean the interior angles.

75. Definition. A triangle is a polygon having three sides.

Triangles may be classified according to their sides or according to their angles.

Triangles Classified by Sides

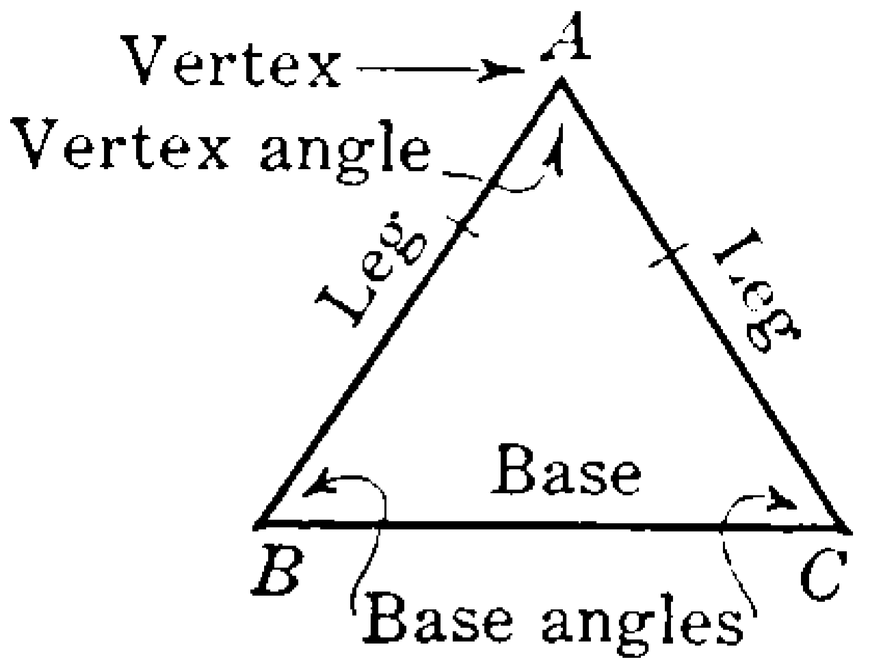
Classified according to their sides, the three kinds of triangles are: (*a*) isosceles triangles, (*b*) equilateral triangles, and (*c*) scalene triangles.

76. An isosceles triangle is a triangle that has two equal sides.

The equal sides of an isosceles triangle are called the legs, and the third side is called the base.

The vertex of an isosceles triangle usually means the vertex opposite the base. We shall call the other two vertices the ends of the base.

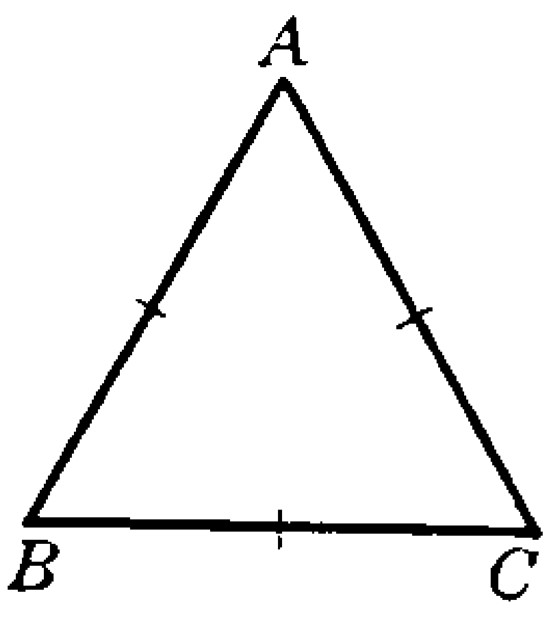
The vertex angle of an isosceles triangle is the angle at the vertex. The other two angles are called the base angles.



$AB = AC$

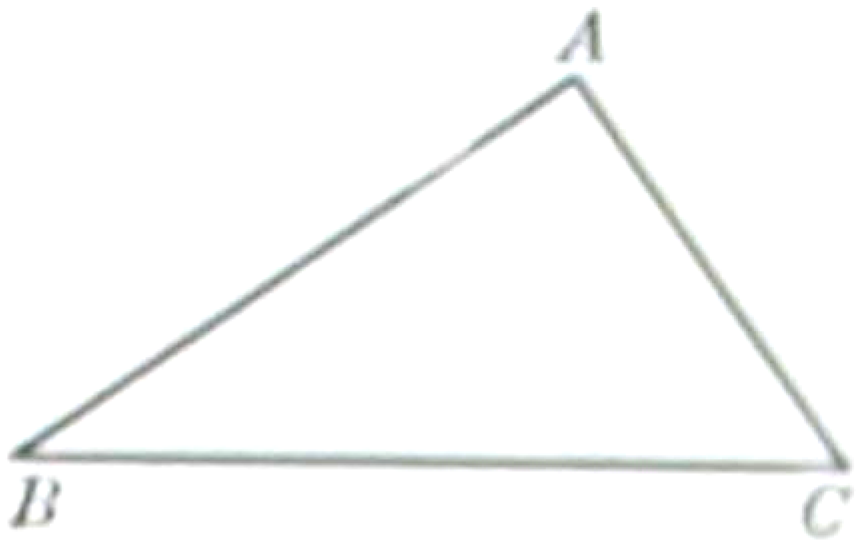
$\triangle ABC$ isosceles
AB and AC the legs
BC the base
A the vertex
$\angle A$ the vertex angle
B and C the ends of the base
$\angle B$ and $\angle C$ the base angles

77. An equilateral triangle is a triangle with three equal sides.



$AB = BC = CA$

78. A scalene triangle is a triangle with no two sides equal.



$AB \neq BC \neq$
$CA \neq AB$

When we speak of a triangle, we mean a scalene triangle unless other conditions make it isosceles or equilateral. In a scalene triangle we think of the side on which the triangle rests as the base. The triangle may, of course, be made to rest on any side.

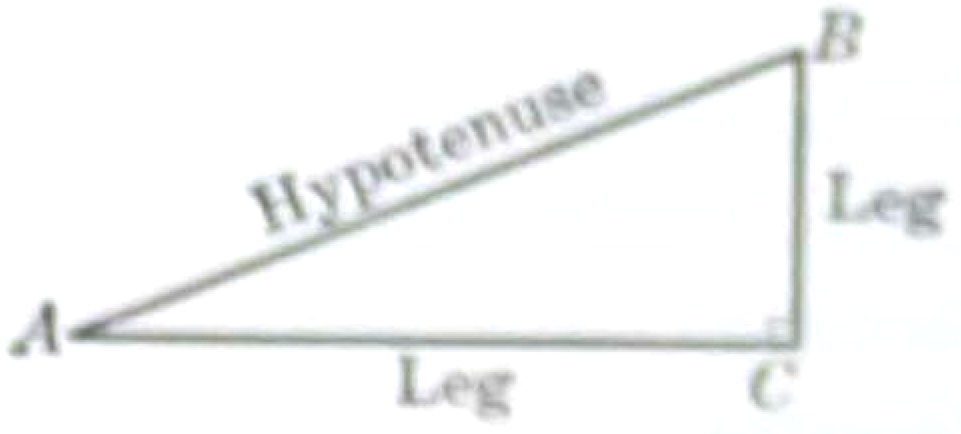
Triangles Classified by Angles

Classified according to their angles, the three kinds of triangles are: (*a*) right triangles, (*b*) obtuse triangles, and (*c*) acute triangles.

79. A right triangle is a triangle that has a right angle.

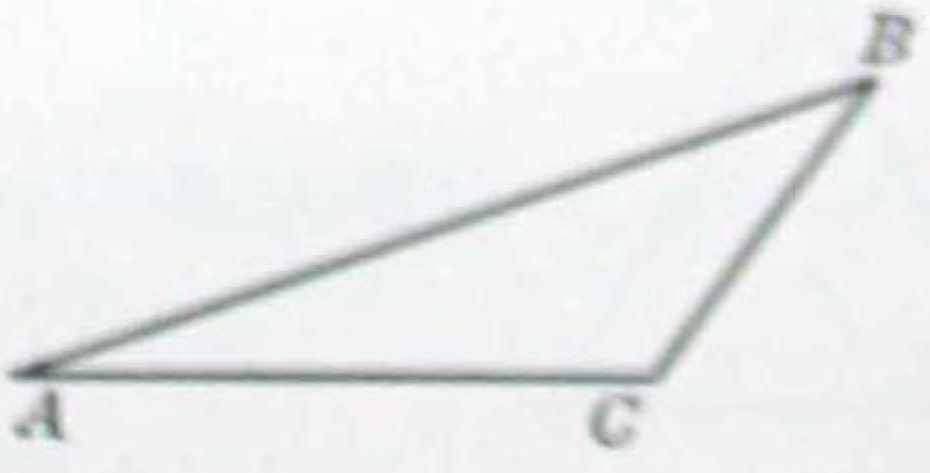
The legs of a right triangle are the sides of the right angle.

The hypotenuse of a right triangle is the side opposite the right angle.

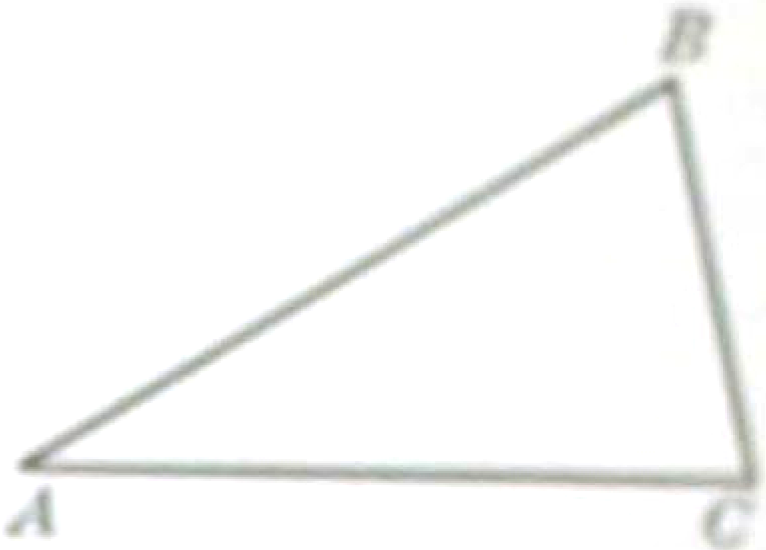


$\angle C$ a right angle
$\triangle ABC$ a right triangle
AC and BC the legs
AB the hypotenuse

80. An obtuse triangle is a triangle that has an obtuse angle.



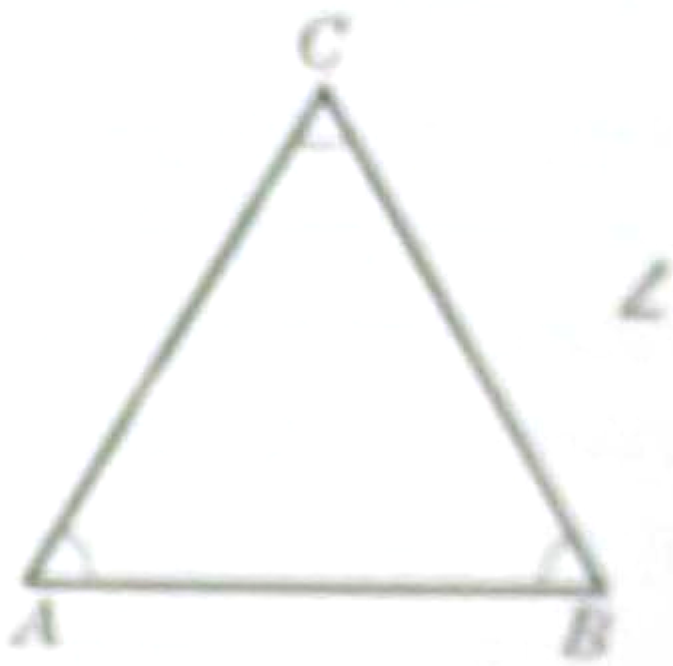
81. An acute triangle is a triangle in which all the angles are acute.



$\angle A$, $\angle B$, and $\angle C$ acute
$\triangle ABC$ an acute triangle

As a special case of an acute triangle, we have an equiangular triangle.

82. An equiangular triangle is a triangle in which all the angles are equal.



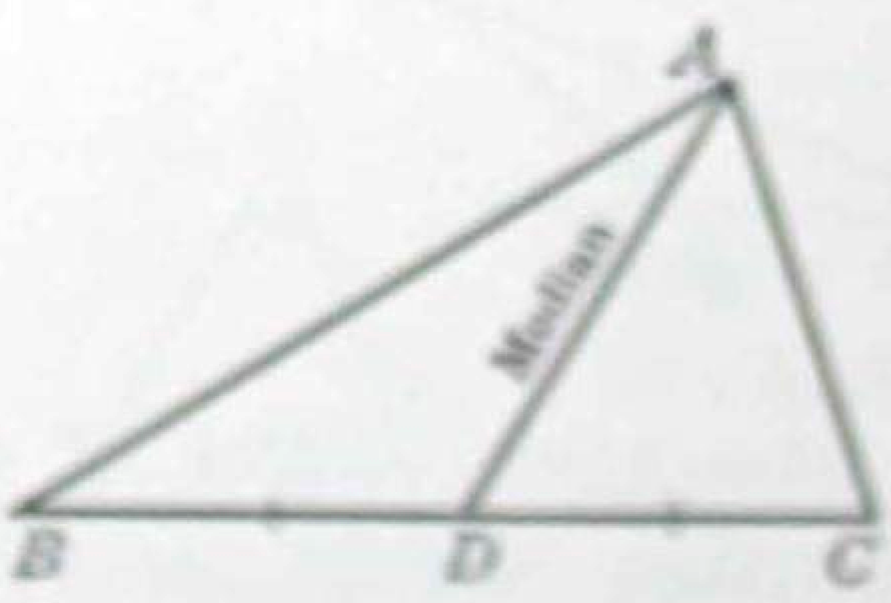
$\angle A = \angle B = \angle C$

83. An oblique triangle is a triangle that does not have a right angle.

Obtuse and acute triangles are *oblique* triangles.

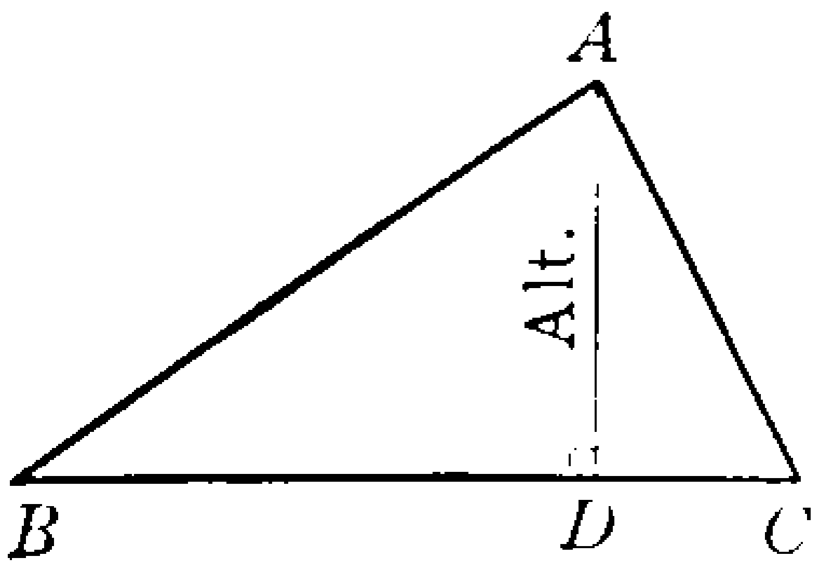
Secondary Parts of a Triangle

84. A median of a triangle is a line segment from a vertex to the midpoint of the opposite side.

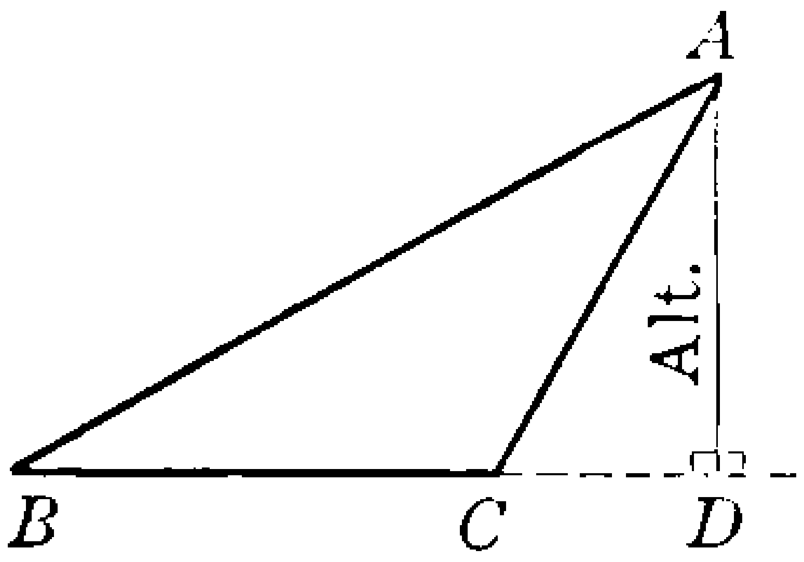


$BD = DC$
AD a median of $\triangle ABC$

85. An altitude of a triangle is a line segment from a vertex of the triangle perpendicular to the opposite side.



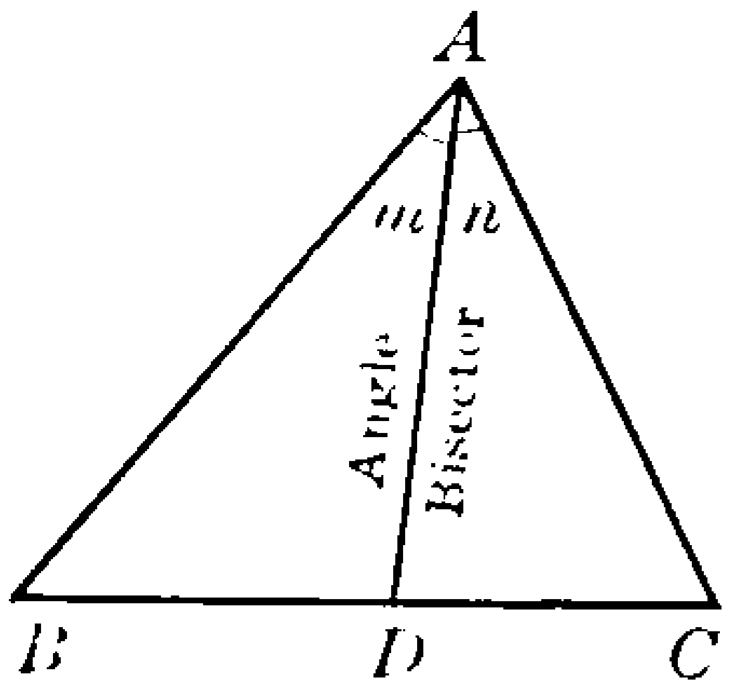
$AD \perp BC$
AD an alt.
of $\triangle ABC$



AD falls
outside
obtuse $\triangle ABC$

86. To bisect means to divide into two equal parts.

87. An angle bisector of a triangle is a line segment bisecting an angle of the triangle and extending to the opposite side.

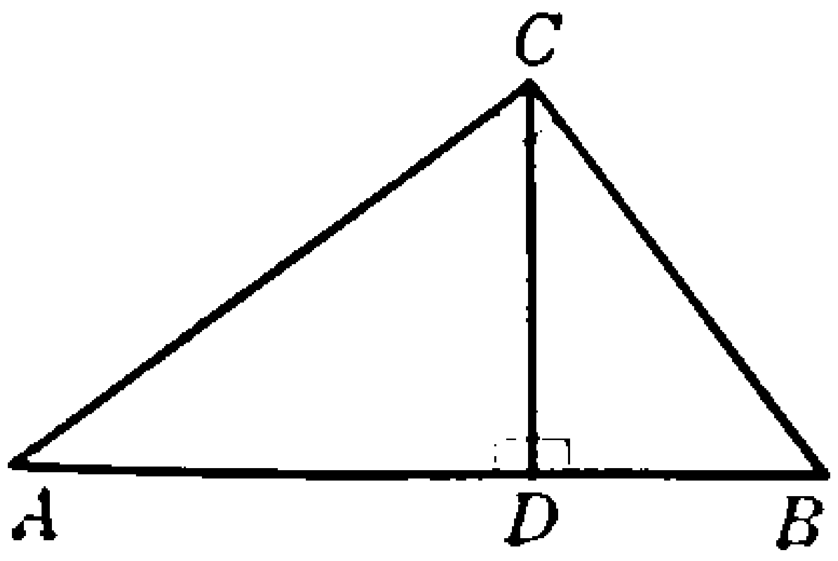


$\angle m = \angle n$
AD an angle
bisector of
$\triangle ABC$

EXERCISES

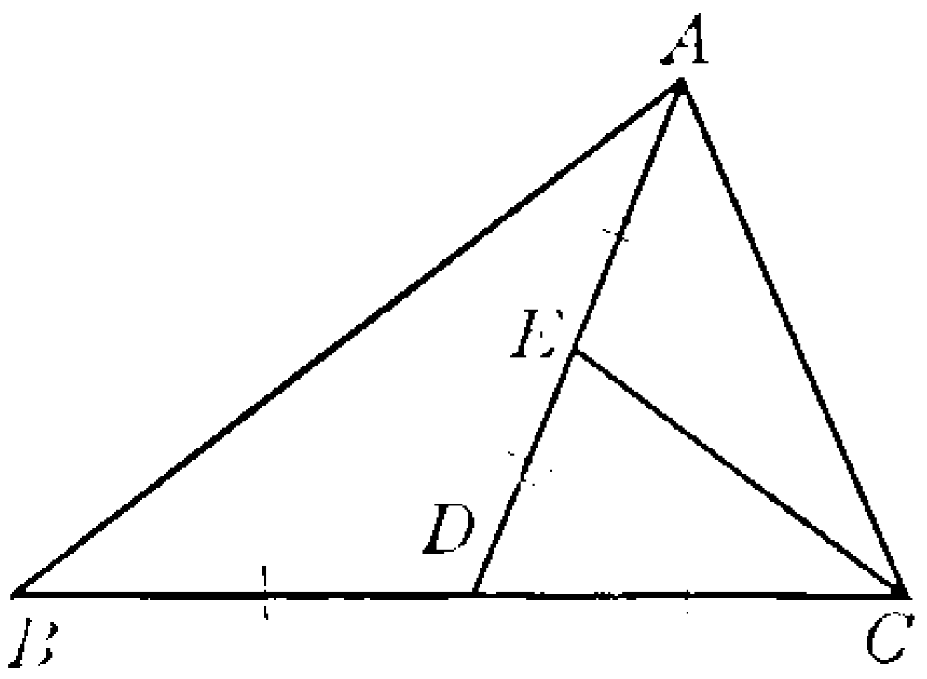
In each of the following exercises, you are to give a description, or draw a figure, or make a generalization. Make your descriptions exact, your figures general.

Illustrative Example. Draw a figure illustrating the altitude to the hypotenuse of a right triangle.



$\angle C = 90°$
$CD \perp AB$
CD the alt.
to hypotenuse of
rt. $\triangle ABC$

Illustrative Example

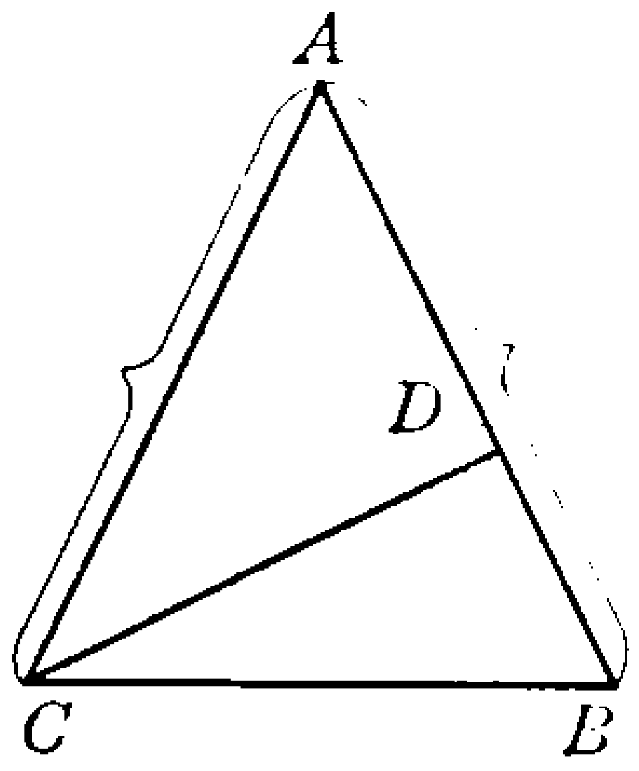


$BD = DC$
$AE = ED$
Describe CE.

Description. CE is a line segment connecting a vertex of a triangle with the midpoint of a median.

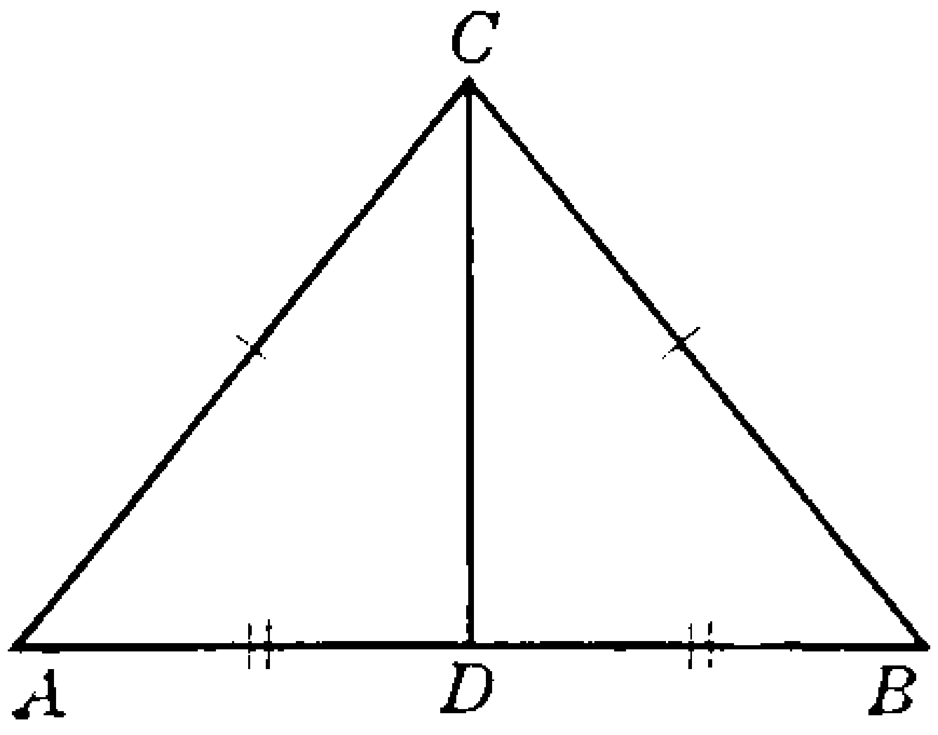
1. Draw a figure illustrating an isosceles right triangle. Write out the known conditions.

2.



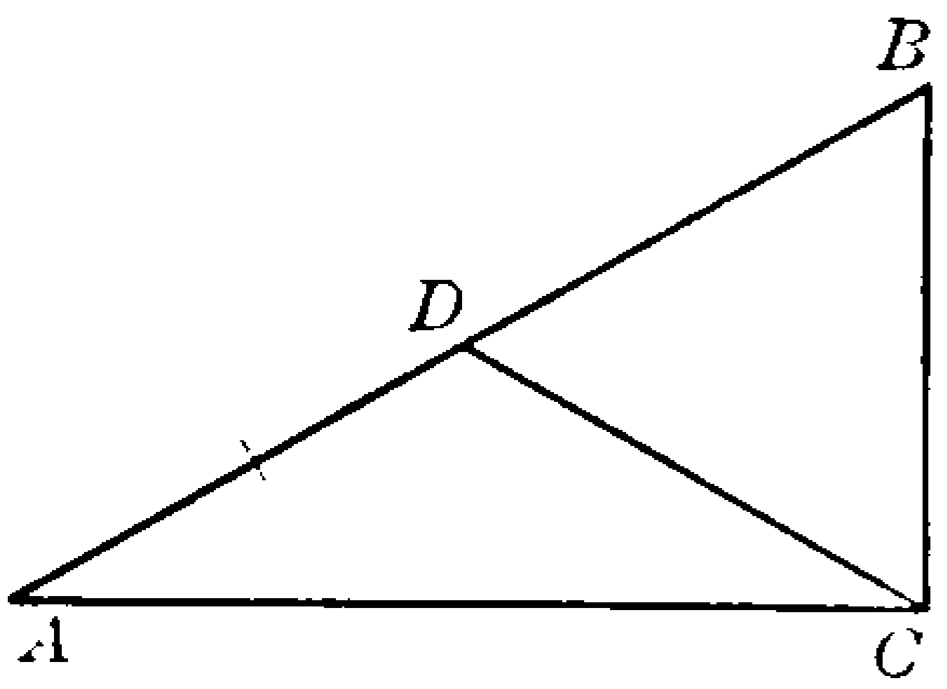
$AB = AC$
$CD \perp AB$
Describe CD.

3.



$AC = CB$
$AD = DB$
Describe CD.

4.

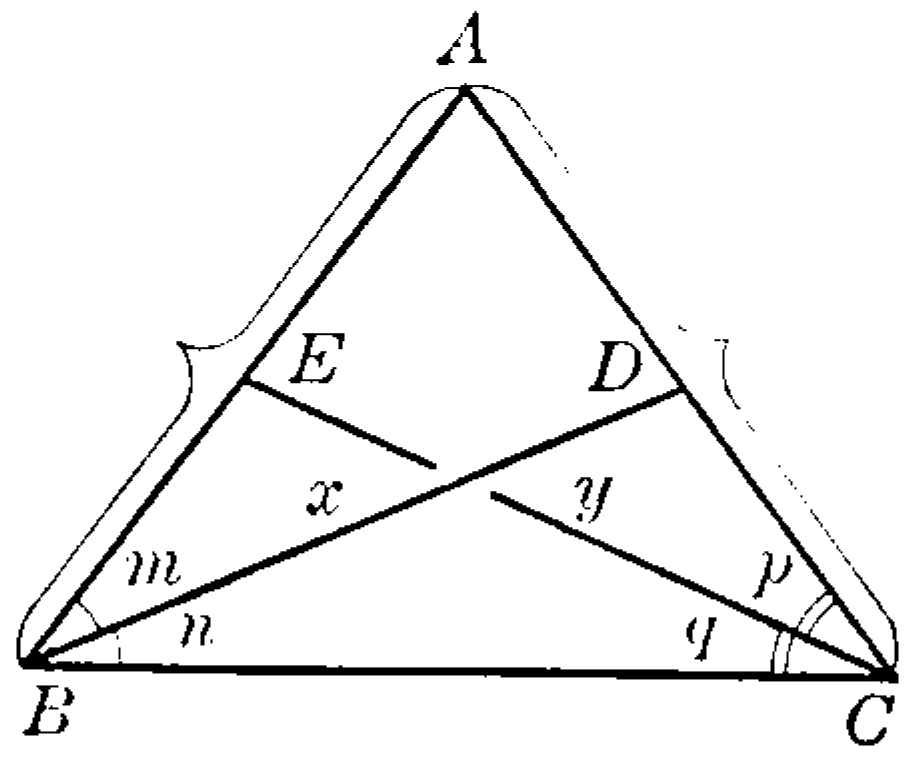


$\angle C = 90°$
$AD = DB$
Describe CD.

5. Draw a figure illustrating the bisector of a base angle of an isosceles triangle.

6. Draw a figure illustrating an altitude of a triangle which is also a median.

Illustrative Example. Make a generalization from the following set of conditions:

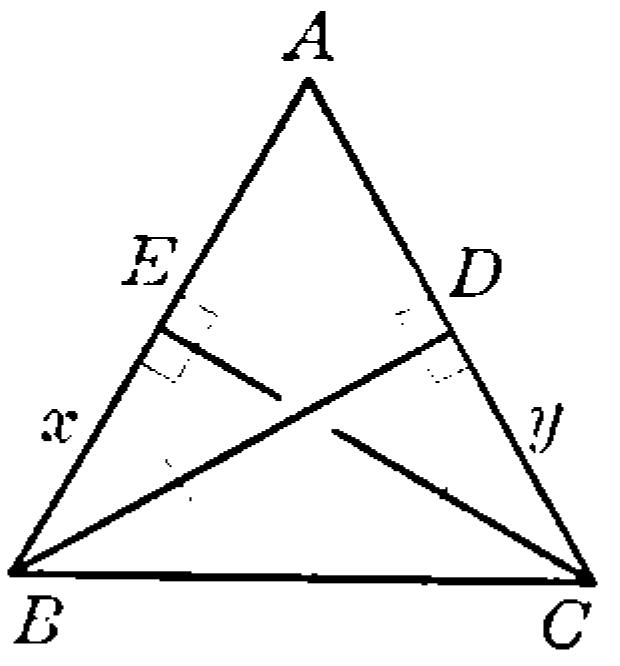


Given:
$\angle m = \angle n$
$\angle p = \angle q$
$AB = AC$
Prove:
$BD = EC$

Generalization. If a triangle is isosceles, then the bisectors of the base angles are equal.

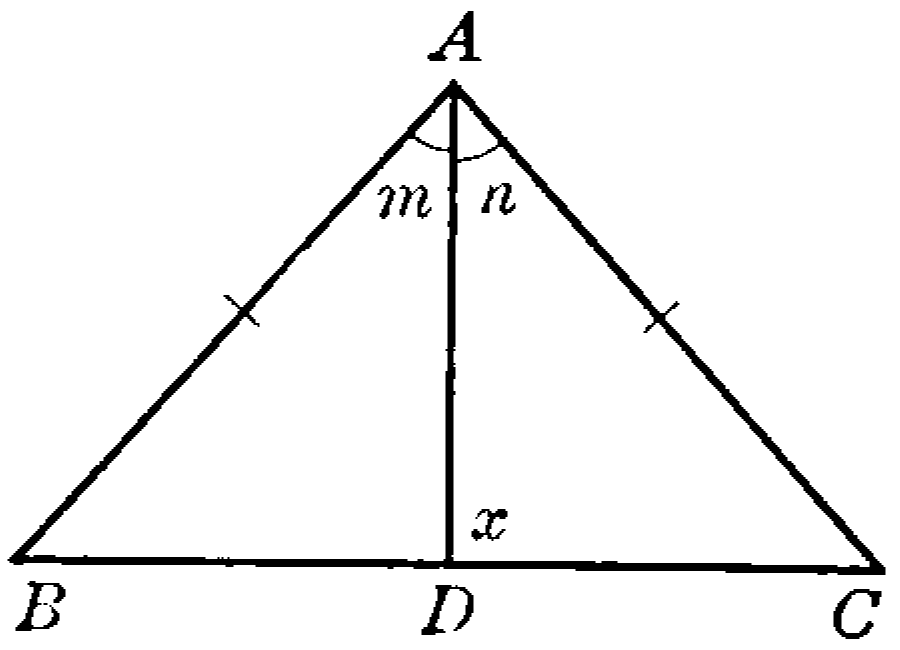
Make a generalization for each of the following sets of conditions:

7.



Given:
$BD = CE$
$BD \perp AC$
$CE \perp AB$
Prove:
$AB = AC$

8.



Given:
$AB = AC$
$\angle m = \angle n$
Prove:
$AD \perp BC$

Illustrate the following propositions:

9. If a triangle is equiangular, then it is equilateral.

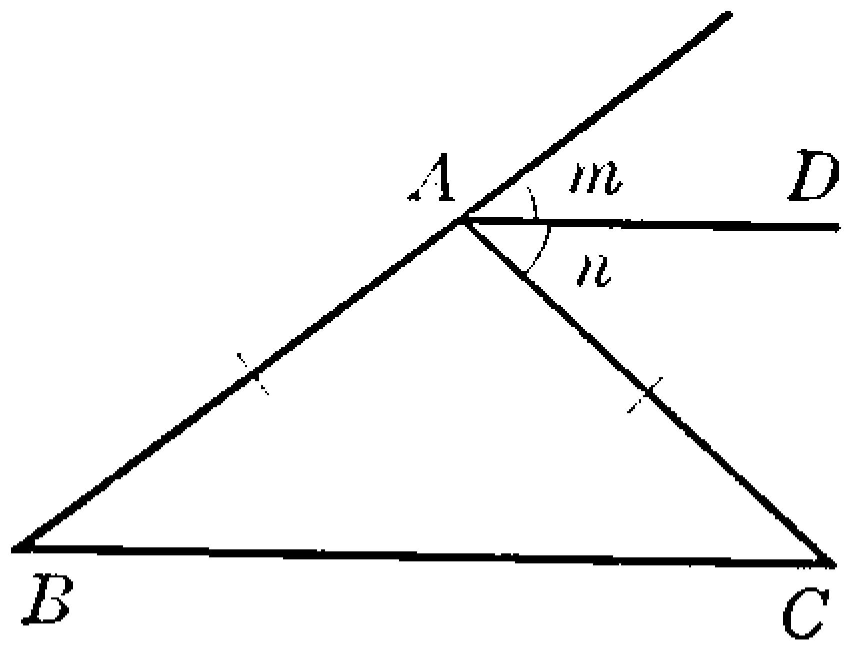
10. In an isosceles triangle, the medians to the legs are equal.

11. If an altitude of a triangle is also a median, the triangle is isosceles.

12. In an isosceles triangle, the altitude to the base bisects the base.

SUPERIOR WORK

S1.



Given:
$AB = AC$
$\angle m = \angle n$
Describe AD.

S2. Draw two kinds of figures illustrating two isosceles triangles having a common base.

S3. Draw a figure illustrating a median of a triangle, prolonged its own length through the midpoint of the side, and its new end joined with the other two vertices.

Illustrate the following propositions:

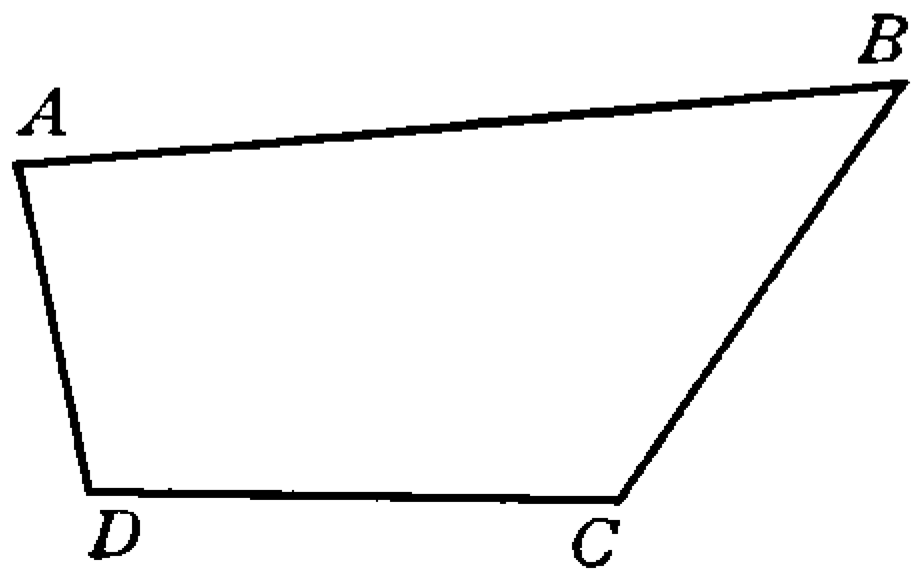
S4. If one leg of an isosceles triangle is prolonged its own length through the vertex, and its new end is joined to the other end of the base, then a right triangle is formed.

S5. If the median of a triangle equals one-half the side that it bisects, then the triangle is a right triangle.

Propositions about Quadrilaterals

To understand general propositions about quadrilaterals and to form such propositions in your own words, you need to know the definitions that follow.

88. A quadrilateral (quad.) is a polygon with four sides.



ABCD a quadrilateral

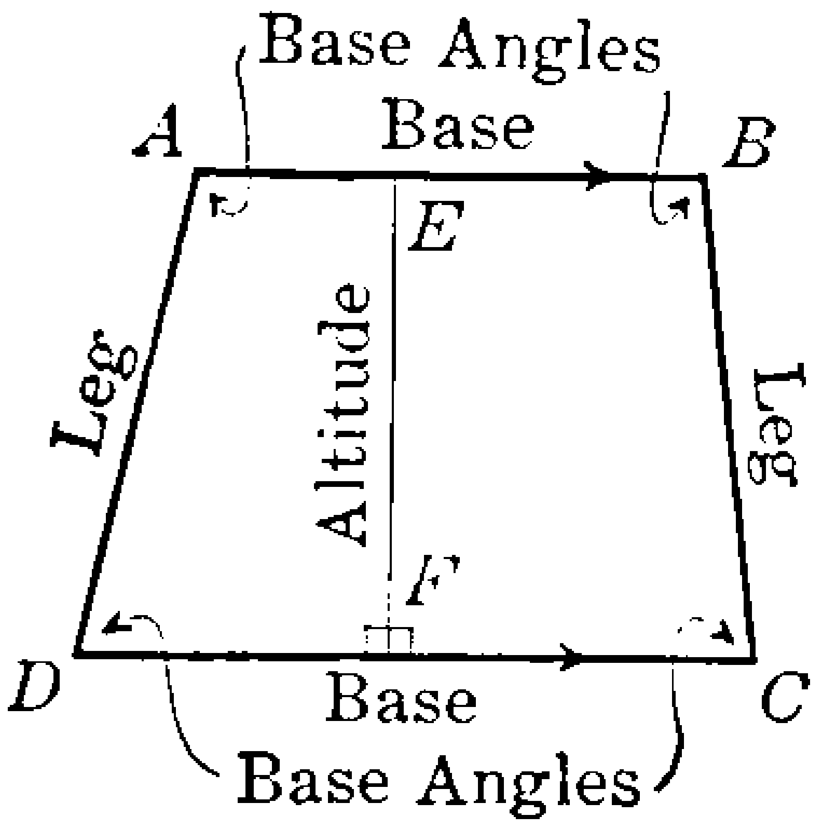
89. A trapezoid (trap.) is a quadrilateral that has two, and only two, sides parallel.

The bases of a trapezoid are the two parallel sides.

A pair of base angles of a trapezoid are the two angles adjoining one base.

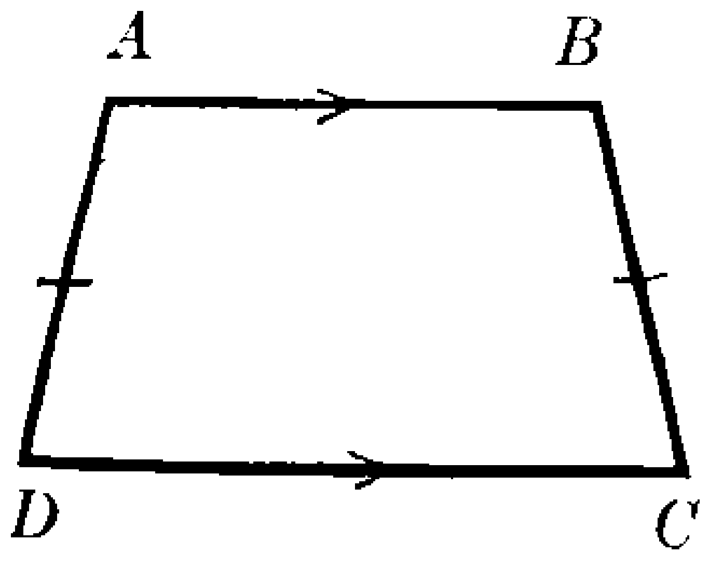
The legs of a trapezoid are the two nonparallel sides.

An altitude of a trapezoid is a line segment included between the bases of the trapezoid and perpendicular to one of them.



$AB \parallel DC$
$AD \nparallel BC$
$ABCD$ a trapezoid
AB and DC the bases
$\angle A$ and $\angle B$ a pair of base $\measuredangle$
$\angle D$ and $\angle C$ a pair of base $\measuredangle$
AD and BC the legs
$EF \perp DC$
EF an altitude

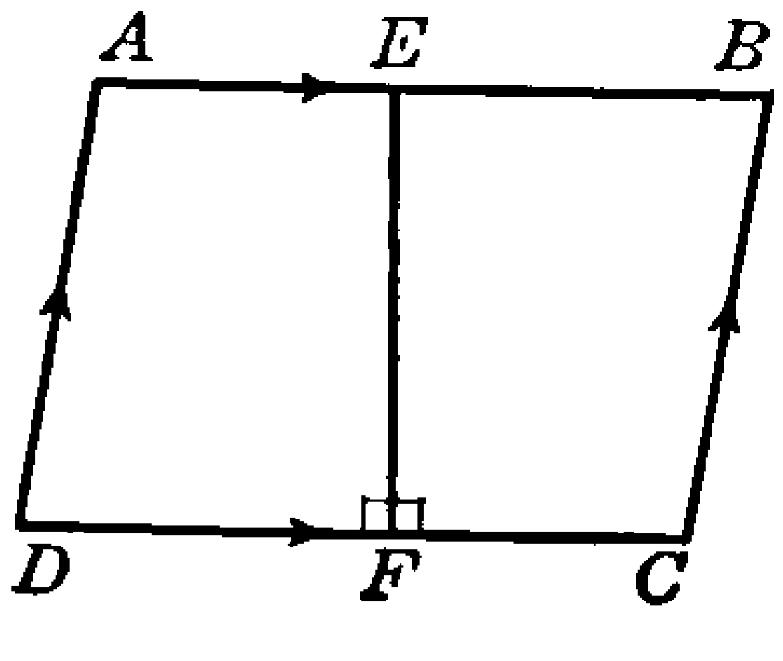
90. An isosceles trapezoid is a trapezoid with equal legs.



$AB \parallel DC$
$AD = BC$
$ABCD$ an isosceles trapezoid

91. A parallelogram (▱) is a quadrilateral whose opposite sides are parallel.

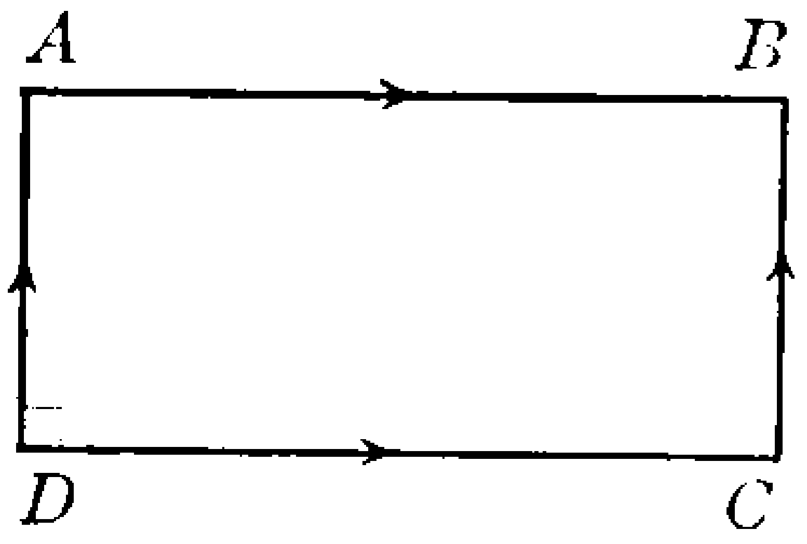
The side on which the parallelogram rests is considered the base.



$AB \parallel DC$
$AD \parallel BC$
$ABCD$ a ▱
$EF \perp DC$
EF an alt.

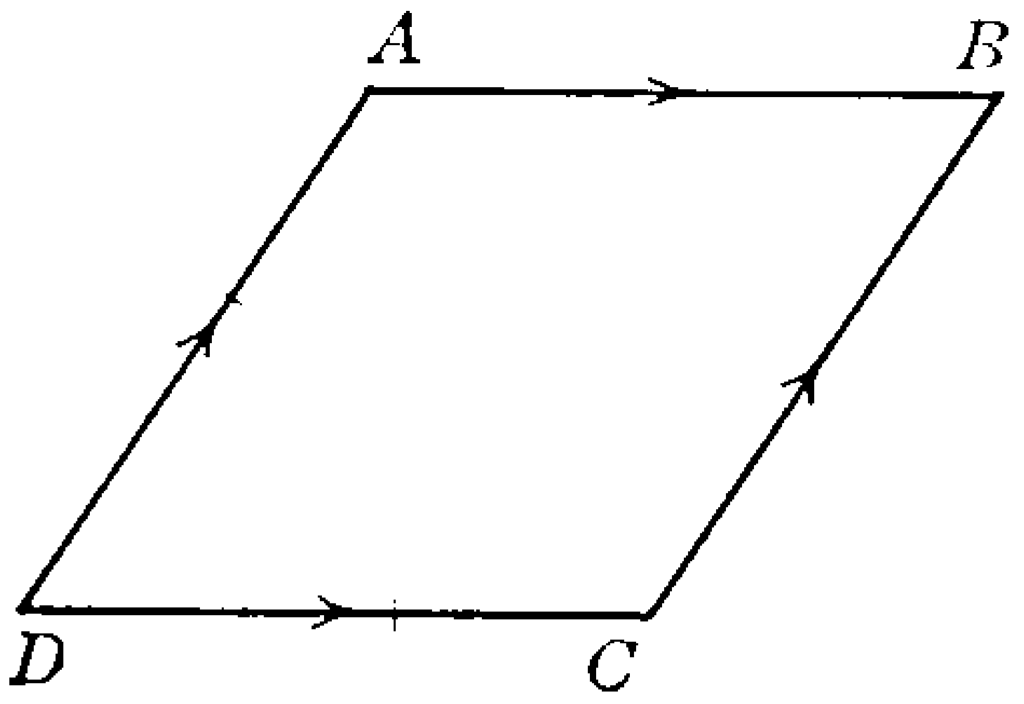
An altitude of a parallelogram is a line segment included between two parallel sides and perpendicular to one of them.

92. A rectangle (▭) is a parallelogram with a right angle.

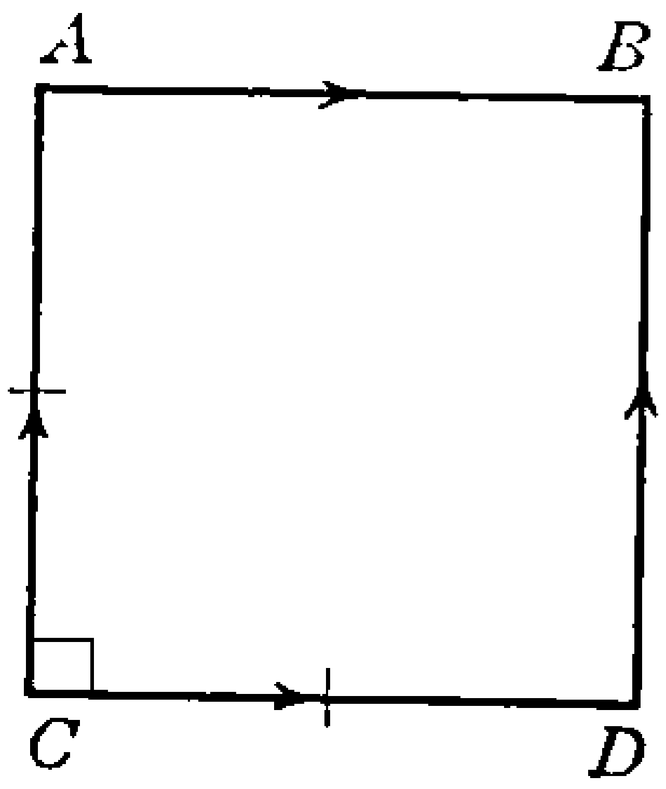


$ABCD$ a ▱
$\angle D = 90°$
$ABCD$ a rect.

93. A rhombus is a parallelogram with two adjacent sides equal.



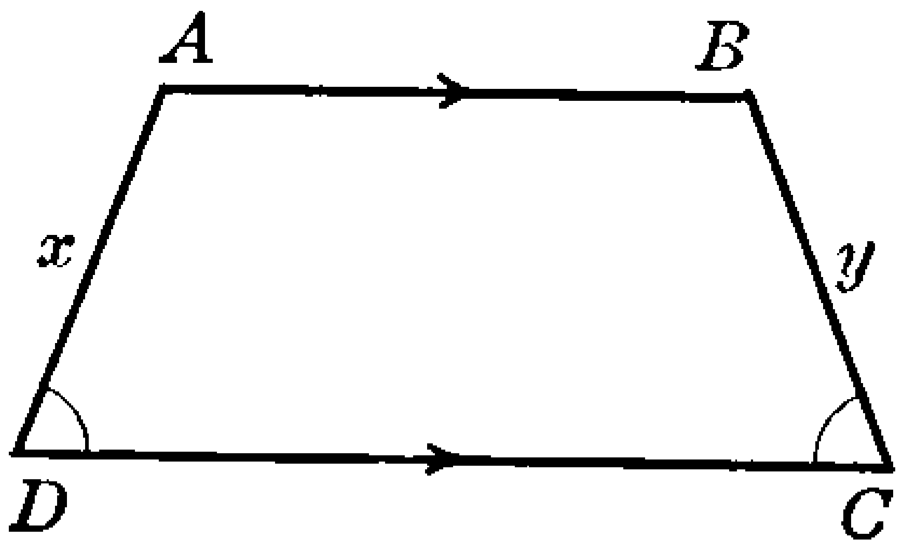
94. A square is a rectangle with two adjacent sides equal.



EXERCISES

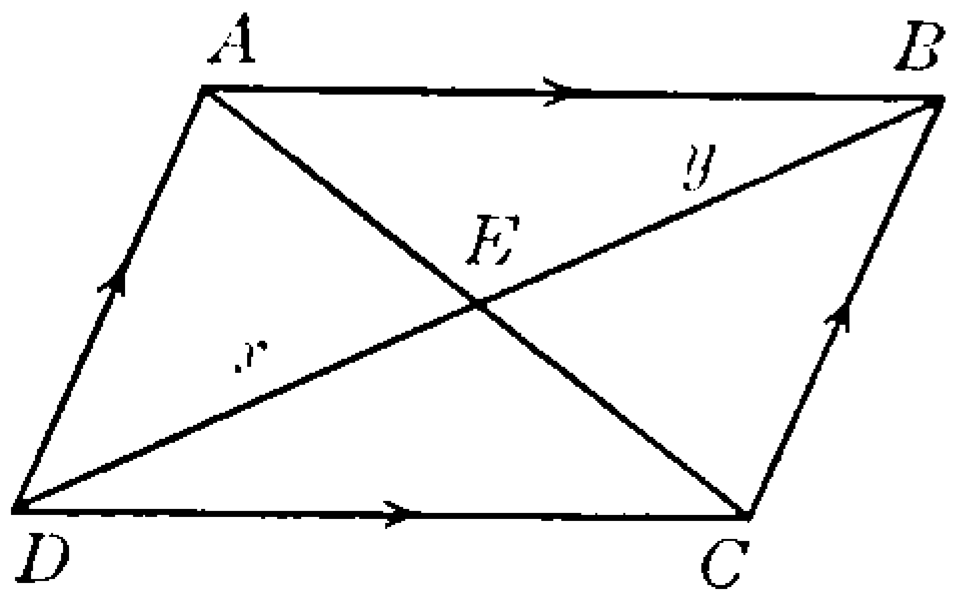
In each of the following exercises you are to draw a figure, to make a generalization, or to give a description.

Illustrative Example. Draw a figure to illustrate the following proposition: If a pair of base angles of a trapezoid are equal, the trapezoid is isosceles.



Given:
$AB \parallel DC$
$\angle D = \angle C$
Prove:
$x = y$

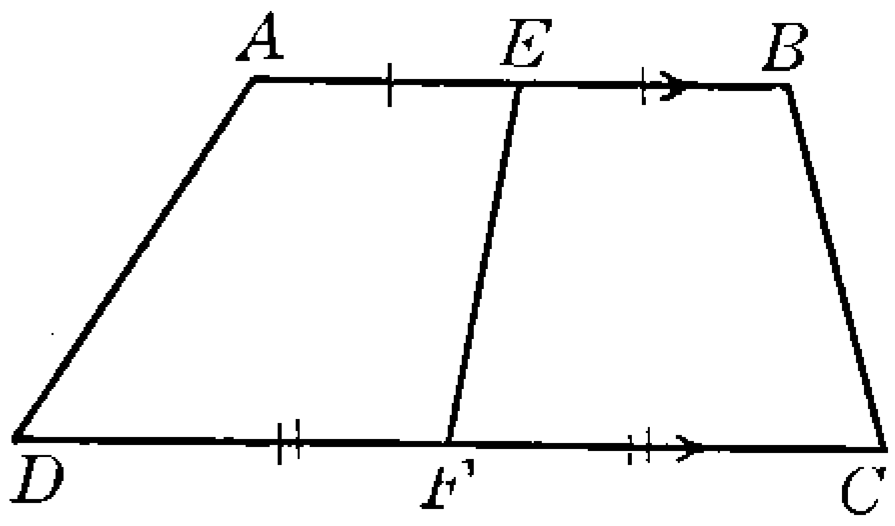
Illustrative Example. Make a general proposition from the following set of given conditions.



Given:
$AB \parallel DC$
$AD \parallel BC$
Prove:
$DE = EB$

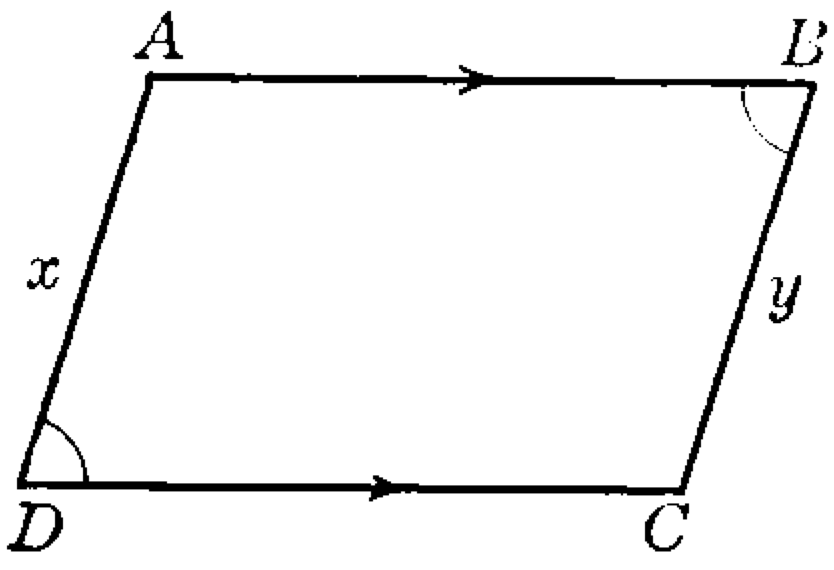
General Proposition. One diagonal of a parallelogram bisects the other diagonal.

1. Draw a figure illustrating the legs of a trapezoid prolonged to meet.
2. Draw a figure illustrating a trapezoid in which one leg is also an altitude of the trapezoid.
3.



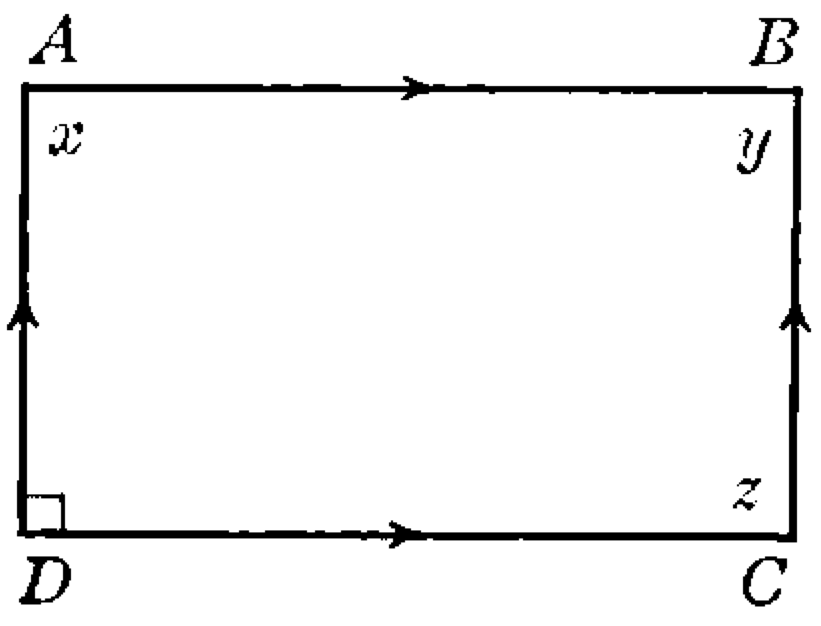
Given:
$AB \parallel DC$
$AE = EB$
$DF = FC$
Describe EF.

4. Draw a figure illustrating a quadrilateral with the diagonals bisecting each other.
5. Make a general proposition from this information:



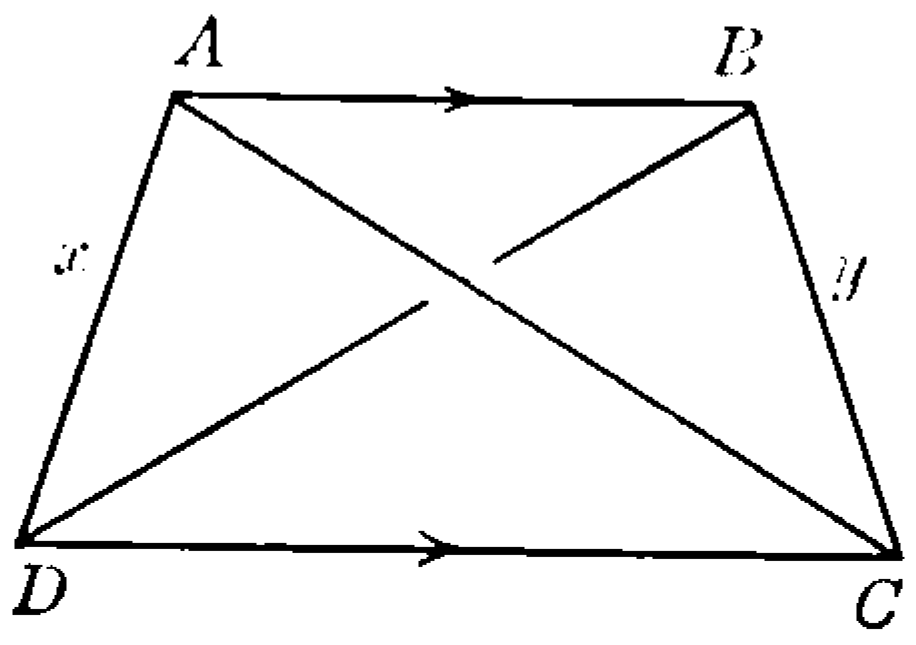
Given:
$AB \parallel DC$
$\angle B = \angle D$
Prove:
$x \parallel y$

6. Make a general proposition from this information:



Given:
$AB \parallel DC$
$AD \parallel BC$
$\angle D = 90°$
Prove:
$\angle x = \angle y =$
$\angle z = 90°$

7. Illustrate the following proposition: Any two altitudes of a trapezoid are equal. Beside the figure, write what is given and what you are asked to prove.
8. Make a general proposition from this information:



Given:
$AB \parallel DC$
$AC = BD$
Prove:
$x = y$

SUPERIOR WORK

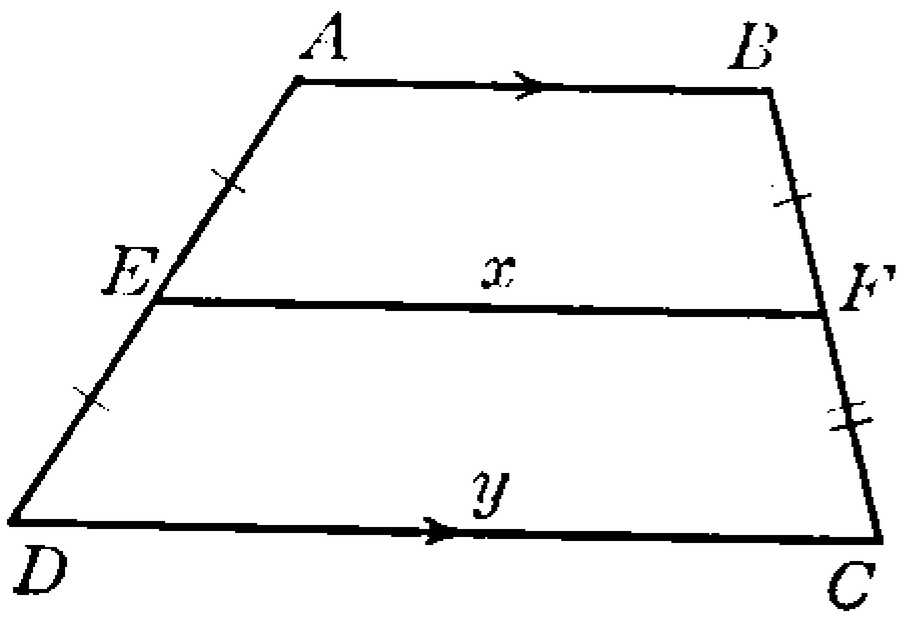
Illustrate each of the following propositions. Beside each figure, write what is given and what you are asked to prove.

S1. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram.

S2. If the diagonals of a rhombus are equal, then the rhombus is a square.

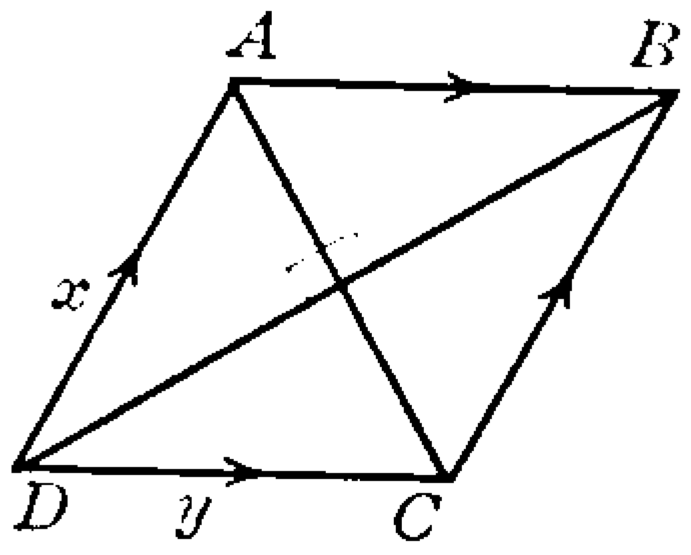
Make a general proposition from the information given in each of the following:

S3.



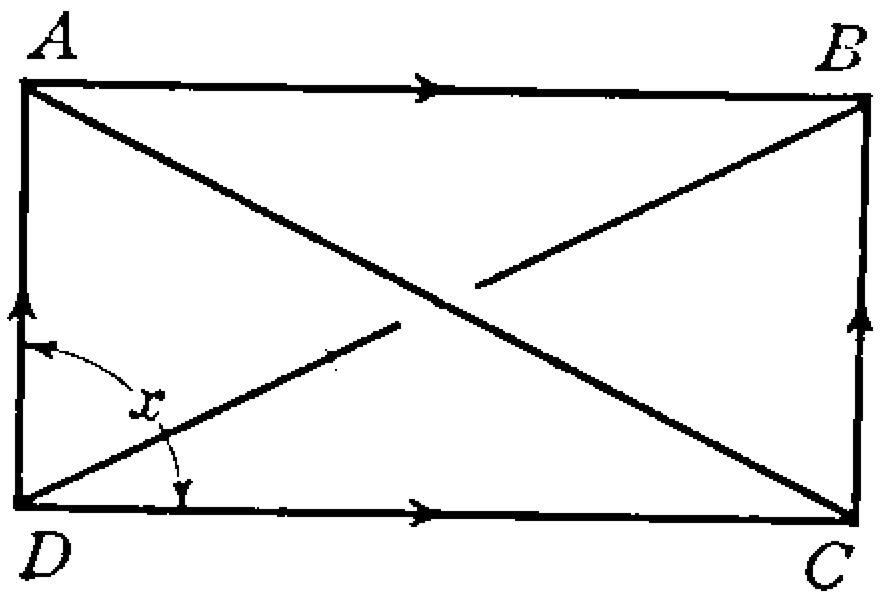
Given:
$AB \parallel DC$
$AE = ED$
$BF = FC$
Prove:
$x \parallel y$

S4.



Given:
$AB \parallel DC$
$AD \parallel BC$
$BD \perp AC$
Prove:
$x = y$

S5.



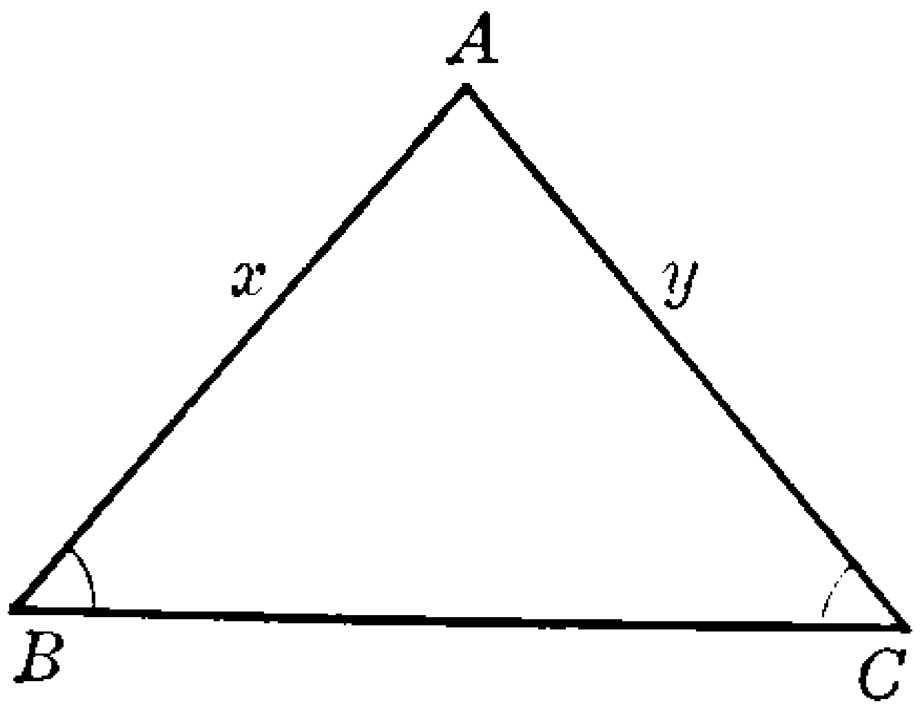
Given:
$AB \parallel DC$
$AD \parallel BC$
$AC = BD$
Prove:
$\angle D = 90°$

95. If a proposition has one condition in the hypothesis and one condition in the conclusion, the converse of the proposition is formed by interchanging the hypothesis and the conclusion. An example of such a converse is given below.

General Proposition. If two sides of a triangle are equal, the angles opposite are equal.

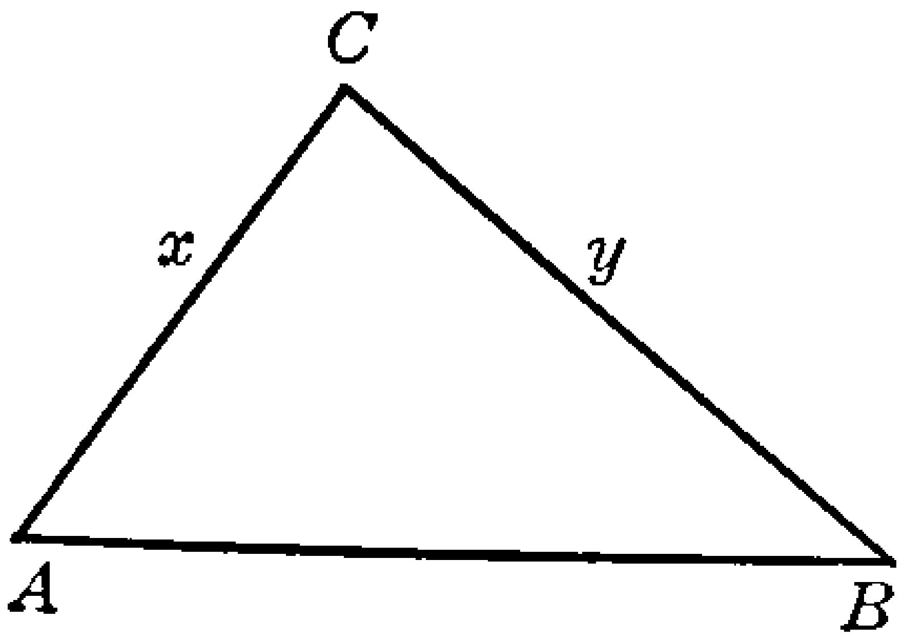
Converse. If two angles of a triangle are equal, the sides opposite are equal.

This converse may be illustrated by the accompanying figure.



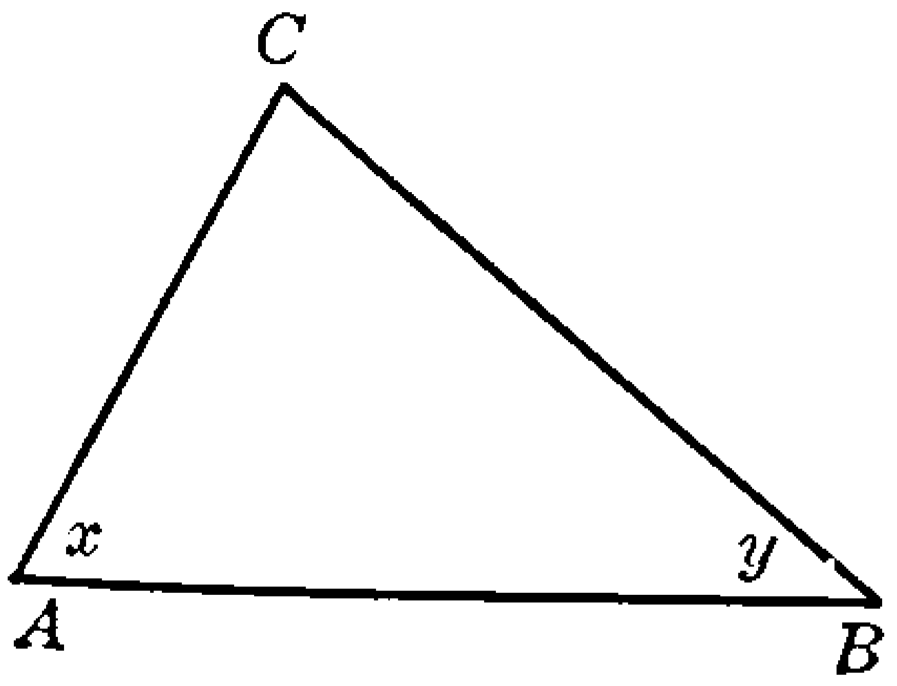
Given:
In $\triangle ABC$
$\angle B = \angle C$
Prove:
$AB = AC$

Sometimes you are given the illustration for a proposition, and you are asked to illustrate the converse of it and form the general proposition that you have illustrated.



Given:
In $\triangle ABC$
$\angle A > \angle B$
($\angle A$ greater than $\angle B$)
Prove:
$CB > CA$

Illustration of Converse.



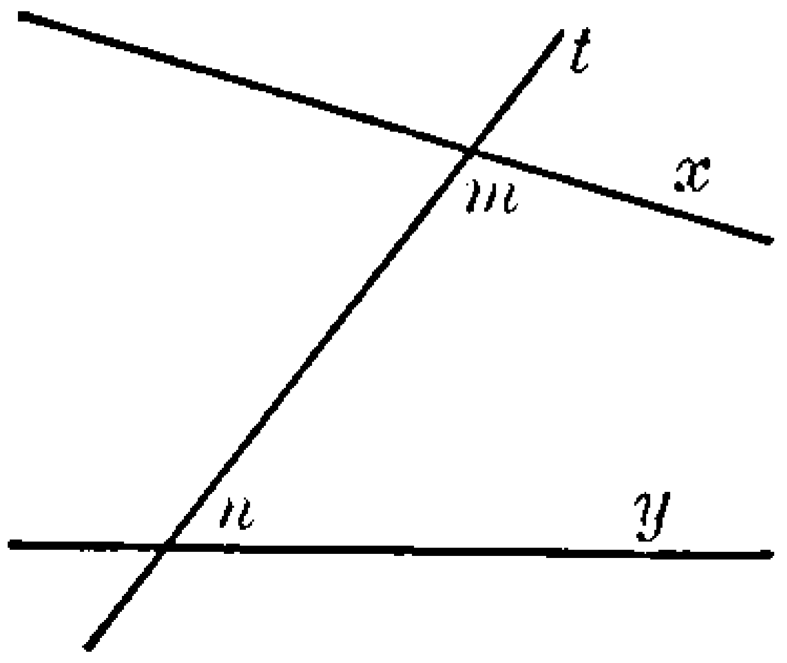
Given:
In $\triangle ABC$
$CB > CA$
Prove:
$\angle A > \angle B$

General Converse. If two sides of a triangle are unequal, then the angles opposite are unequal and the greater angle is opposite the greater side.

Sometimes you are given a proposition without illustration, and you are asked to form the converse of it and illustrate it. Try to do this with the following proposition.

Illustrative Example. If two nonparallel lines are cut by a transversal, the interior angles on the same side of the transversal are not supplementary.

General Converse. If two lines are cut by a transversal, and the interior angles on the same side of the transversal are not supplementary, then the lines are not parallel.



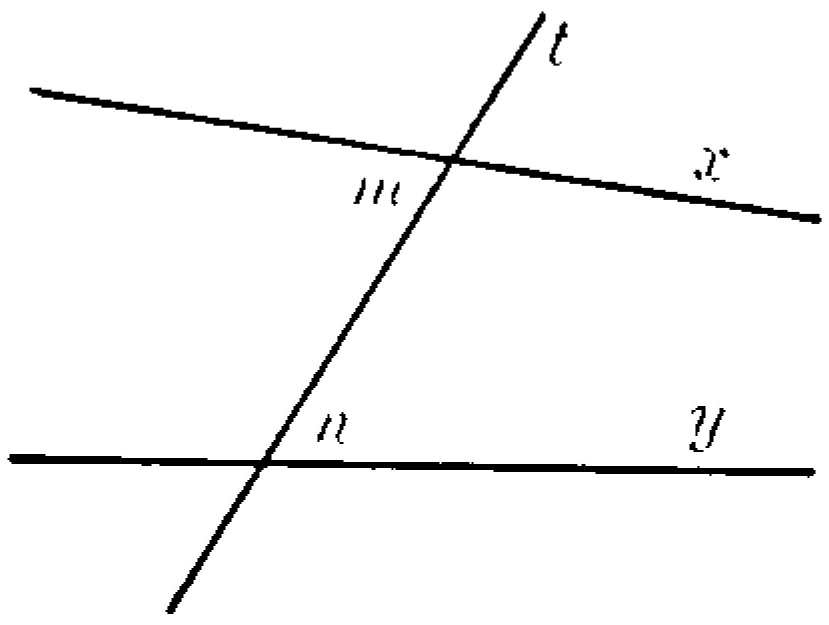
Given:
Lines x and y cut by the trans. t
$\angle m + \angle n \neq 180°$
Prove:
$x \nparallel y$
(x not parallel to y)

EXERCISES

1. In a right triangle, the acute angles are complementary. Form the general converse of this proposition and illustrate it.
2. If the noncommon sides of two adjacent angles form a straight line, then the angles are supplementary. Form the general converse of this and illustrate it.

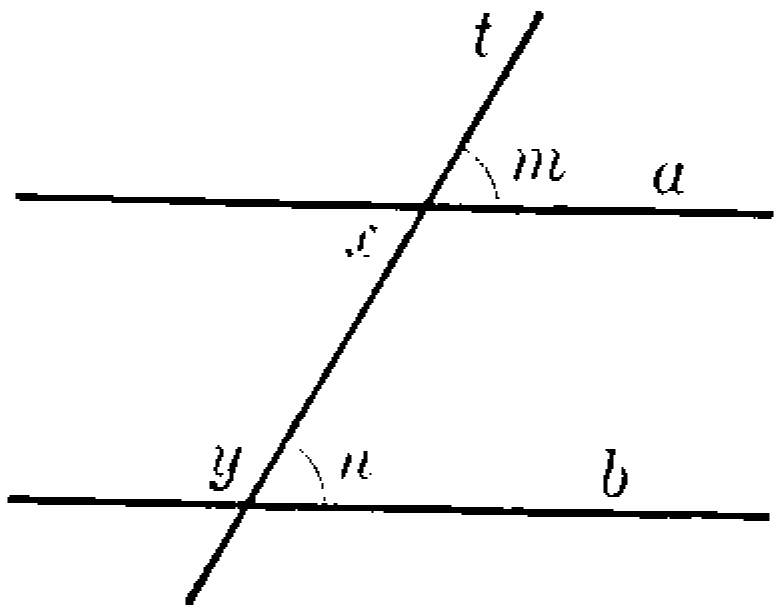
In each of the following illustrate the converse and form the general proposition that your figure illustrates.

3.



Given:
Lines x and y cut by the trans. t
$\angle m \neq \angle n$
Prove:
$x \nparallel y$
(x not parallel to y)

4.

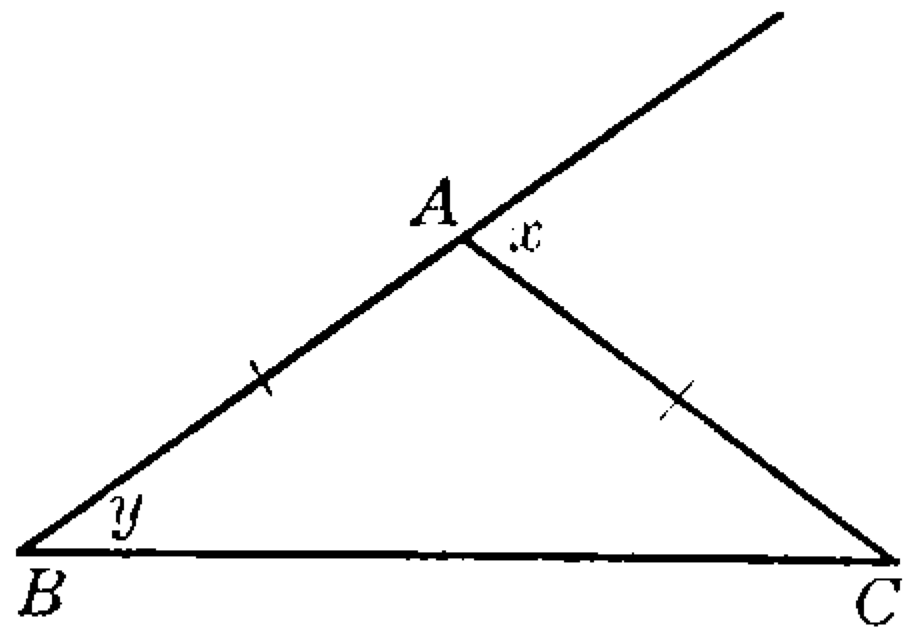


Given:
a and b cut by the trans. t
$\angle m = \angle n$
Prove:
$\angle x + \angle y = 180°$

5. State the general converse of the following proposition, and illustrate your statement: If two lines are cut by a transversal, and the interior angles on the same side of the transversal are not supplementary, then the alternate interior angles are not equal.

6. If two intersecting lines are cut by a transversal, the interior angles on the same side of the transversal are not supplementary. Form the general converse of this and illustrate it.

7. Illustrate the converse of the following and state the general proposition that *your* figure illustrates.



Given:
$\angle x$ an ext. $\angle$ of $\triangle ABC$
$AB = AC$
Prove:
$\angle x = 2 \cdot \angle y$

8. Write the converse of each of the following propositions:
   (*a*) If a polygon can have two diagonals, then it is a quadrilateral.
   (*b*) If the sum of the angles of a polygon is 180°, the figure is a triangle.
   (*c*) Vertical angles are equal.
   (*d*) If a figure is a trapezoid, then it is a quadrilateral.
   (*e*) If a triangle is equilateral, it is isosceles.
   (*f*) Parallel lines are in the same plane.

9. Write the converse of each of the following propositions:
   (*a*) An integer ending in 7 is not a perfect square.
   (*b*) If the sum of the digits of a number is divisible by 9, then the number itself is divisible by 9.
   (*c*) If $2x + 3 = 17$, then $x = 7$.
   (*d*) If $4x - 1 = 19$, then $x \neq 6$.
   (*e*) If $x = -5$, then $x^2 = 25$.
   (*f*) If $xy \neq 0$, then $x \neq 0$.

SUPERIOR WORK

Opposite Propositions

S96. If one of two statements must be true and the other false, then each statement is said to be the *opposite* of the other. For example, the statement that $x = 5$ is the opposite of the statement that $x \neq 5$. If two lines a and b are in the same plane, the statement that a and b are parallel is the opposite of the statement that a and b intersect. The statement that $x = 6$ and the statement that $x > 6$ are not opposites, because they may both be false.

If the hypothesis and the conclusion of one proposition are the opposites of the hypothesis and the conclusion of another proposition, then each proposition is said to be the opposite of the other proposition.

Example 1. If two sides of a triangle are equal, the angles opposite are equal.

The opposite of this proposition is: If two sides of a triangle are not equal, then the angles opposite are not equal.

Example 2. If two nonparallel lines are cut by a transversal, the interior angles on the same side of the transversal are not supplementary.

The opposite of this proposition is: If two parallel lines are cut by a transversal, the interior angles on the same side of the transversal are supplementary.

Example 3. If a man is in Detroit, then he is in Michigan. Form the opposite of the converse (the opposite-converse) of this proposition.

The converse is: If a man is in Michigan, then he is in Detroit.

The opposite of this converse is: If a man is not in Michigan, then he is not in Detroit.

Write the opposite of each proposition below:

S1. In a right triangle, the acute angles are complementary.

S2. If the noncommon sides of two adjacent angles form a straight line, then the angles are supplementary.

S3. If two lines are cut by a transversal and the corresponding angles are not equal, the lines are not parallel.

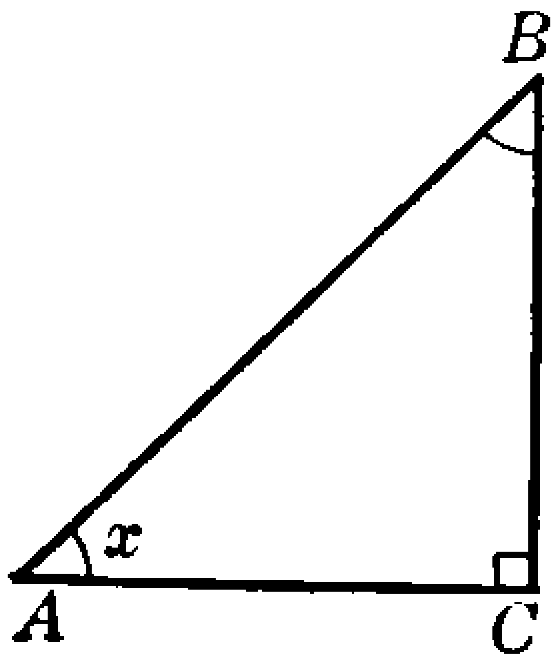
S4. Write the opposite of each proposition in Ex. 8 on page 64.

S5. Write the opposite of the converse of each proposition in Ex. 9 on page 64.

Multiple Converses

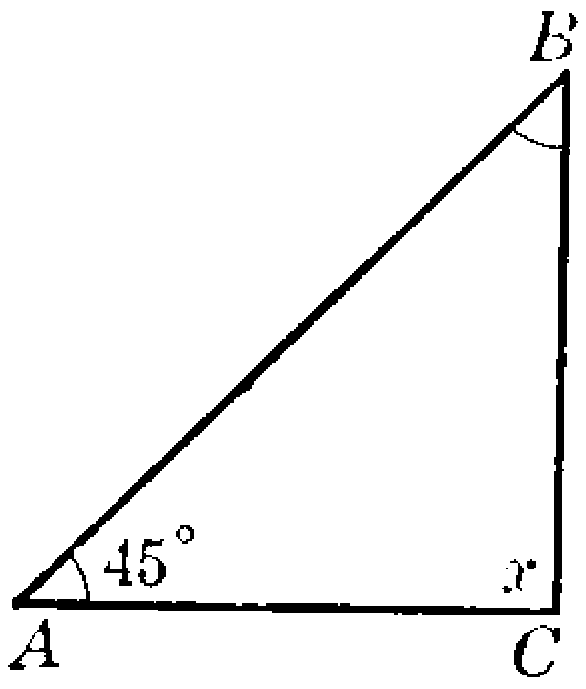
97. If a proposition has more than one condition in the hypothesis or in the conclusion, then we may have more than one converse. We may form a new converse by interchanging one condition in the hypothesis with one condition in the conclusion. Examples are given below.

Illustrative Example. Form two converses of this proposition:



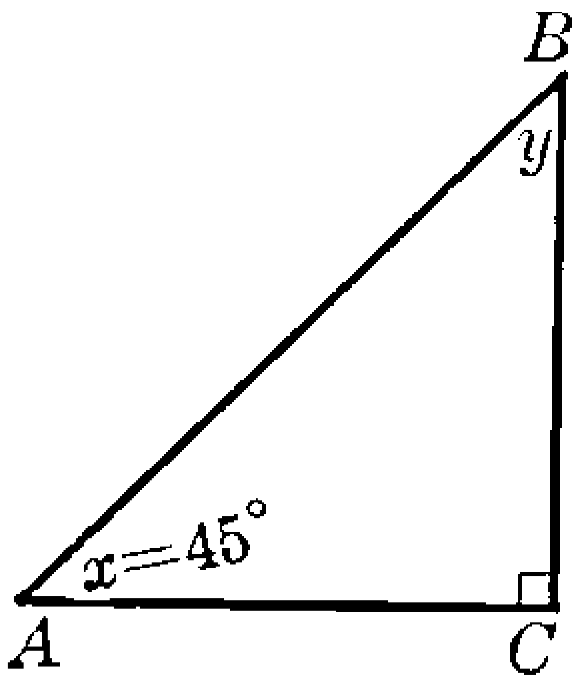
Given:
In $\triangle ABC$
$\angle C = 90^\circ$
$\angle A = \angle B$
Prove:
$\angle A = 45^\circ$

By interchanging the statement $\angle C = 90^\circ$ with $\angle A = 45^\circ$, we get the following converse:



Given:
In $\triangle ABC$
$\angle A = 45^\circ$
$\angle A = \angle B$
Prove:
$\angle C = 90^\circ$

By interchanging the statement $\angle A = \angle B$ with $\angle A = 45^\circ$ (in the original proposition), we get another converse:

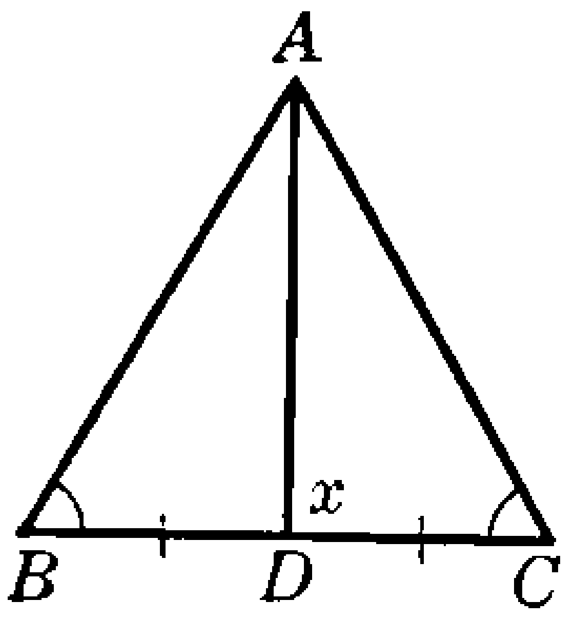


Given:
In $\triangle ABC$
$\angle C = 90^\circ$
$\angle A = 45^\circ$
Prove:
$\angle A = \angle B$

EXERCISES

1. Write the general proposition that is illustrated in each of the three figures in the Illustrative Example above.

2.

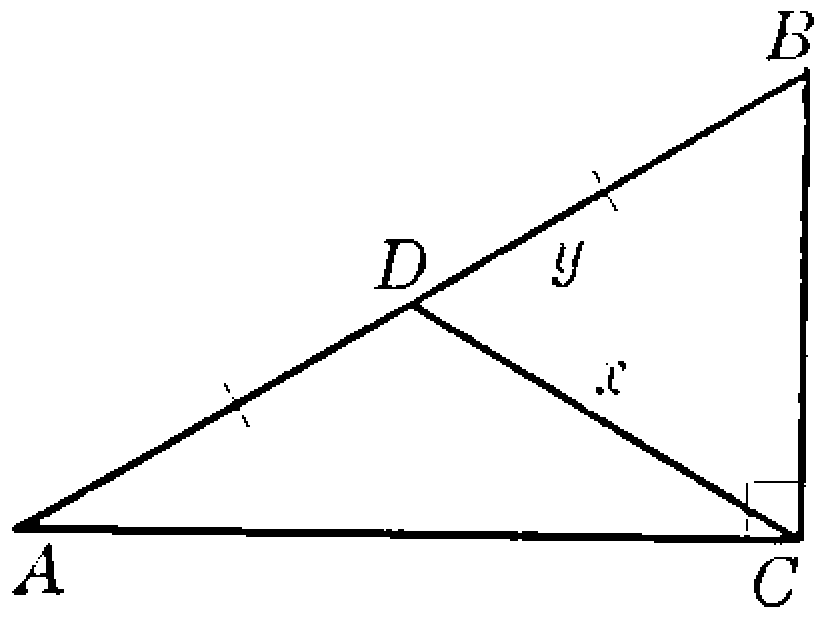


Given:
In $\triangle ABC$
AD a median
$\angle B = \angle C$
Prove:
$AD \perp BC$

(*a*) Form the converse of this illustration so that the conclusion is that AD is a median. State the converse as a general proposition in words.

(*b*) In the same way, form another converse having for its conclusion that $\angle B = \angle C$

3. Start with this general proposition: The median to the hypotenuse of a right triangle equals half the hypotenuse.



Given:
In $\triangle ABC$
$\angle C = 90°$
$AD = DB$
Prove:
$CD = \frac{1}{2}AB$

(*a*) Form the converse of this proposition, drawing the conclusion that $\angle C = 90°$. State the general proposition and illustrate it.
(*b*) Form another converse.

4. Start with this general proposition: In an isosceles triangle, the bisector of the vertex angle is perpendicular to the base. State two converses in words and illustrate them.

5. Start with this proposition: If the diagonals of a trapezoid are equal, the trapezoid is isosceles. Form two general converses of this and illustrate them.

6. Start with this proposition: If two lines are perpendicular to the same line, they are parallel. Write the converse of this proposition. Is more than one converse possible?

7. Write two converses of each of the following propositions:
(*a*) If $x = 2$ and $y = 3$, then $xy = 6$.
(*b*) If $x = 2$ and $y = 0$, then $xy = 0$.

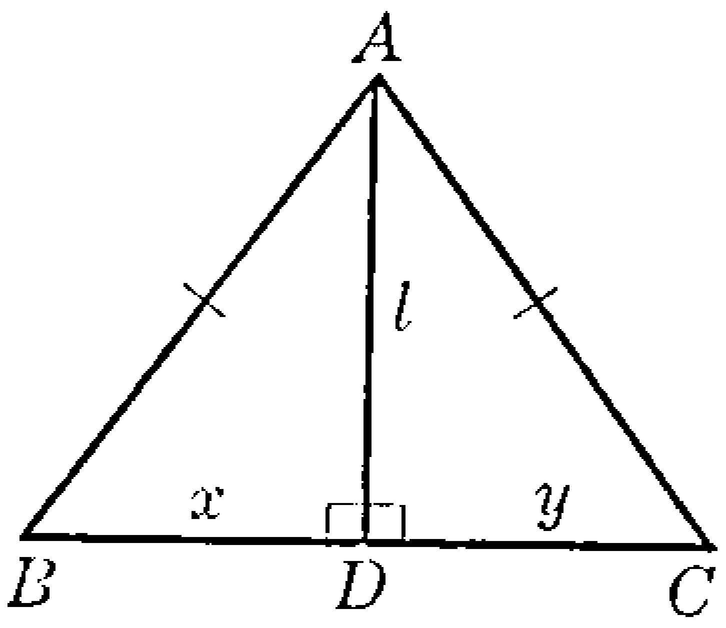
SUPERIOR WORK

S1. Not many propositions are so simple that they have only one condition in the hypothesis. Often a statement that seems to have only one condition really has another hidden condition. For example, the statement that a triangle is isosceles has two conditions: namely, (1) a triangle and (2) two equal sides.

Similarly, the statement that a line segment is an altitude of a triangle really contains two conditions: (1) a line perpendicular to one side of the triangle and (2) a hidden condition that this line passes through the opposite vertex.

An illustration of a proposition that contains hidden conditions is given below:

Proposition. The altitude to the base of an isosceles triangle bisects the base.



Given:
In $\triangle ABC$
$AB = AC$
$l \perp BC$
l passes through A
(hidden condition)
Prove:
$BD = DC$

(*a*) State the converse of this proposition, placing the hidden condition in the conclusion. Write the general proposition.

(*b*) Form the opposite of the original proposition by "opposing" the hidden condition and the conclusion.

S2. *Proposition.* Any two right angles are equal.

At first, it would seem that the converse of this is the proposition that any two equal angles are right angles. By illustrating the original proposition, you will discover that, actually, this is not true.

(*a*) Write the converse of the proposition by interchanging *one* condition in the hypothesis with the condition in the conclusion.

(*b*) Write the opposite of the original proposition.

S3. Whenever it is possible to form more than one converse of a proposition it is also possible to form more than one opposite of the proposition (S96). Each time we replace a condition in the hypothesis and a condition in the conclusion by their opposites, we get a new opposite of the original proposition.

Form two opposite propositions of the proposition in the Illustrative Example on page 65. State them as general propositions and illustrate them.

S4. In like manner, form two opposite propositions of the proposition in Ex. 2 on page 65.

S5. Form two opposite propositions of the proposition in Ex. 5 above.

Everyday Reasoning

98. In everyday life indirect reasoning is often used to reach a conclusion.

For example, if Joan and Bruce are the only persons who could have done a certain thing, and it is proved that Joan did not do it, then we reason indirectly that Bruce must have done it.

As a second example, consider the case of the lawyer who tells the jury that his client cannot be guilty of the crime of which she is accused because at the time the crime was committed the client was in another city attending a banquet in her honor. He tells the jury that its only possible verdict is not guilty. Analysis of the lawyer's reasoning may be stated briefly as follows:

1. Either the woman is guilty or not guilty.
2. If she is guilty, she was at the scene of the crime when it was committed.
3. But this is impossible because she was in another city at that time.
4. Therefore she is not guilty.

The method used in indirect reasoning is this:

1. Make a list of all possible conclusions, one of which must be correct.
2. Show that all conclusions except one are impossible.
3. To prove a conclusion false, assume for the sake of argument that it is true; then see whether it contradicts any known fact.

Let us listen to the following conversation. Examine it and see if you can tell how often indirect reasoning is used and whether or not any of the conclusions are false.

George: Well, Ann, I see that John and Walter are running for president of the Student Council.

Ann: I'm going to vote for Walter.

George: I believe in the secret ballot and so I didn't ask how you intend to vote. But, since you told me anyway, would you mind telling me also why you are against my friend John?

Ann: No, I don't mind telling you that I think your friend John is selfish.

George: What makes you think so?

Ann: I saw him buy a double dessert for lunch yesterday.

George: Well?

Ann: Do you think that a generous person would treat himself to extra dessert after hearing such a stirring appeal in the assembly that very morning? Our principal begged us to deny ourselves extra luxuries so that we could contribute more to the Community Chest. Why, I almost felt ashamed to buy my lunch, after hearing that so many needy and unfortunate people depend on us to share a few of the good things with them.

George: Ann, do you think that a selfish person would take the blame for his friends and even pay for their damages?

Ann: Did John do this?

George: Yes. Last term there was a little accident in the laboratory. When Mr. Williams accused some boys of carelessness, John took all the blame and paid for the damages.

Ann: He was probably just showing off.

George: Absurd. If that were so, John would have taken the blame right there in the "lab" in front of the others. But he didn't do that. He straightened everything out with Mr. Williams after school and told nobody.

Ann: Now, George, that's ridiculous because you're contradicting yourself. If what you tell me is true, how come you know about it?

George: Pure accident. At the end of the term I was helping Mr. Williams make up the list of fines when I came across the record of John's payment. Then I fished out the whole truth.

Ann: John's father must give him a big allowance.

George: Wrong again. John earns his spending money.

Ann: From what I know about him, John is either rich or selfish—

George: —or he likes his dessert.

George and Ann are using indirect reasoning in the following ways:

1. Ann has decided to vote for Walter, not because she thinks that Walter will make a good president of the Student Council but because she thinks John is selfish.
2. Ann again reasons indirectly that John is selfish because if he were generous he would not spend extra money that could help the needy.
3. Now George reasons indirectly that if John were selfish he would not pay for his friends' damages.
4. Again George tries to prove to Ann that his friend was not showing off because he did his good deed secretly.
5. Then Ann thinks that George is not telling the truth. If this were a secret, she reasons, then George would not know it.
6. Finally, Ann is ready to clinch her argument. She would like to say that John is either rich or selfish. Then, since George himself told how John earns his spending money, he is not rich and must therefore be selfish. But right here George catches Ann making a common mistake in indirect reasoning. She has assumed that only two things are possible: either John is rich or he is selfish. George therefore points out that there is a third possibility. In life situations there may be many possibilities, all of which must be considered before a conclusion can be reached.

Indirect reasoning is also useful in geometry. Often the simplest way to prove that a thing is true is to show that it cannot be false. In the indirect proof of the following theorem, we reason that if the conclusion were false, then we should have to contradict a proposition that we have already accepted.

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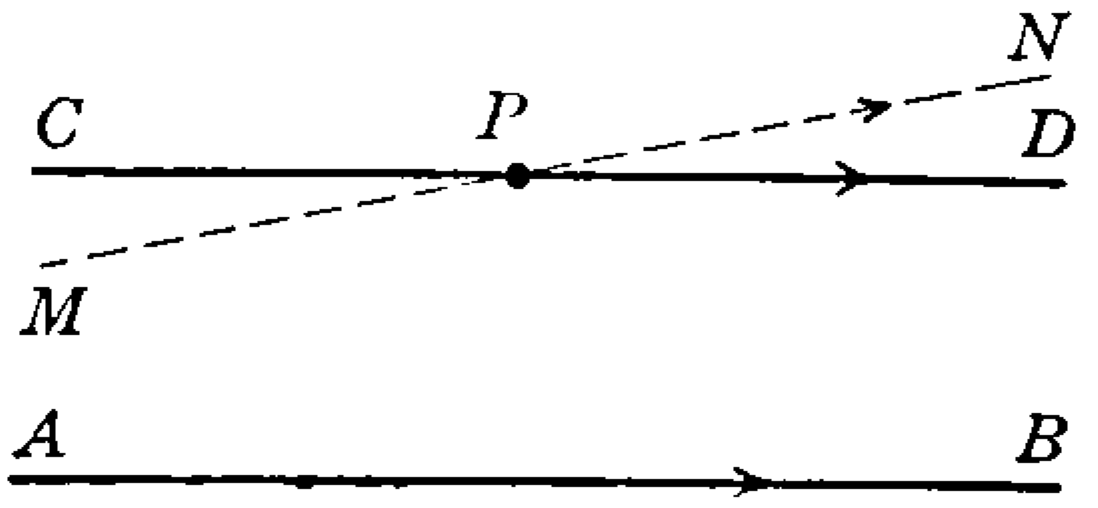
99. THEOREM. Through a given point not on a given line, only one parallel can be drawn to the given line.



Given: CD passes through P and $CD \parallel AB$

Prove: CD the only line through P parallel to AB

1. Either CD is the only line through P parallel to AB, or there is another line through P parallel to AB
2. *Assume* that MN is another line through P parallel to AB

3. Since CD is also parallel to AB,	3. Given.
4. Then MN CD	4. If 2 lines are to the same line, they are to each other.
5. But this is false	5. $\|$s are lines in the same plane that do *not* meet.
6. Hence, CD is the only line through P AB	6. This is the only remaining possibility.

In this kind of proof, all the possible conclusions are stated in the first step. (In this proof there are only two.) Then all the possibilities are eliminated except one—and that one must be true. The false conclusions are eliminated by showing that they contradict some accepted proposition. In this proof the false conclusion contradicted the definition of parallel lines. This kind of indirect proof is often called proof by *elimination* or *exclusion*. Another name for it is *reductio ad absurdum* from Latin words meaning "reduction to absurdity."

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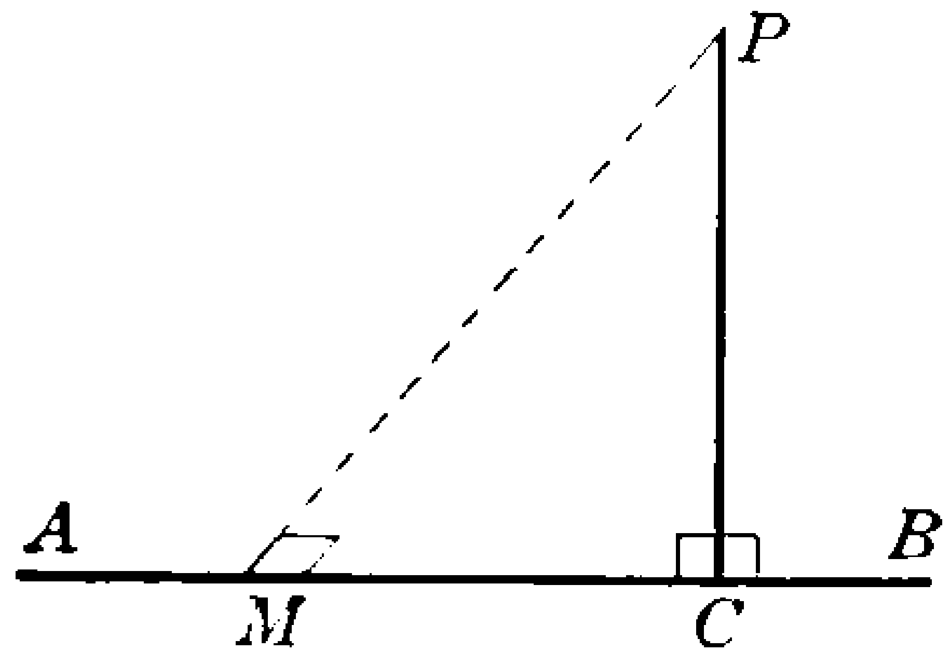
100. THEOREM. Through a given point, only one perpendicular can be drawn to a given line.



Given: PC passes through P
$PC \perp AB$
Prove: PC the only line through P perpendicular to AB

INDIRECT PROOF

1. Either PC is the only line through $P \perp AB$, or there is another line through $P \perp AB$
2. *Assume* that PM is another line through $P \perp AB$

3. Since PC is also $\perp AB$,	3. Given.
4. Then PM PC	4. Why?

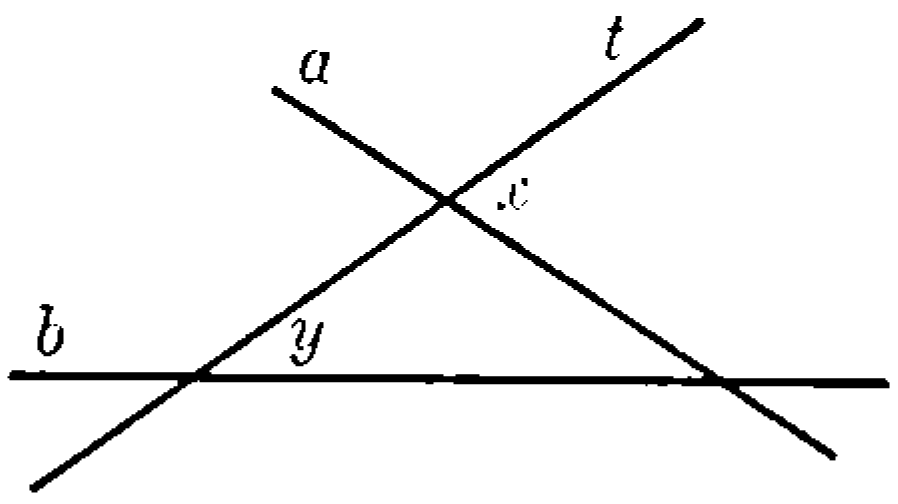
Complete the proof.

In an indirect proof the figure does not usually give a true picture of what we are assuming. The reason for this is, of course, that the assumption that we try to picture turns out to be false. We must therefore try to imagine how the figure would be if the assumption were true.

EXERCISES

Use indirect reasoning in proving the following propositions:

1. If two intersecting lines are cut by a transversal, then the corresponding angles are not equal.



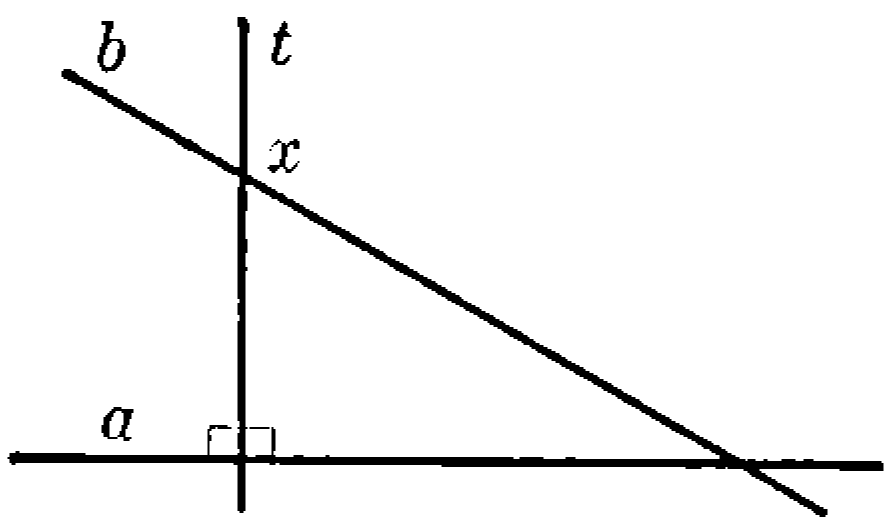
Given:
Intersecting lines a and b cut by the trans. t
Prove:
$\angle x \neq \angle y$

INDIRECT PROOF

1. Either $\angle x \neq \angle y$ or $\angle x = \angle y$
2. *Assume* $\angle x = \angle y$

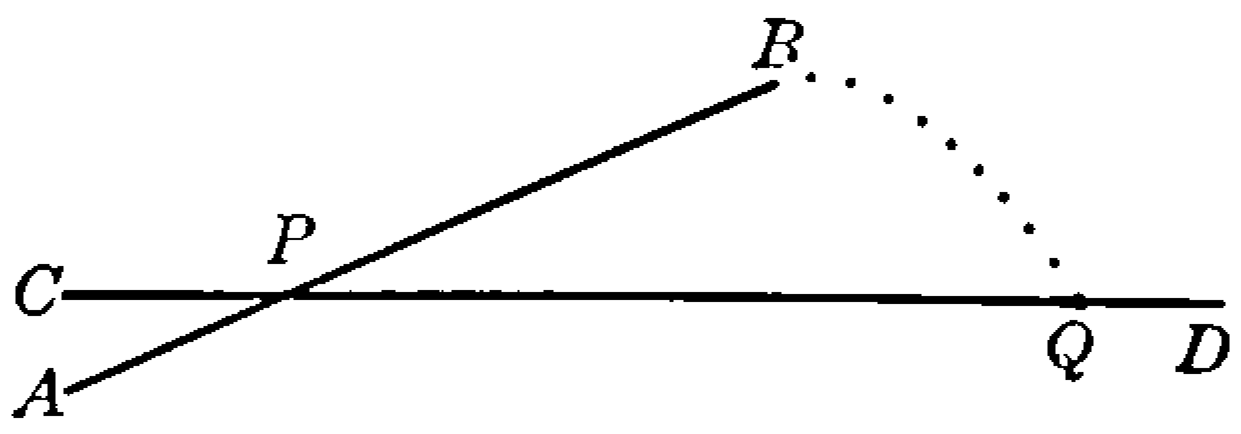
Complete the proof.

2. If a line is perpendicular to one of two intersecting lines, then it is not perpendicular to the other.



Given:
Intersecting lines a and b
$t \perp a$
Prove:
t not $\perp b$

3. Two lines can intersect in only one point.
Given: AB and CD intersect in P
Prove: AB and CD cannot intersect in any other point

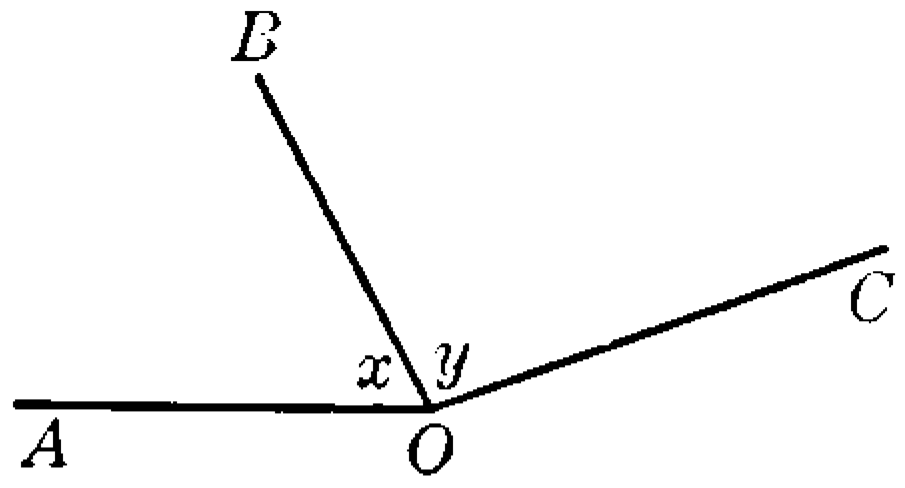


INDIRECT PROOF

1. Either AB and CD can intersect in some other point or not
2. *Assume* that AB and CD can intersect in some other point Q

Complete the proof.

4. If a figure is not a rectangle, then it is not a square.
5. If the sum of three angles is not 180°, they cannot be the angles of a triangle.
6. If the noncommon sides of two adjacent angles are not in a straight line, the angles are not supplementary.



Given:
$\angle x$ and $\angle y$ adj. $\measuredangle$ AOC not a st. line
Prove:
$\angle x + \angle y \neq 180°$

7. If two angles of a triangle are not equal, the bisector of the third angle is not perpendicular to the opposite side.
8. If $xy = -6$ and x is negative, then y is positive.
9. If $x^2 - 12 = x$, then $x \neq 5$.
10. If a man does not live in California, then he does not live in San Francisco.
11. You cannot start the motor of a car, but the lights burn normally. Prove that the battery is not at fault.

SUPERIOR WORK

S101. We have seen that a mathematical proposition is a conditional statement. It states that if a certain condition is true, then a certain conclusion must also be true. Another kind of statement is an *alternative* proposition which states that one of several things is true. For example, "you must pay or I will sue" is an alternative proposition. We have here a choice between two alternatives, so that if we reject one, then we must accept the other. More definitely, it means:

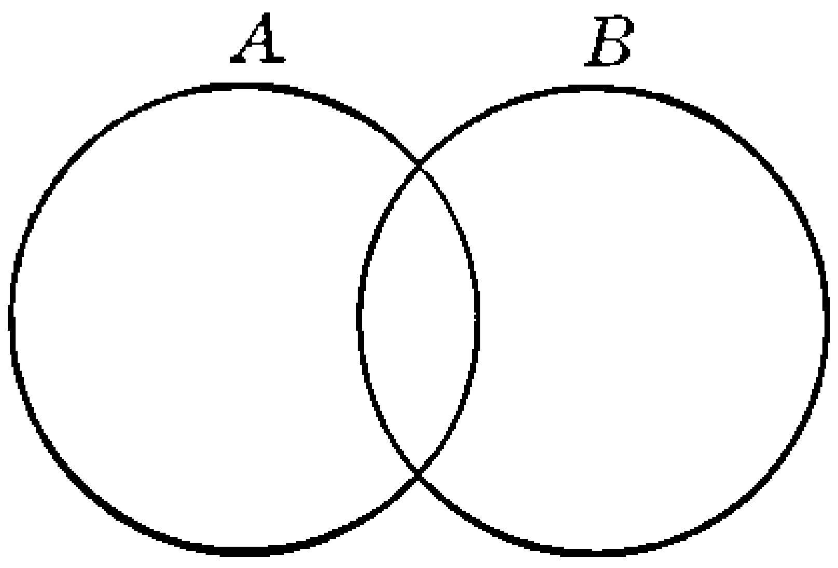
(1) If you do not pay, then I will sue, and

(2) if I am not to sue, then you must pay.

We see that each of these propositions is the opposite-converse of the other. Since they both arise from the same alternative proposition, we assume that they are equally valid.

S102. Axiom. A proposition and its opposite-converse are equally valid.

An alternative proposition, "either A or B," may be pictured by two intersecting circles. The proposition states that a point is either in circle A or in circle B (or in both). The two conditional propositions that arise are:



(1) If a point is not in circle A, then it is in circle B.

(2) If a point is not in circle B, then it is in circle A.

Here again, we see that proposition (2) is the opposite-converse of proposition (1).

In indirect proof, we state the possible conclusions as an alternative proposition. We then disprove the false alternatives.

S1. In each case, state the two conditional propositions that arise from the given alternative proposition:

(*a*) Two lines in the same plane either meet or are parallel.

(*b*) Two angles that have their sides respectively parallel are either equal or supplementary.

(*c*) In the equation $xy = 0$, either $x = 0$ or $y = 0$.

(*d*) On a school day, John is either ill or in school.

(*e*) Two angles are either acute or not complementary.

S2. In each case, one of the two conditional propositions that come from an alternative proposition is given. State the alternative proposition and the other conditional proposition.

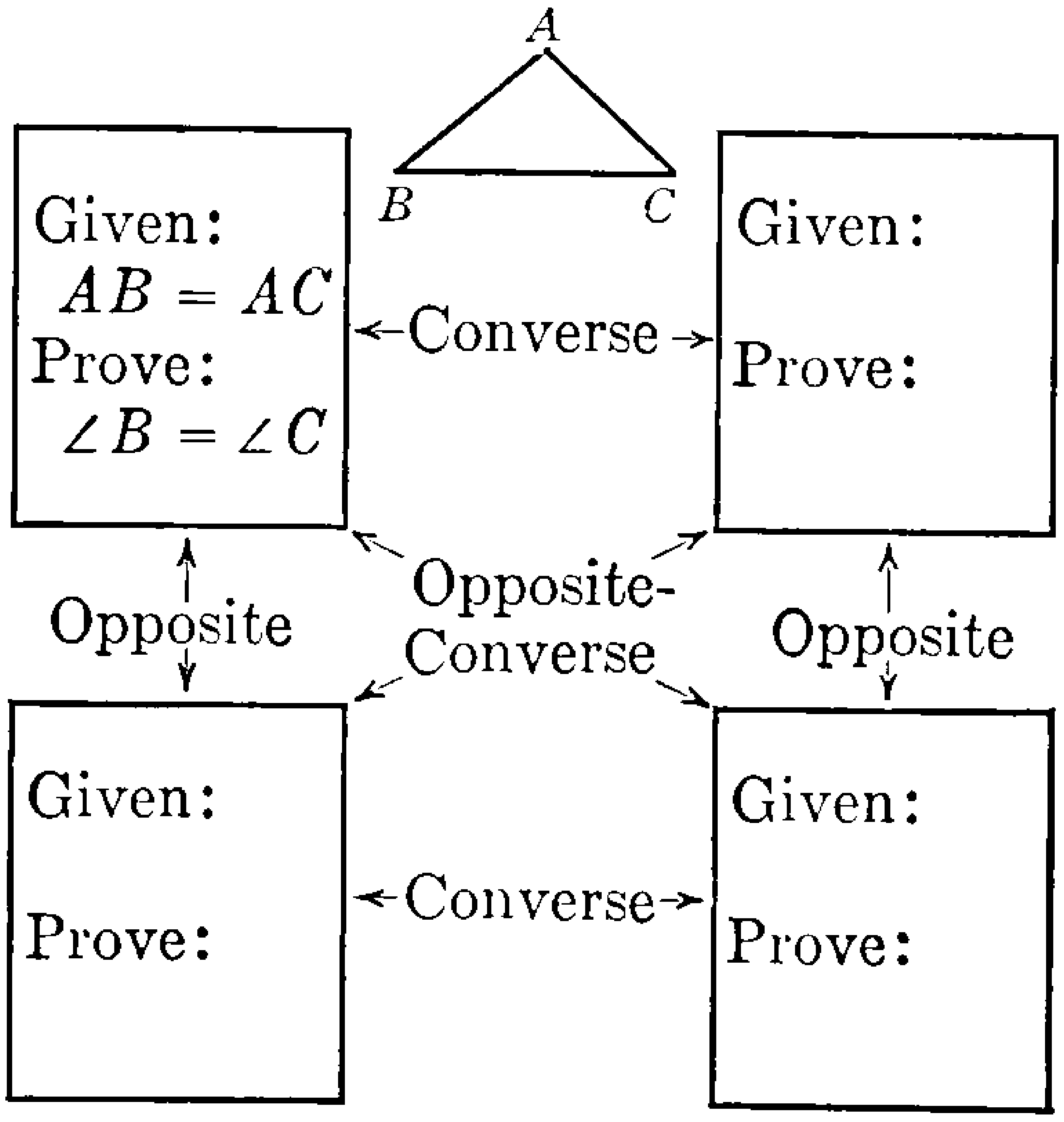
(*a*) If two angles are vertical, then they are equal.

(*b*) If two angles are right angles, then they are equal.

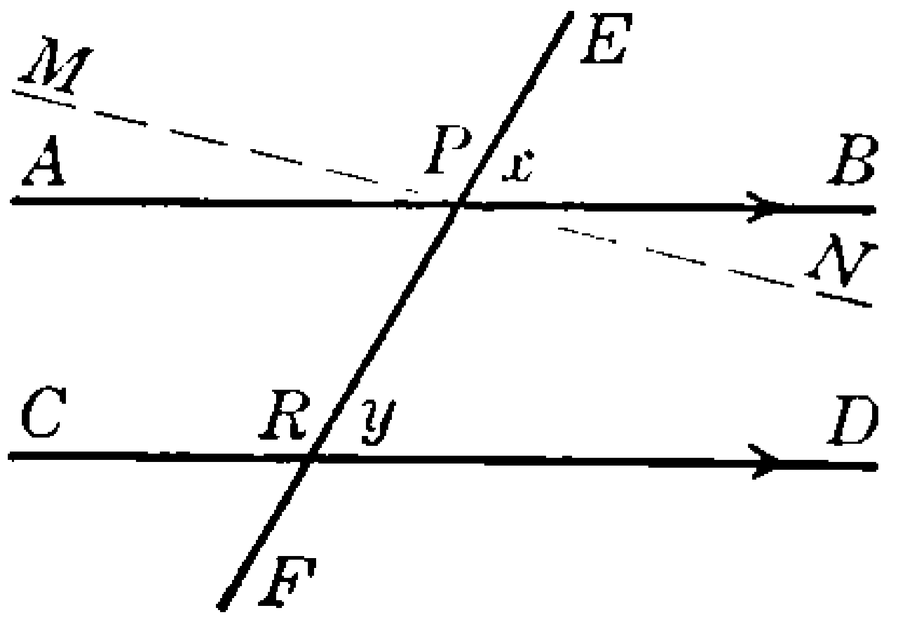
(*c*) If two lines are parallel, they are in the same plane.

(*d*) If a point is inside circle B, then it is inside circle A.

S3. The chart shows how a proposition is related to the converse, to the opposite, and to the opposite-converse. Copy the chart and fill in the hypothesis and the conclusion in the three empty boxes. *Do not mark this book.*



S4. The proposition that only one parallel can be drawn to a given line through an external point (§ 99) is usually called the *parallel postulate*. If we assume this proposition, we can prove indirectly that if two parallels are cut by a transversal, the corresponding angles are equal.



Given:
$AB \parallel CD$
Trans. EF
Prove:
$\angle EPB = \angle PRD$

PROOF

1. Either $\angle EPB = \angle PRD$ or not
2. Assume $\angle EPB \neq \angle PRD$
3. Then draw MN through P so that $\angle EPN = \angle PRD$

Complete this proof.

S5. A famous assumption about parallel lines is the following: If two lines are cut by a transversal and the sum of the interior angles on the same side of the transversal is less than two right angles, then these lines meet on that side of the transversal.

This assumption is usually known as Euclid's parallel postulate, or Euclid's fifth postulate, and is one of the most famous postulates in geometry. Euclid was once head of a school in Alexandria, Egypt. In addition to geometry, his writings include books on optics, logic, and music.

A space in which Euclid's parallel postulate, or any other form of it, is true is called *Euclidean space*. In the nineteenth century several geometries were developed in which this postulate about parallel lines was not assumed. These geometries have to do with non-Euclidean space. In non-Euclidean space, if two parallels are cut by a transversal, the corresponding angles are not necessarily equal, two intersecting lines may both be parallel to a third line, the sum of the angles of a triangle is not 180°, and so on. Scientists are now engaged in measuring the space between the stars to find out whether non-Euclidean geometry fits the universe more accurately than the geometry we study. Even if this is found to be true, however, our Euclidean geometry is simpler, and it is accurate enough for measurement on the earth.

Using Euclid's postulate, prove indirectly that if two parallels are cut by a transversal, the corresponding angles are equal.

S6. Based on the proposition that if two angles have their sides respectively parallel, they are either equal or supplementary, prove indirectly: If two triangles have their sides respectively parallel, then the angles of one triangle are equal respectively to the angles of the other triangle.

Help: All the angles whose sides are parallel cannot be supplementary. Why? Two pairs of these angles cannot be supplementary. Why?

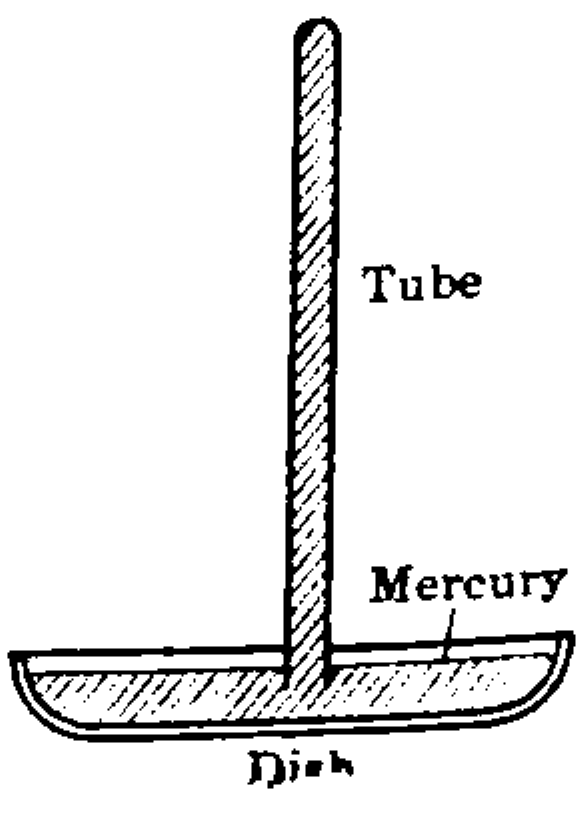
S7. The final score of a ten-inning baseball game was New York 9–Boston 4. Prove that the game was played in Boston.

HONOR WORK

H1. Over two thousand years ago, Aristotle wrote that heavier objects fall faster than lighter objects even if we disregard air resistance. This statement, of course, is false. It seems that Aristotle never thought of experimenting with falling objects; he preferred to depend entirely on logic. Many years later, Galileo proved that Aristotle was wrong. Some people think that he proved it experimentally by dropping stones from the Leaning Tower of Pisa. We are not sure that he really did this, but we *are* sure that he reasoned it out correctly by logic. In his *Dialogues Concerning Two New Sciences*, he gives the following indirect proof:

1. *Assume* that each falling body acquires a definite speed fixed by nature—the heavier the body, the greater the speed.
2. Then a large stone would have a speed of, say 8 feet per second after a certain time; and a small stone would have a lesser speed, say 4 feet per second, after the same time.
3. Now tie the two stones together. The faster stone will hasten the slower one, and the slower stone will retard the faster one. The new speed will therefore be greater than 4 but less than 8.
4. But the new mass is heavier than the large stone and should therefore have a speed greater than 8 feet per second.
5. This contradicts the original assumption.

Experiment. Fill a glass tube about 10 inches long with mercury. Holding the tube closed with your thumb, invert the tube in a dish of mercury. Now remove your thumb. The mercury does not run into the dish but remains in the tube.



The ancient scientists explained this result by saying that if the mercury ran down into the dish, it would create a vacuum in the tube and "nature abhors a vacuum."

Can you prove indirectly that nature does not abhor a vacuum by explaining what happens if the experiment is performed with a 36-inch tube?

H2. Nine men suspected of a crime made the following statements:

Ames: Egan is guilty.

Bates: Ames is untruthful.

Cole: I did it.

Dolan: It was either Cole or Hale.

Egan: Don't believe Bates.

Flynn: Cole is telling the truth.

Gates: Flynn is perjuring himself.

Hale: I am innocent, but Flynn is not telling the truth.

Ives: Hale is right but Ames is wrong.

A stool pigeon who considers it "against his code of ethics to snitch" nevertheless gives the reliable information that only three of the suspects are telling the truth. By the method of elimination determine the guilty person.

SUMMARY

103. A general proposition is a word statement that applies to a whole group of things. In order to understand a general proposition better or to prove it, we usually draw a figure and write out what is given and what we must prove. We call this illustrating the general proposition. Sometimes we have an illustration and we wish to write the general proposition that is so illustrated.

In order that we may write and understand general propositions more easily, we have learned special names of certain polygons. We have classified triangles according to their sides as scalene, isosceles, and equilateral triangles; and according to their angles as acute, obtuse, and right triangles. We have classified quadrilaterals according to the sides that are parallel or equal. The most important kinds of quadrilaterals are trapezoids, parallelograms, rectangles, rhombuses, and squares.

We form the converse of a proposition by interchanging a condition in the hypothesis with a condition in the conclusion. If the hypothesis or the conclusion has more than one condition, then we can form more than one converse.

Sometimes the easiest way of proving a proposition is by an indirect method. In its simplest form, we begin this kind of proof by saying that what we wish to prove is either true or false. We then assume that it is false and continue to argue on that basis until we come to a contradiction. This eliminates the false assumption and proves indirectly that the only remaining possibility is true. This indirect method is called proof by contradiction, proof by elimination, or *reductio ad absurdum*.

We have proved indirectly that through a given point not on a given line, only one parallel can be drawn to the given line; and that through *any* point only one perpendicular can be drawn to a given line.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

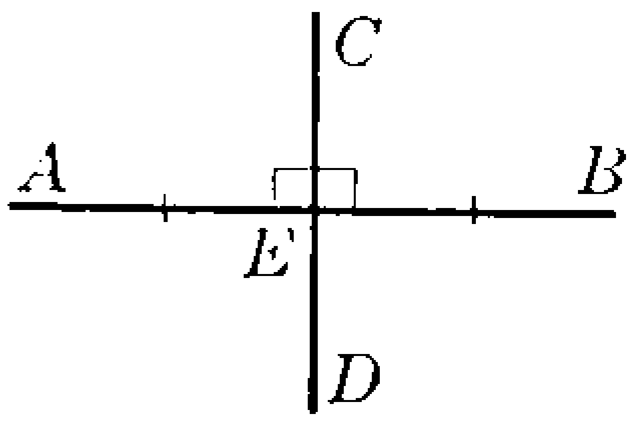
general proposition § 68
converse (of any proposition) § 97
bisect § 86
polygon § 71
vertex of a polygon § 72
diagonal of a polygon § 73
exterior angle of a polygon § 74
isosceles triangle (isos. △) § 76
    legs § 76, vertex angle § 76
    base § 76, base angles § 76
equilateral triangle § 77
scalene triangle § 78
right triangle (rt. △) § 79
    legs § 79, hypotenuse § 79
oblique triangle § 83
    obtuse triangle § 80
    acute triangle § 81
equiangular triangle § 82
median of a triangle § 84
altitude (alt.) of a triangle § 85
angle bisector of a triangle § 87
quadrilateral (quad.) § 88
trapezoid (trap.) § 89, § 90
    bases § 89, base angles § 89
    legs § 89, altitude § 89
parallelogram (▱) § 91
    altitude § 91

rectangle (▭) § 92
rhombus § 93
square § 94
indirect proof § 98
  proof by contradiction § 98
  proof by elimination § 98
  reductio ad absurdum § 98

REVIEW EXERCISES

1. Illustrate the following: A median to a leg of an isosceles triangle.

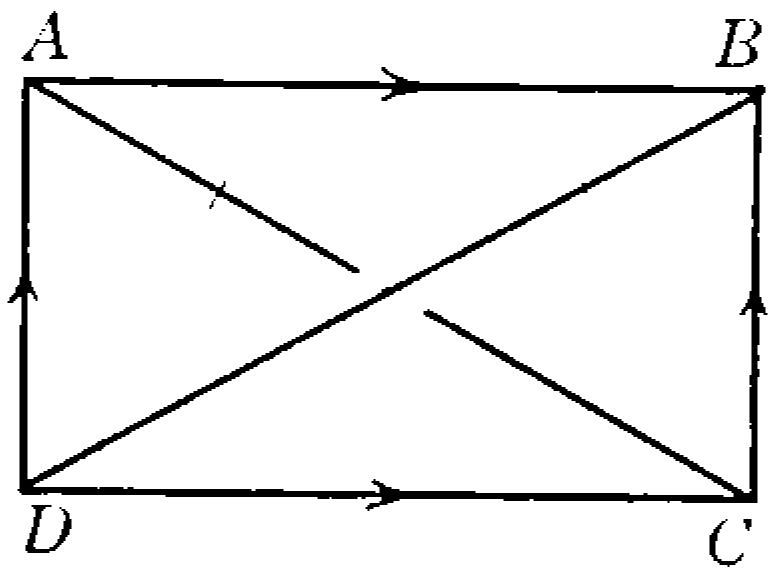
2.



$CD \perp AB$
$AE = EB$
Describe CD in general terms.

3. Draw a figure illustrating a quadrilateral with the diagonals perpendicular to each other.

4. Describe the figure:

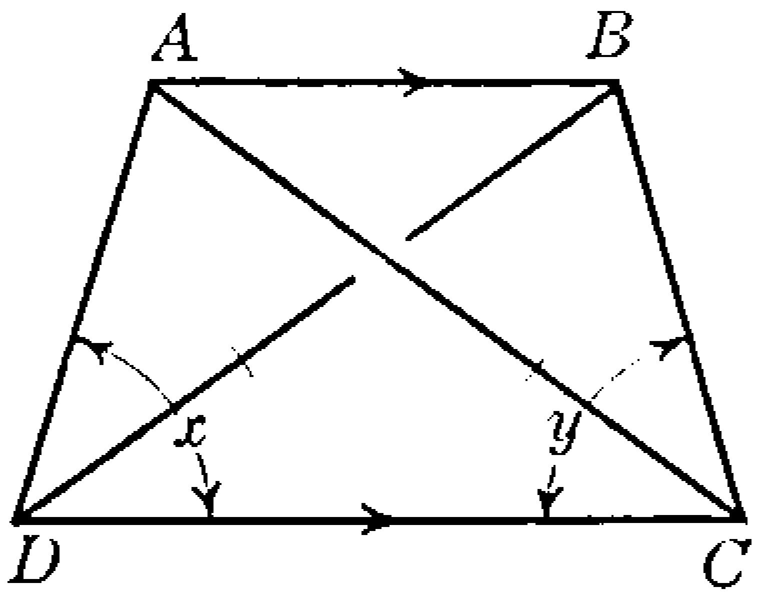


$AB \parallel DC$
$AD \parallel BC$
$AC = BD$

5. Illustrate the following proposition: If two angles of a quadrilateral are supplementary, then the other two angles are supplementary.

6. Illustrate the following: The diagonals of a parallelogram bisect each other.

7. Write the general proposition that this figure illustrates:



Given:
In trap. $ABCD$
$AC = BD$
Prove:
$\angle D = \angle C$

8. (*a*) Write the converse of this proposition: If an integer is a perfect square, the last digit is not 3.

(*b*) Write a converse of this proposition: If a point is equidistant from the sides of an angle, it is on the bisector of the angle.

(*c*) Write two general converses of the proposition that is illustrated in Ex. 7.

9. Write an indirect proof for the following proposition: If one angle of a quadrilateral is acute, the figure is not a rectangle.

10. Prove indirectly that a triangle can have only one right angle or obtuse angle.

11. Prove indirectly: If an integer that is a perfect cube ends in 2, then its cube root ends in 8. *Hint:* There are 10 possibilities, and you can quickly eliminate 9 of them.

SUPERIOR WORK

S1. Write the opposite of each of the following propositions:

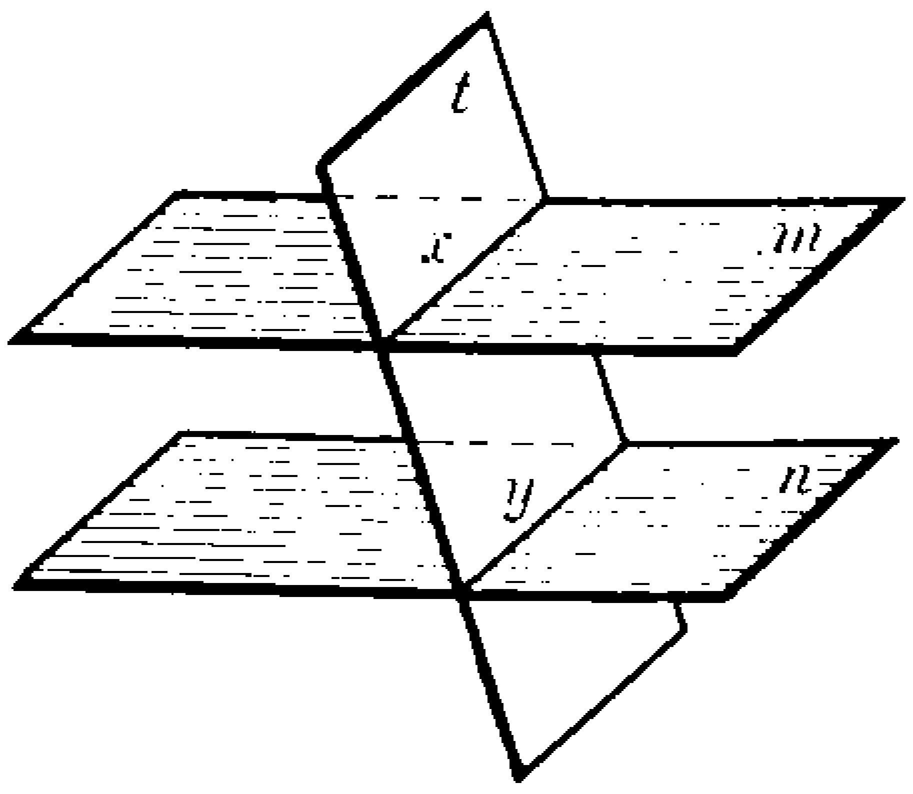
(*a*) If n is an integer, then $2n$ is an even integer.

(*b*) If the sum of the digits of a number is divisible by 9, then the number itself is divisible by 9.

S2. Write a converse of the following proposition and illustrate it: If two isosceles triangles have the same vertex angle, then their bases are parallel.

S3. Write the opposite of the following proposition and illustrate it: If the denominator of a fraction in its lowest terms contains no factors other than 2 or 5, then the fraction can be expressed exactly as a terminating decimal.

S4. *Theorem.* If two parallel planes are cut by a third plane, the intersections are parallel. (See page 53, S10.)



Given: Plane $m \parallel$ plane n
Plane l cuts planes m and n in x and y
Prove: $x \parallel y$

INDIRECT PROOF

1. Either $x \parallel y$ or $x \nparallel y$
2. *Assume* $x \nparallel y$
3. Then x and y meet in some point P — 3. Why?

Complete this proof by showing that P is in both planes m and n.

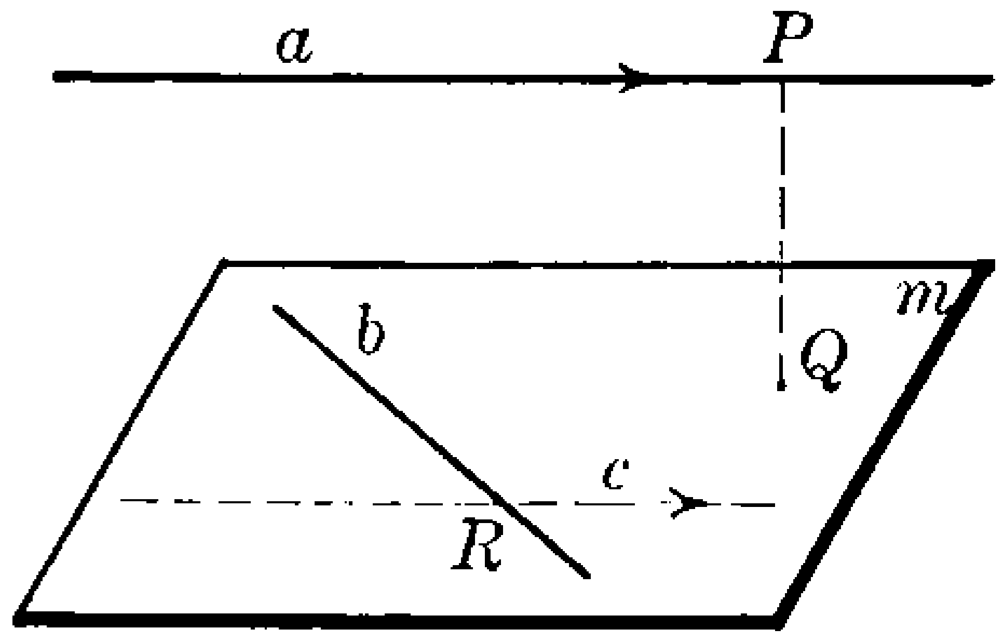
S5. *Theorem.* Two planes perpendicular to the same line are parallel.

Prove this indirectly. Assume that the two planes meet. Through a point on their intersection, and through the given line, pass a plane. Now prove that this would result in two perpendiculars in the same plane to the same line from a given point.

S6. *Theorem.* A plane containing one, and only one, of two parallel lines is parallel to the other line.

Prove this indirectly by assuming that the plane meets the other line.

S7. *Theorem.* Through one of two skew lines, it is possible to pass a plane that is parallel to the other. (See page 53, S10).



Given: Lines a and b not in the same plane
Prove: There is a plane that passes through b and is parallel to a

PROOF

1. Through any pt. R in line b, draw a line c parallel to a
2. Pass a plane m through b and c
3. Plane m is parallel to line a — 3. Why?

From a point P in line a (in the figure above), let PQ be a line perpendicular to plane m. As the point P moves along the line a, will Q the foot of the perpendicular ever reach the line b? Will PQ then be a common perpendicular to a and b? State this conclusion as a general proposition.

HONOR WORK

H1. Criticize the following definitions, paying particular attention to reversibility and redundancy:

(*a*) An acute triangle is one having an acute angle.

(*b*) An obtuse triangle is one having one obtuse angle and two acute angles.

(*c*) A rectangle is a parallelogram in which the angles are right angles.

(*d*) A rectangle is a quadrilateral that has two right angles.

(*e*) A rectangle is a quadrilateral that has three right angles.

(*f*) A rectangle is a quadrilateral that has four right angles.

(*g*) An altitude of a trapezoid is a line segment included between the bases of the trapezoid and perpendicular to them.

H2. Form one converse, one opposite, and one opposite-converse of each of the following:

(*a*) *Z* cigarettes do not irritate the throat.

(*b*) If a mature person smokes *Z* cigarettes moderately, his throat is not irritated by them.

(*c*) John is necessary to the team.

H3. (*a*) The most common fallacy in the proof by elimination is the failure to list all the possible conclusions before disproving any of them. After assassinating Caesar, Brutus said to the mob, "Had you rather Caesar were living, and die all slaves, than that Caesar were dead, to live all freemen?"

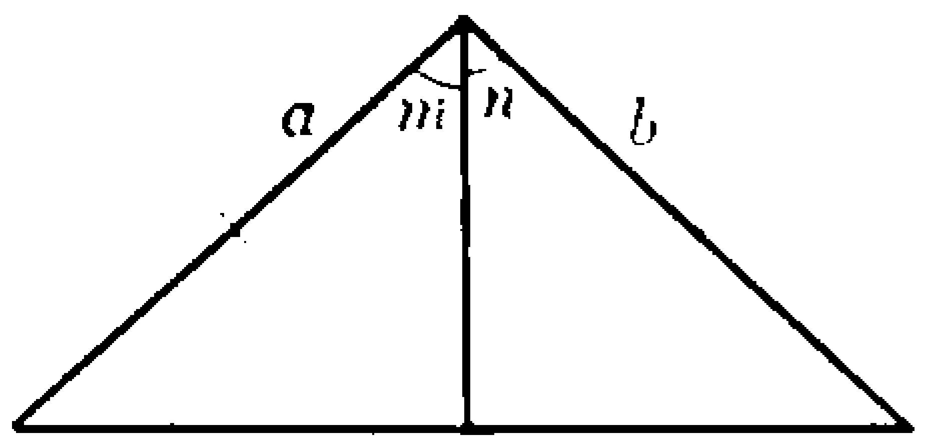
Did Brutus list all the alternatives?

(*b*) Again, Brutus said, "Who is here so vile that will not love his country? If any speak, for him have I offended." State this as an alternative proposition in your own words. Where is the fallacy? Can you cite a similar fallacy from some current speech or article?

TEST

1. Draw a figure illustrating two line segments joining the midpoints of the opposite sides of a quadrilateral.

2. Describe the following figure:

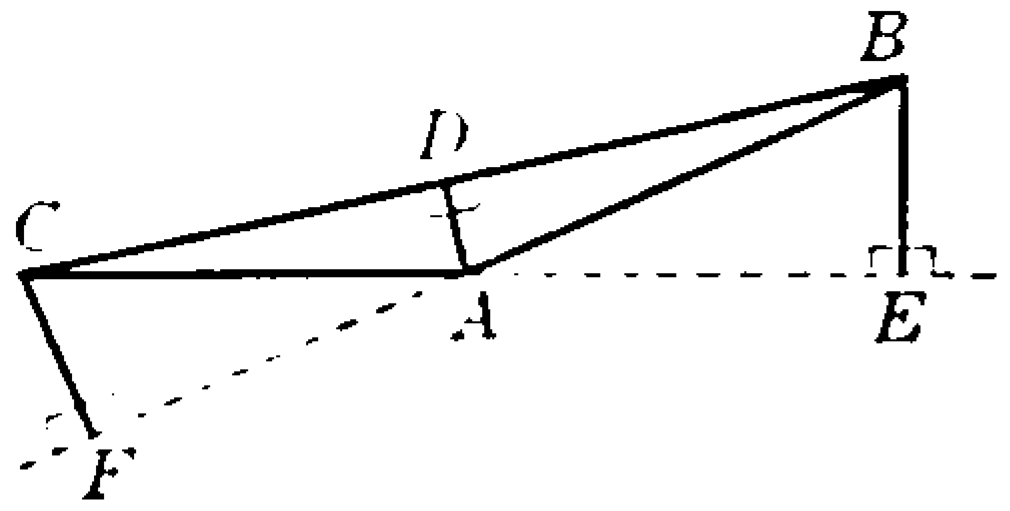


$a = b$

$\angle m = \angle n$

3. Draw a figure illustrating two isosceles triangles in which the base of one triangle is a leg of the other.

4. Describe the following figure:



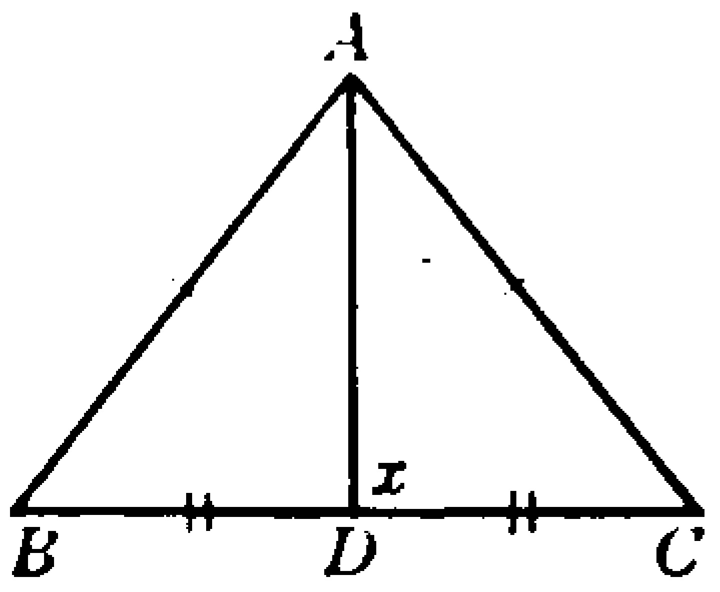
$AD \perp BC$

$BE \perp CA$

$CF \perp BA$

5. Illustrate the following proposition: If a line bisects one leg of a trapezoid and is parallel to a base, then it bisects the other leg.

6. Generalize the following proposition:



Given:
$AB = AC$
$BD = DC$
Prove:
$AD \perp BC$

7. Illustrate the converse of the following proposition: If a transversal to two lines makes a pair of corresponding angles supplementary, then the interior angles on the same side of the transversal are equal.

8. Illustrate the converse of the following proposition: If two angles of a triangle are not equal, then the sides opposite these angles are not equal.

9. Write the converse of the following proposition: If each of two numbers is a perfect square, then their product is a perfect square.

10. Prove the proposition that from an external point only one perpendicular can be drawn to a given line. Show that if two perpendiculars were possible, it would contradict the theorem that the sum of the angles of a triangle is 180°.

11. Prove the following proposition indirectly: If a transversal to two lines makes the corresponding angles unequal, then another transversal to these lines will make the alternate interior angles unequal.

12. A boy bought some pigeons for $2. For a white pigeon he paid 50 cents, and for a blue pigeon he paid 75 cents. Explain with the aid of indirect reasoning how many blue pigeons he bought.

4

Induction and Deduction

Congruent Triangles

PROOF BY INDUCTION

Induction

104. In everyday reasoning we often argue from observation of a number of cases that a certain conclusion is valid. For example, from observation of a large number of birds we arrive at the conclusion that all birds have wings. This kind of reasoning from observation is called *inductive reasoning.*

In geometry we may use the same kind of reasoning in proving a proposition. For example, to prove the proposition that vertical angles are equal, we can draw a number of pairs of vertical angles and measure them. We then see that in each *particular* case they are equal. From these we arrive at the *general* proposition that vertical angles are equal. This is the method of *induction.* By this method we arrive at a general proposition from a number of particular cases.

By another method of reasoning (which we have already used), we may reason that a pair of vertical angles is a special case of the more general proposition about supplements of the same angle. This is the method of *deduction.* By this method we reason that a particular proposition is true because it is part of a more general proposition that is true.

By *induction* we reason from the particular to the general.

By *deduction* we reason from the general to the particular.

In demonstrative geometry we generally use the method of deduction, but we can often use the inductive method to discover new propositions. Certainly, if geometry is to improve our thinking, we must learn something about the inductive method, a method that is important both in scientific work and in everyday thinking.

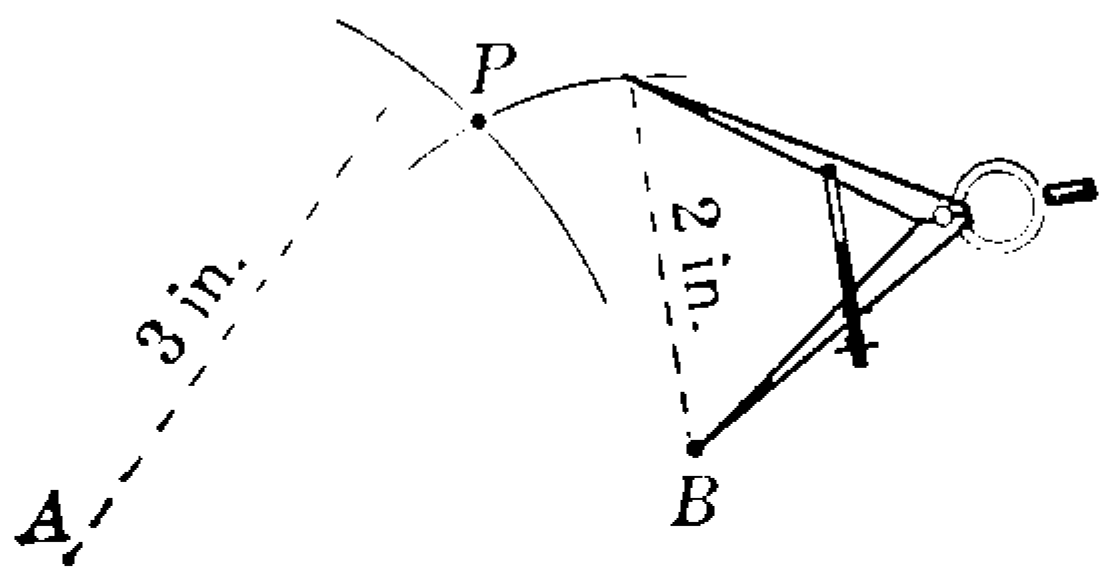
Experimentation

105. By the inductive method we argue from a number of particular facts. We discover these facts by experimenting. Some people think that experiments are made only in a laboratory, but that is not true. By observing or measuring carefully anywhere we may find out many useful facts.

The following simple tools are useful in experimenting with geometric figures: (1) a straight-edge scale ruler, (2) a protractor, (3) a compass, and (4) a small right triangle. A protractor usually provides a straight-edge scale (marked ruler).

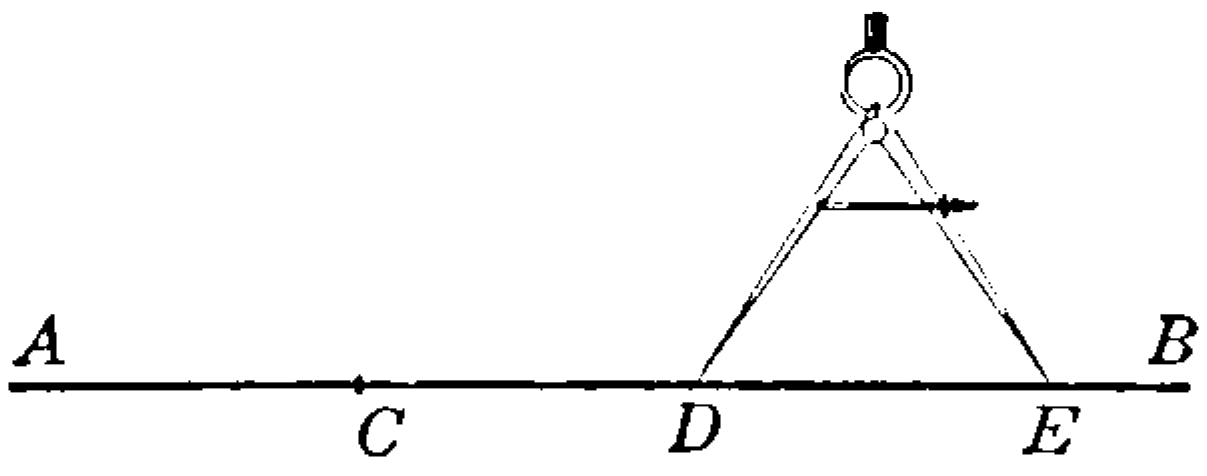
106. You may use a compass in a number of ways:

(*a*) With a compass you can locate a point P that is a given distance (say 3 inches) from a given point A, and another distance (say 2 inches) from a given point *B*.

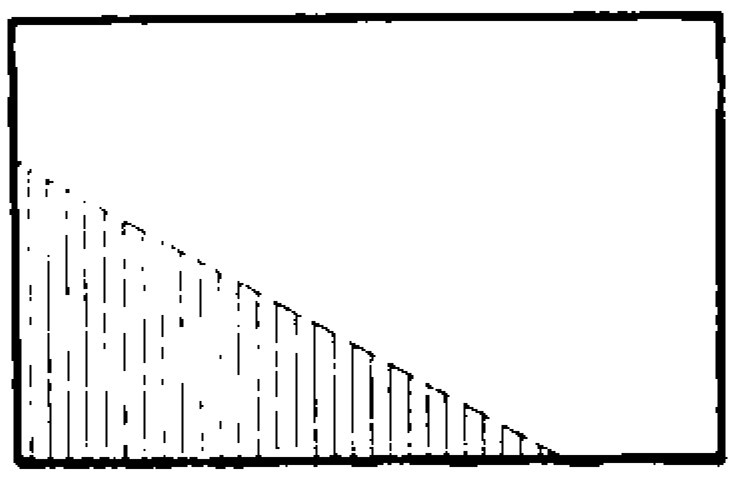


(*b*) With a compass you can find out quickly whether or not two line segments are equal.

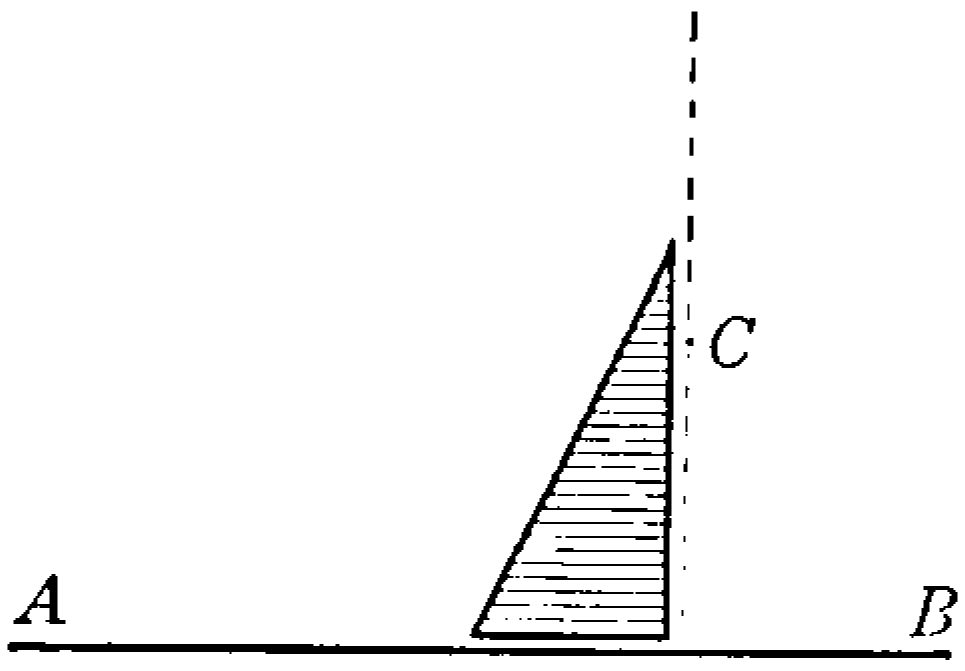
(*c*) Use a compass to divide a line segment into a number of equal parts by approximation. For example, draw a straight line *AB*. To divide *AB* into three equal parts, set the span of your compass at a distance *AC* that is approximately one-third of *AB*. Then mark off this span three times from *A* to *B*. The span will usually be too short or too long, so that you will miss the end point *B* by a distance *EB*. Next, lengthen or shorten the span by one-third of *EB* and mark it off again. You should now have a closer approximation. Practice until you can make this division accurately and quickly.



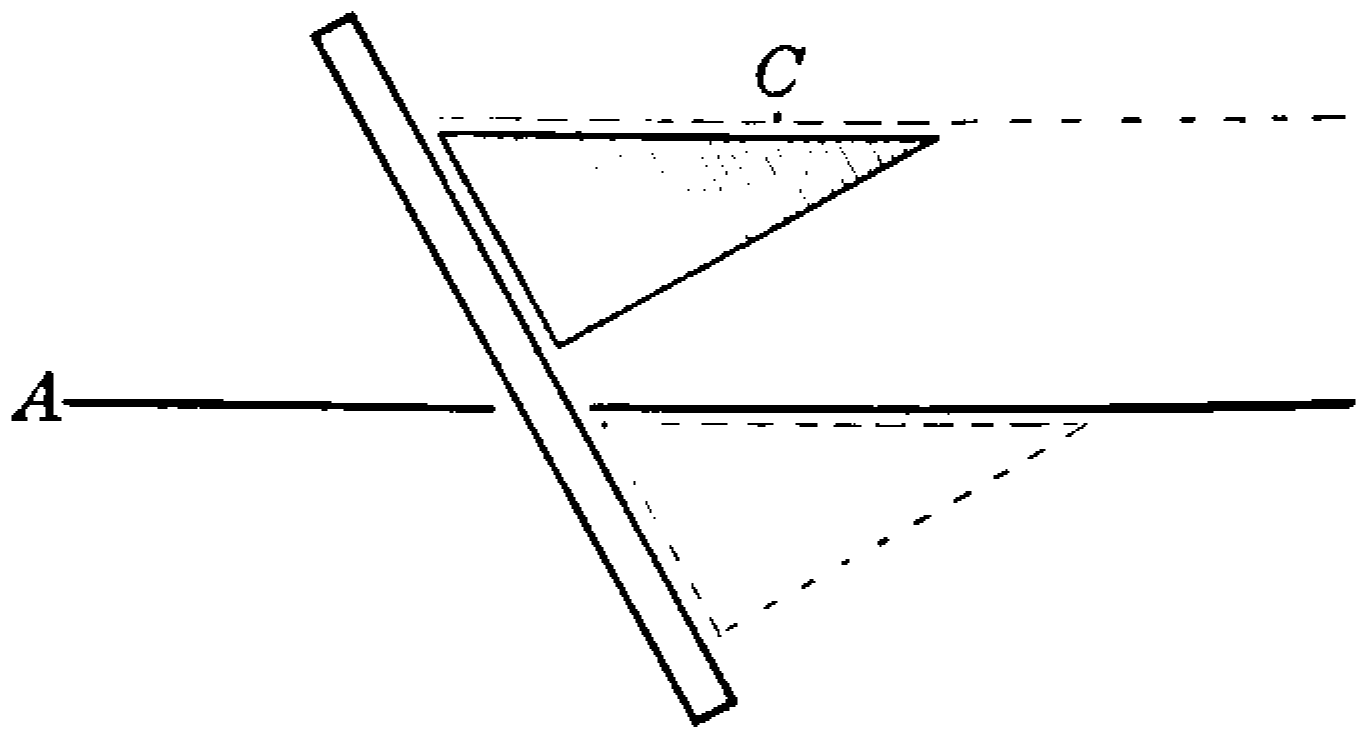
107. Make a small right triangle by cutting off a corner from a stiff rectangular card. The triangle will be more useful if the acute angles are 30° and 60°.



(*a*) Can you use your triangle (or the square corner of your protractor) to draw a line through a given point *C* perpendicular to a given line *AB*?



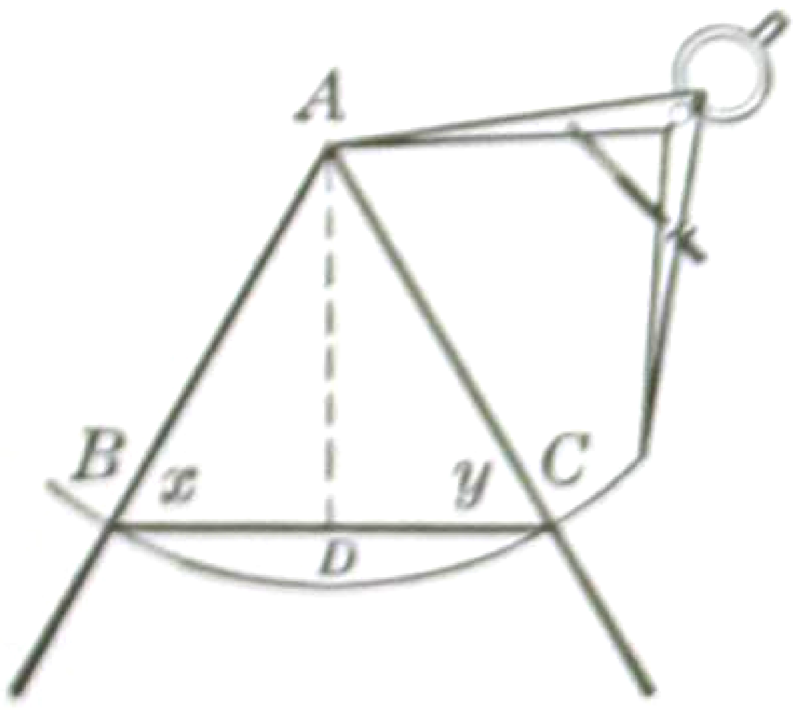
(*b*) Can you use the triangle to draw a line through a given point *C* parallel to a given line *AB*? To do this, place your triangle so that one side falls along *AB*, and then place a straightedge (an unmarked ruler) along another side of the triangle. Now shift the triangle along the straightedge (which should be held firmly against the paper) until the side of the triangle that was in contact with *AB* goes through the point *C*, and draw the line through *C*. What proposition tells you that the line so drawn is parallel to *AB*? How can this method be used to determine whether or not two lines are parallel?



EXERCISES

1. Using a ruler, draw three different isosceles triangles. In each case, measure the base angles with your protractor and write their sizes in the figures.
2. Using a compass, construct an isosceles triangle *ABC*. By the method of ap-

proximation, described in § 106 (*c*), find the midpoint *D* of *BC*. Draw *AD*. Now cut out the triangle *ABC* and fold it along the line *AD*. Do the angles *x* and *y* fit together?

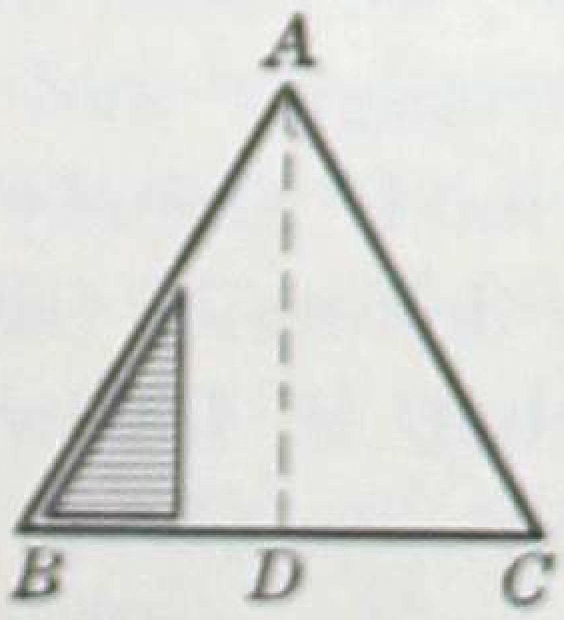


State the general proposition that seems to follow from the experiments in Ex. 1 and Ex. 2.

3. Using a protractor, draw a triangle in which each one of a pair of equal angles is 50°. Measure the sides opposite these angles and write their lengths on the figure.

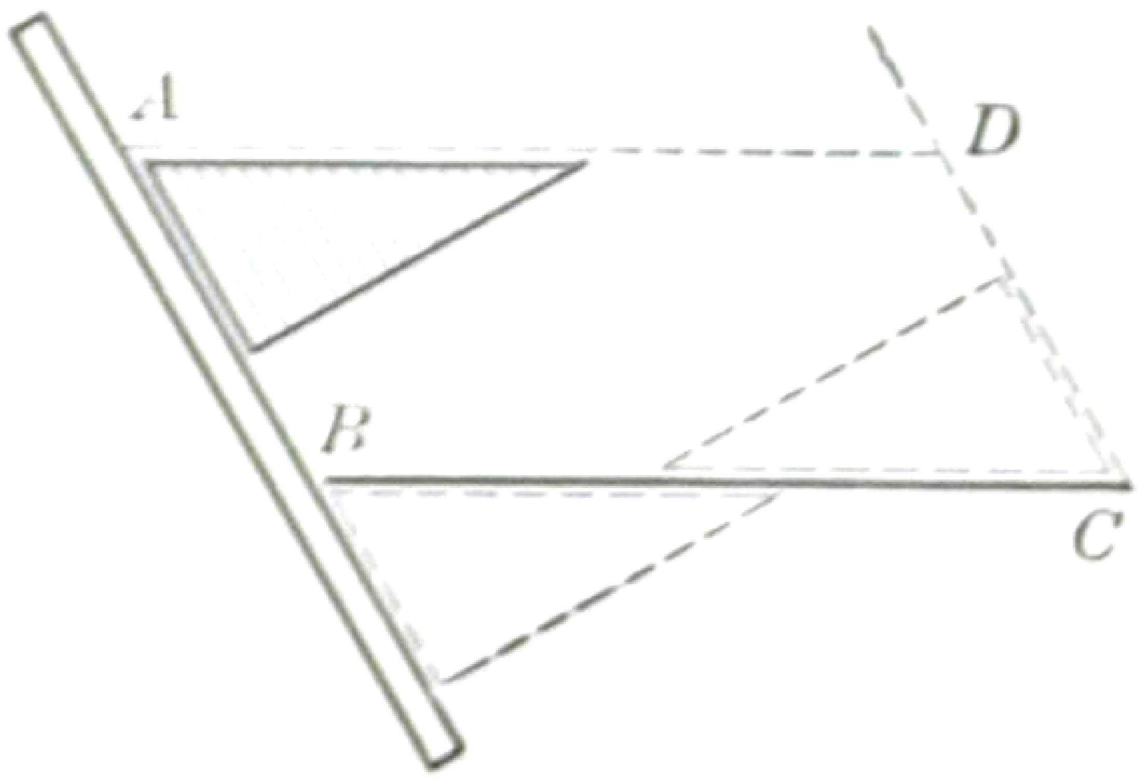
   Repeat this experiment for a triangle with equal angles of 70°, and for a triangle with equal angles of 35°.

4. Draw a triangle *ABC*, making angles *B* and *C* each equal to an acute angle of your cardboard triangle. Cut out the triangle *ABC* and fold it along *AD* so that *B* falls on *C*. Compare the lengths of *AB* and *AC*.

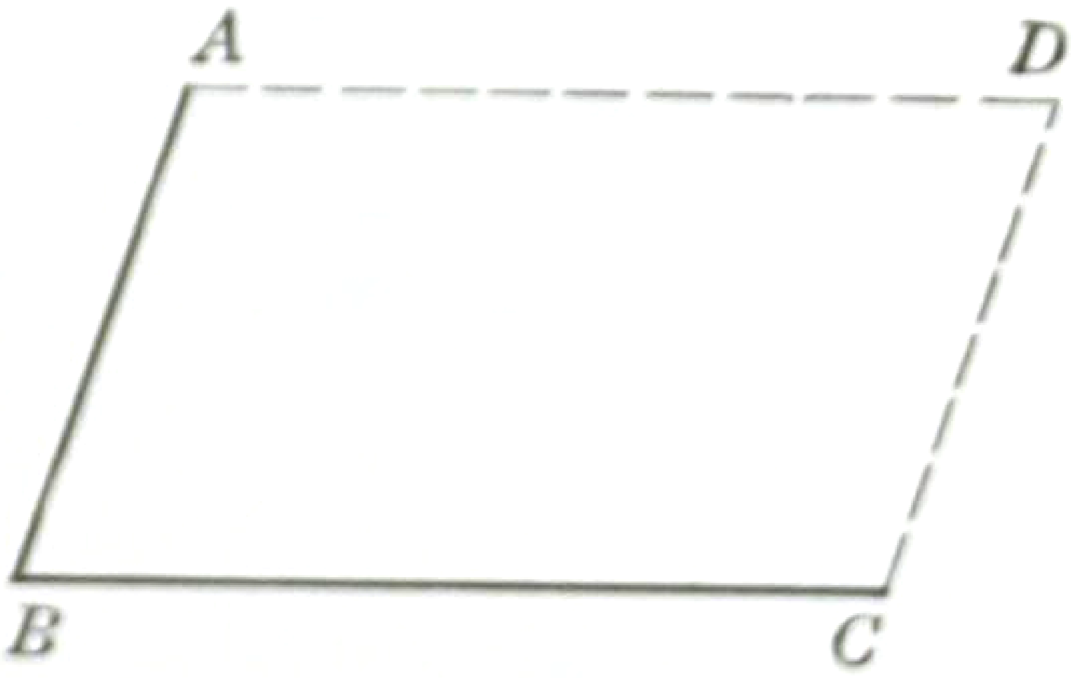


State the general proposition that seems to follow from the experiments in Ex. 3 and Ex. 4.

5. Draw an angle *ABC*. Complete the parallelogram *ABCD* by drawing *AD* parallel to *BC*, and *CD* parallel to *AB*, using the method in § 107 (*b*). Now measure the four sides of the parallelogram and write the measurements in your figure.

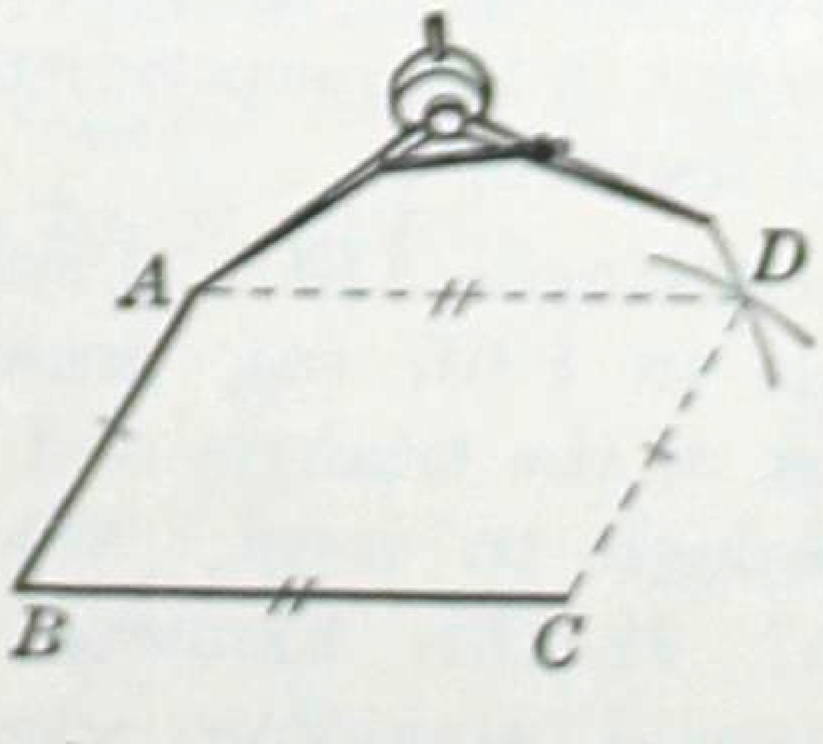


6. Draw an angle *ABC*. Complete the parallelogram *ABCD* with a protractor as follows: Measure $\angle B$ and draw $\angle A$ and $\angle C$ each equal to the supplement of $\angle B$. Using a compass, compare the lengths of the opposite sides of the parallelogram.



State the general proposition that seems to follow from the experiments in Ex. 5 and Ex. 6.

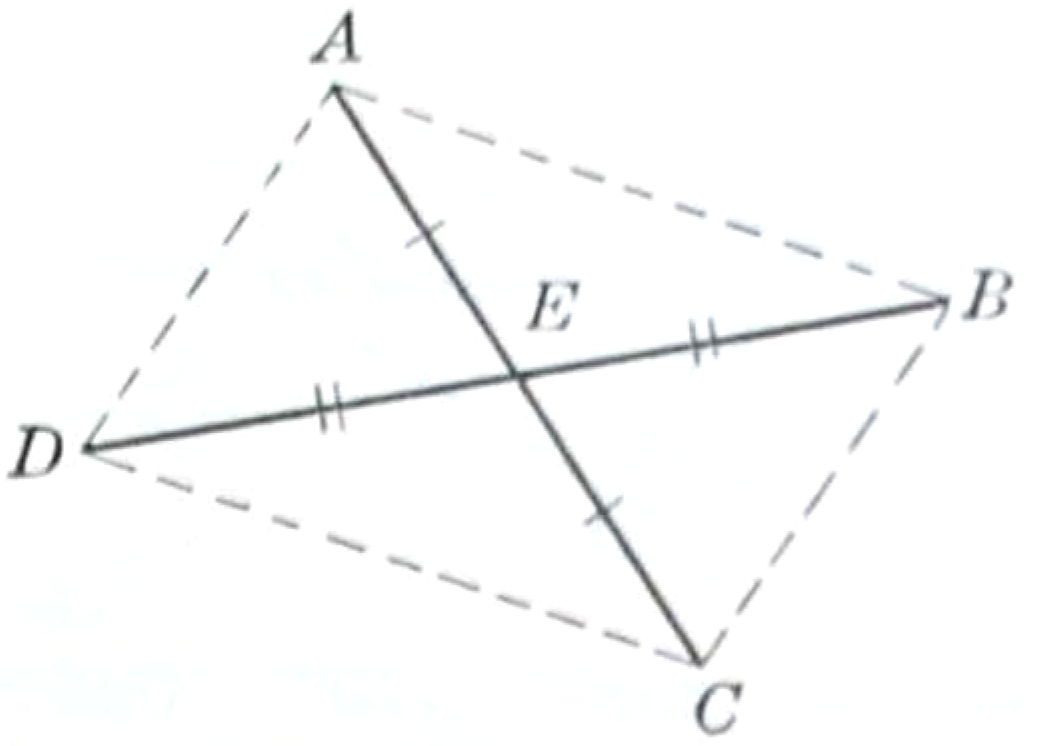
7. Draw an angle *ABC*. Locate a point *D* so that $DC = AB$ and $AD = BC$, as in § 106 (*a*). Draw *AD* and *DC*. Using your triangle and straightedge as in § 107 (*b*), compare the directions of *AD* and *BC*; also the directions of *DC* and *AB*.



Assuming that this result is typical, what general proposition seems to follow?

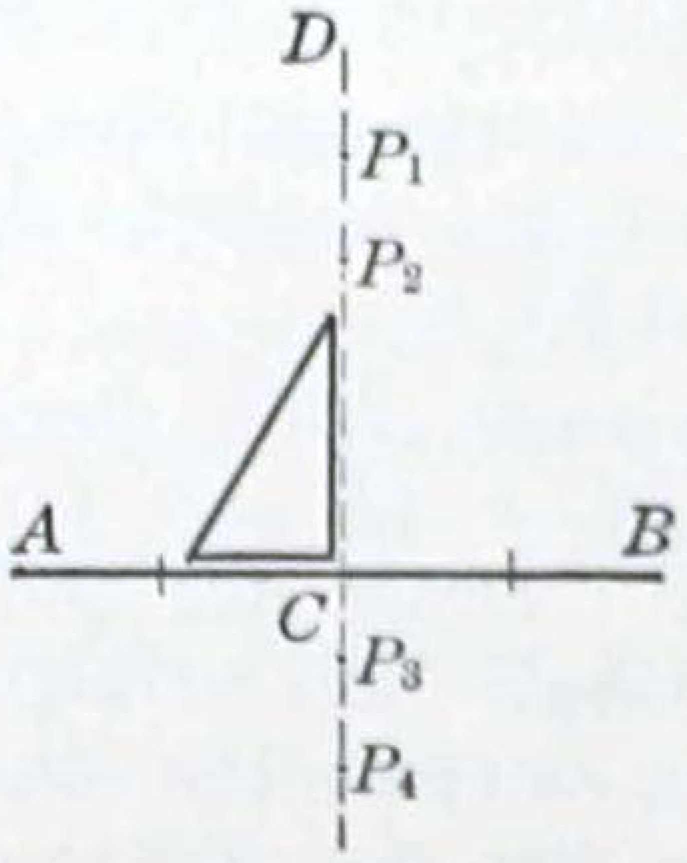
8. Using a scale ruler or compass, draw straight lines *AC* and *DB* so that they bisect each other. Complete the quadrilateral *ABCD*. With a triangle and straightedge determine whether any lines are parallel. Repeat your experiment

several times, varying the lengths of the straight lines AC and DB.



The general proposition that seems to follow by induction is stated below; complete it: If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a —?—.

9. Draw a line segment AB. Using the method of approximation, find the midpoint C. Using a right triangle, draw CD perpendicular to AB. Locate a number of points (P_1, P_2, P_3, etc.) on CD. With a compass compare P_1A with P_1B, P_2A with P_2B, P_3A with P_3B, and so on.



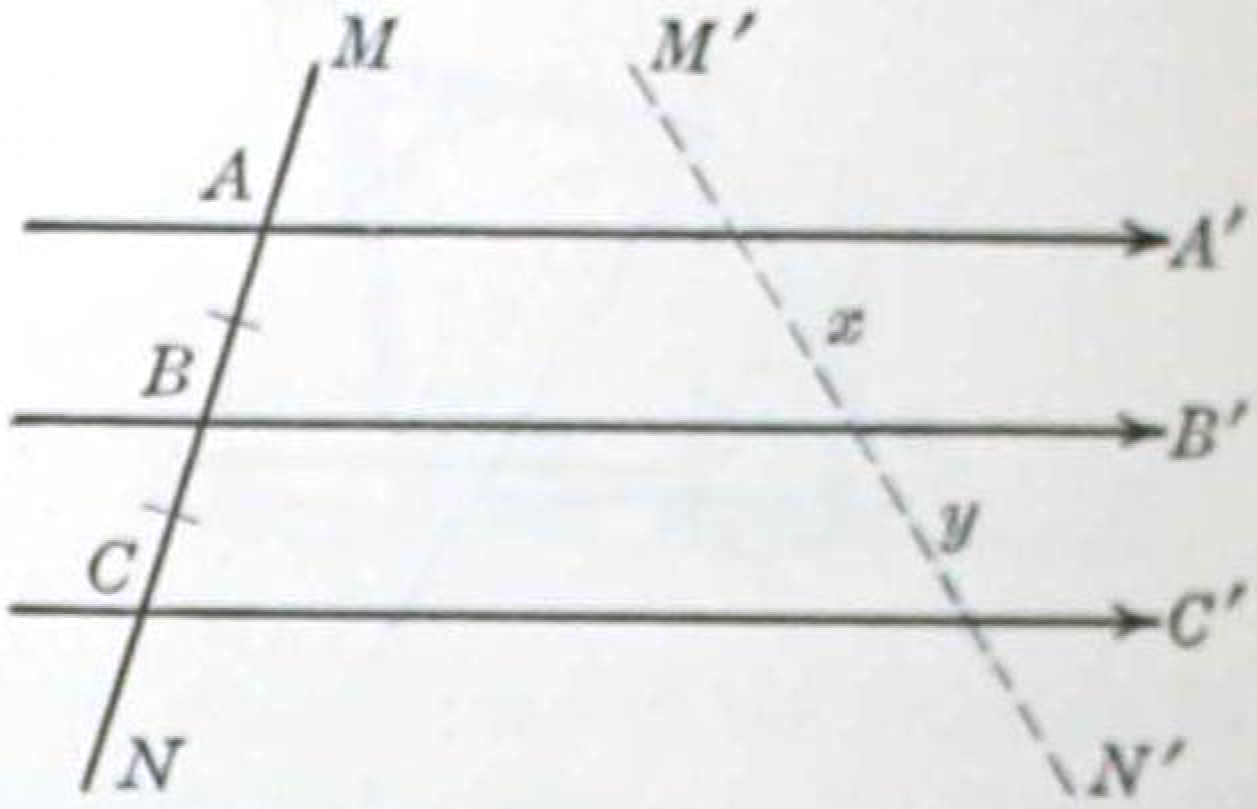
Complete the following general proposition that seems to follow by induction: Every point in the perpendicular bisector of a line segment is . . . .

10. Draw a triangle ABC. Using your triangle as in § 107 (a), draw the three altitudes of the triangle and make them long enough to meet. What do you observe? If this happens repeatedly, what general proposition seems to follow by induction?

SUPERIOR WORK

S1. Draw a triangle with a line segment connecting the midpoints of two sides. Compare this line segment with the third side of the triangle as to length and direction. State this relationship as a general proposition.

S2. On a line MN, lay off $AB = BC$. Through A, B, and C, draw three parallel lines. Now, draw another transversal $M'N'$ and measure the segments x and y on it.



The experiment points to the following general proposition; complete it: If three parallels cut equal segments on one transversal, then . . . .

S3. (a) Draw a triangle and two of its medians. Measure the segments made by each median as it cuts the other median. Can you discover any relationship? State it as a general proposition.

(b) Draw the third median of the triangle. What do you observe about the intersections? State it as a general proposition.

Induction versus Deduction

108. To prove by induction the proposition that the sum of the angles of a triangle is 180°, we could draw a number of different triangles and measure the angles. We have already proved this proposition by deduction (see § 62), where we showed that it follows from more general propositions about parallel lines. Let us now examine some advantages and disadvantages of these two methods.

One disadvantage of the inductive method of reasoning is that the resulting general proposition is not absolutely certain. If we measured the angles of a hundred triangles, for example, and found that in each case the sum of the angles is 180°, it might still be possible that someone could find a triangle in which this is not true. Another drawback in the inductive method is that the method does not tell why a proposition is true.

The inductive method has the great advantage that it leads to discovery more often than the deductive method. Thus, in the last set of exercises you have discovered certain propositions that you will not be ready to prove deductively for some time. Most of our scientific knowledge was obtained by induction. In fact, the inductive method is sometimes called the *scientific method*.

In practical thinking we often use the two methods together. We get our assumptions about space by inductive thinking, and we often discover general propositions inductively by first proving some special cases deductively. The inductive method is based on the assumption that nothing happens without a cause. This assumption is sometimes called the *postulate of knowledge*. In inductive reasoning we conclude that if a proposition is true a great many times, there must be an unknown reason that will make it true generally. The postulate of knowledge gives us a connecting link between the inductive and the deductive methods; because once the unknown reason becomes known, we have a deductive proof of our proposition.

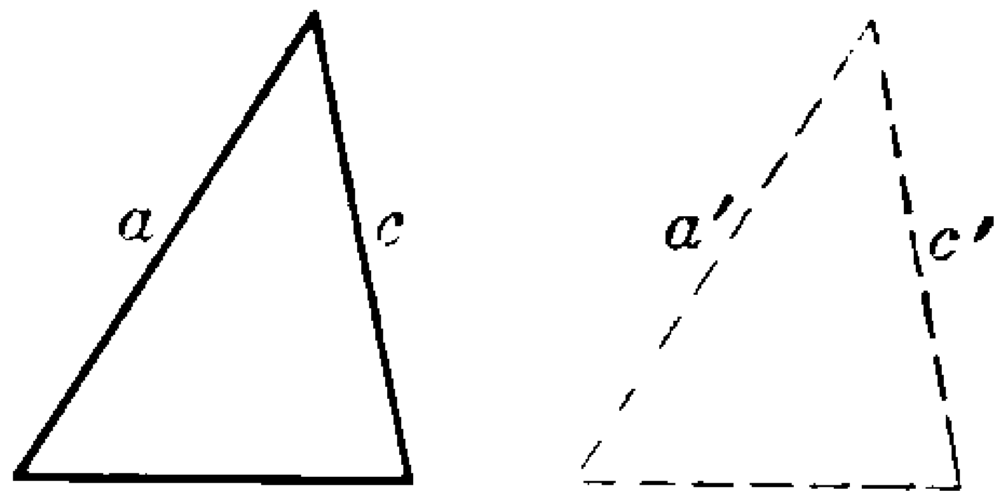
In science a general proposition, called a *hypothesis*, is often set up after a number of experiments. It is a proposition that is assumed in order to explain the results of the experiments. A scientific hypothesis remains an assumption (or a theory) until someone finds an experiment that contradicts it, in which case it is discarded; or until someone finds the unknown reason why it is true, in which case it becomes a deductive theorem.

When the method of attacking a geometric problem is not known, we often reason inductively in the hope of discovering a method. We examine many typical cases and draw a general conclusion.

As shown by the propositions in Chapter 3, we can prove certain statements about angles if we have some information about other angles. At this time, however, we have no way of proving line segments equal, when we are told that certain angles are equal, and vice versa. Our problem now, therefore, is to prove certain line segments equal when we know that certain angles are equal; in other words, to find out how those equal angles are related to equal line segments.

Copying a Triangle

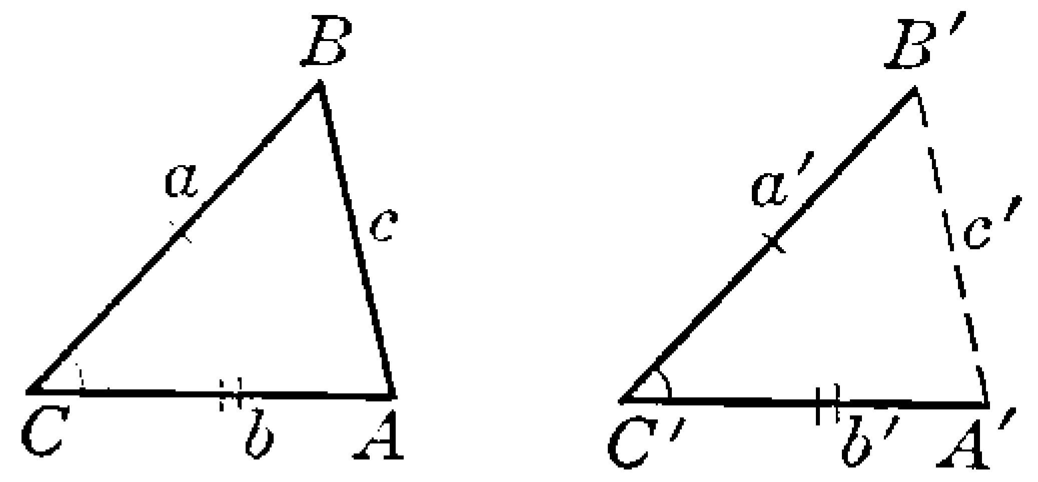
109. The simplest figure in which line segments are connected with angles is the triangle. A triangle has six parts, three sides and three angles. To copy a triangle ABC, how many and which of the six parts must we copy? If it turns out that triangle $A'B'C'$ can be made an exact copy of triangle ABC by merely copying three parts, then we could conclude that the remaining three parts of the two triangles are equal respectively. We would then have arrived at a method that would enable us to prove certain line segments equal, because certain angles are known to be equal.



In labeling triangles, it is often convenient to use capital letters for the vertices, and the corresponding small letters for the sides opposite the vertices.

EXERCISES

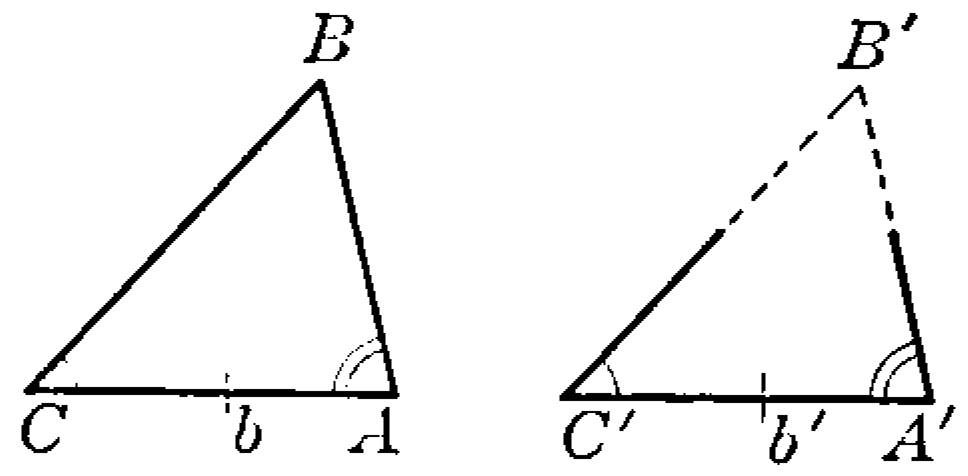
1. (*a*) Draw a triangle ABC having an acute angle at C. Copy angle C and the sides a and b as parts of a new triangle. Now complete the new triangle $A'B'C'$. Measure and compare angle A with angle A', angle B with angle B', and side c with side c'.



(*b*) Repeat part (*a*) several times, varying the size of the angle from acute to obtuse.

Complete the general proposition below, which seems to follow by induction from the experiment above: If two triangles agree in two sides and the included angle, then they agree . . . .

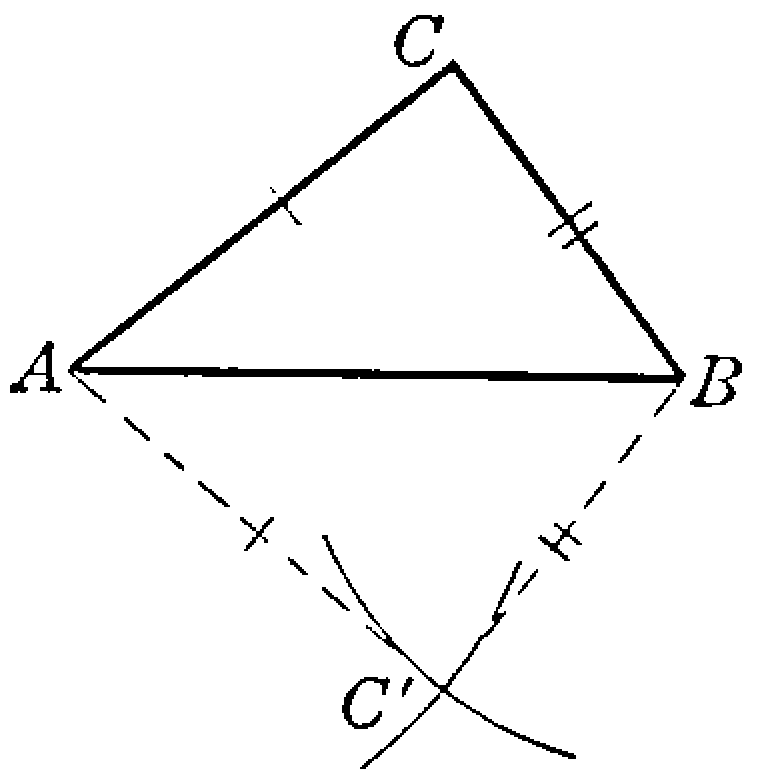
2. (*a*) Draw a triangle ABC having acute angles at A and C. Copy angles A and C, and side b as parts of a new triangle. Complete the new triangle $A'B'C'$. Measure and compare the remaining parts of the two triangles.



(*b*) Repeat part (*a*) several times, varying the size of the angles.

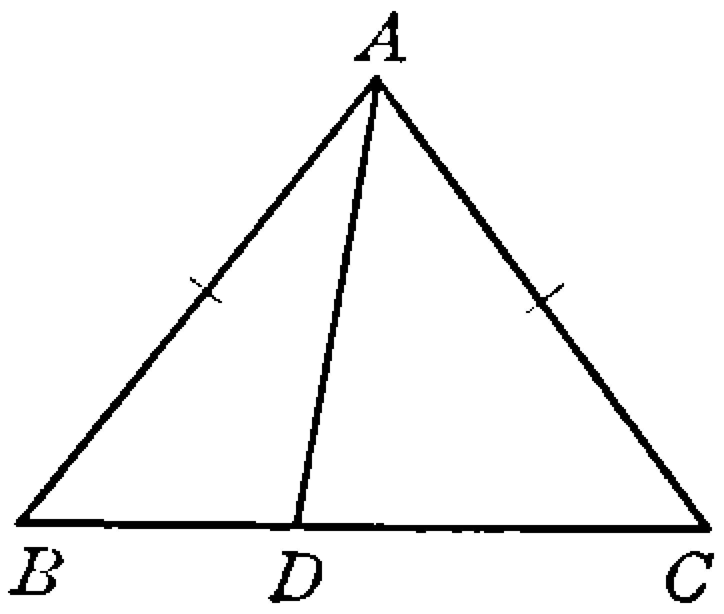
Complete the general proposition, which seems to follow from your experiment: If two triangles agree in . . . .

3. Draw a triangle ABC having three unequal sides. Using the method explained in § 106 (*a*), locate a point C' that is the same distance from A and B as point C. Draw lines connecting C' with A and B. You now have two triangles that agree in three sides. Cut out your figure and fold it along the line AB. Compare the three angles of one triangle with the three angles of the other triangle. What do you discover? State this as a general proposition arrived at inductively.



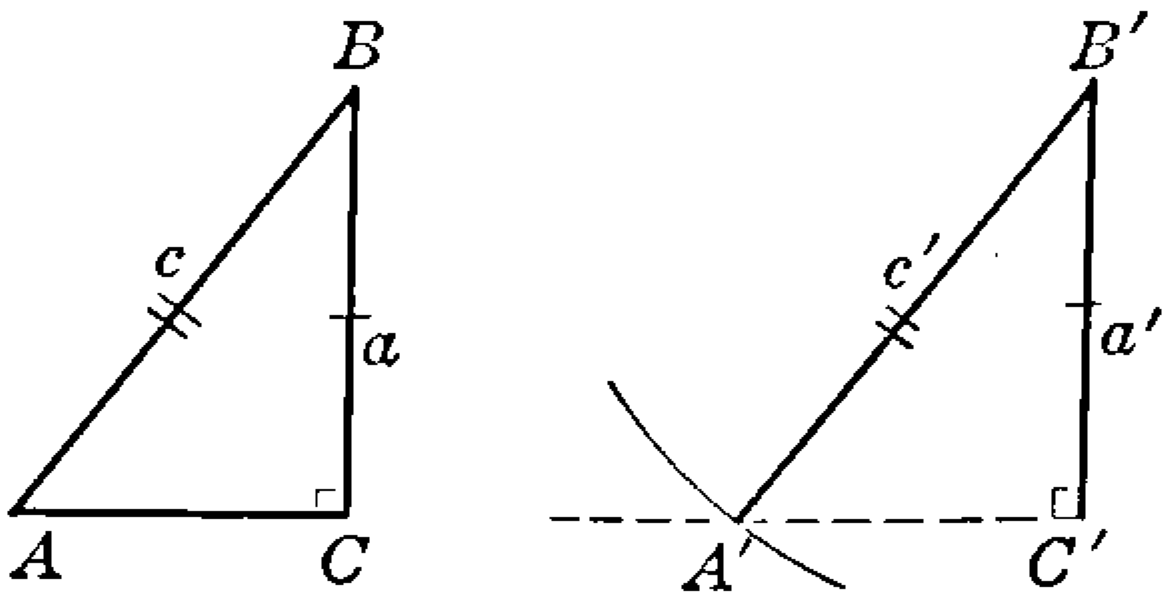
4. Draw a triangle ABC. Draw another triangle that agrees with it in the three angles but not in the sides. Complete the following proposition arrived at inductively: If two triangles agree in three angles, then . . . .

5. Draw an isosceles triangle ABC and a line AD to some point on the base. Measure the parts of the triangles ABD and ACD. Do these triangles agree in three parts? Reasoning inductively, complete the following proposition: If two triangles agree in two sides and a nonincluded angle, then . . . .



6. Draw a triangle ABC having a right angle at C. Copy the right angle C, the leg a, and the hypotenuse c as parts of a new triangle $A'B'C'$. (Use your compass to locate A'.) Measure and compare the remaining parts of the two triangles.

Complete the proposition, which seems to follow by induction from your experiment: If two right triangles agree in the hypotenuse and a leg, then . . . .



Congruent Triangles

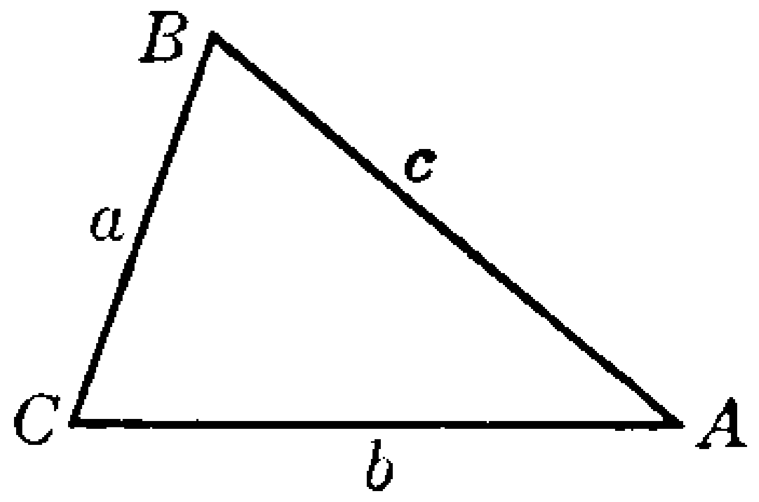
110. By inductive reasoning we have concluded that under certain conditions two triangles that agree in three parts also agree in the remaining parts. We shall assume two of these discoveries and prove the others deductively from propositions that we have previously proved or assumed.

If two triangles agree in all their six parts, they are *congruent*. Congruent means *agreeing*. The symbol for congruence is $\cong$, and it is read "congruent" or "is congruent to."

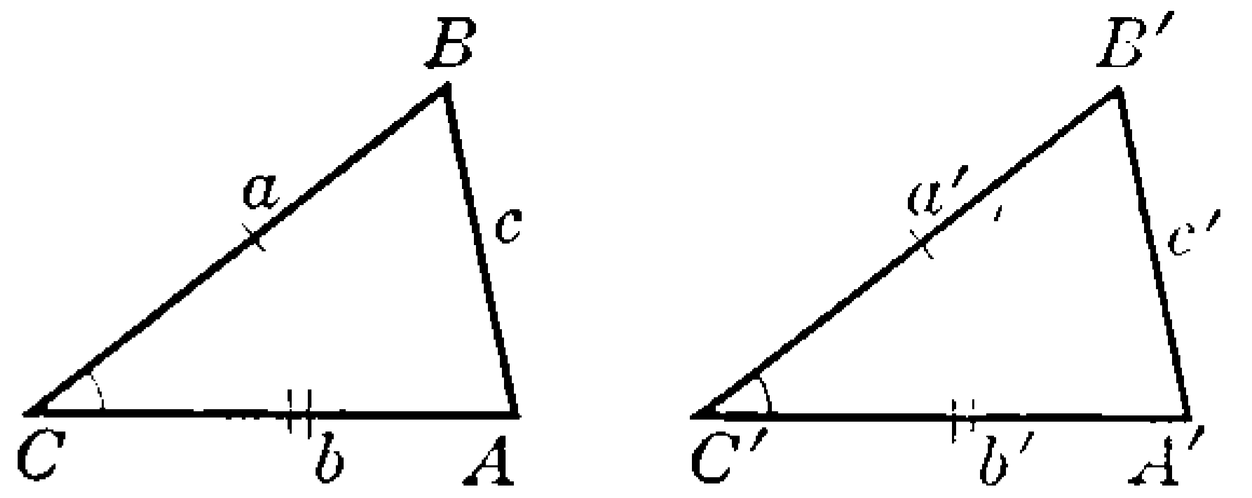
111. Definition. Congruent triangles are triangles that have all the parts of one equal respectively to the corresponding parts of the other.

Consecutive Parts of a Triangle

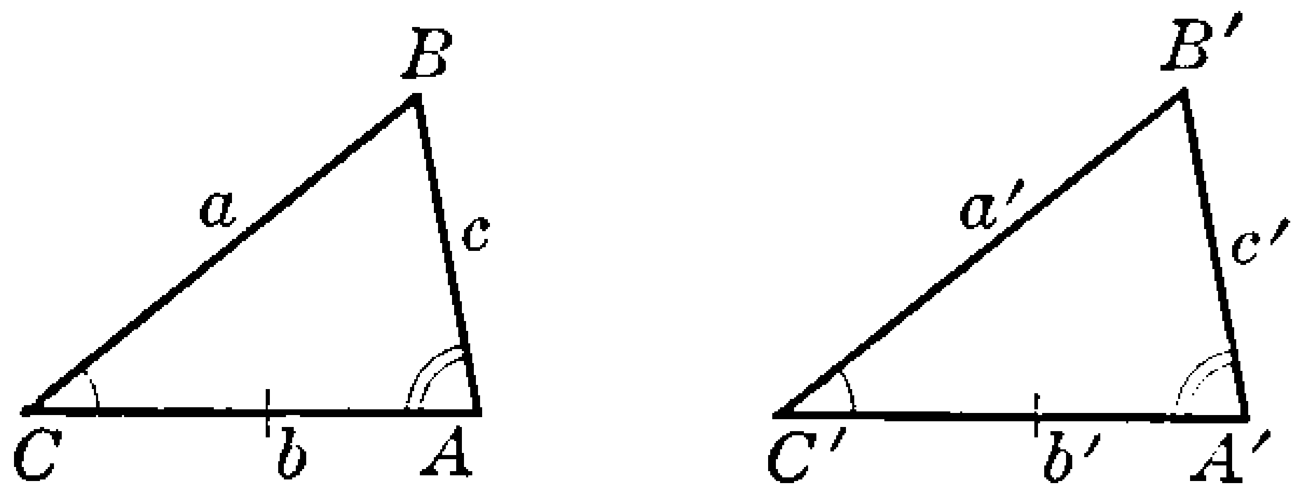
112. In a triangle such as ABC, the angle A and side AB are consecutive parts. In our experimental work, we saw that if we copy three consecutive parts of a triangle, there is only one way to complete the new triangle. Thus, if we copy sides a and b and the included angle C, then we have only one way to draw the third side. We may reason that this is true because the two-point postulate (§ 11) assumes that only one line can be drawn through the two points A and B. However, we assume without further proof that three consecutive parts determine a triangle. We state this as two separate postulates.



113. Congruence Postulate I. If two sides and the included angle of one triangle are equal respectively to two sides and the included angle of another triangle, the triangles are congruent. (This postulate is often abbreviated as s.a.s., meaning "side, angle, side.")



114. Congruence Postulate II. If two angles and the included side of one triangle are equal respectively to two angles and the included side of another triangle, the triangles are congruent. (This postulate is often abbreviated as a.s.a., meaning "angle, side, angle.")

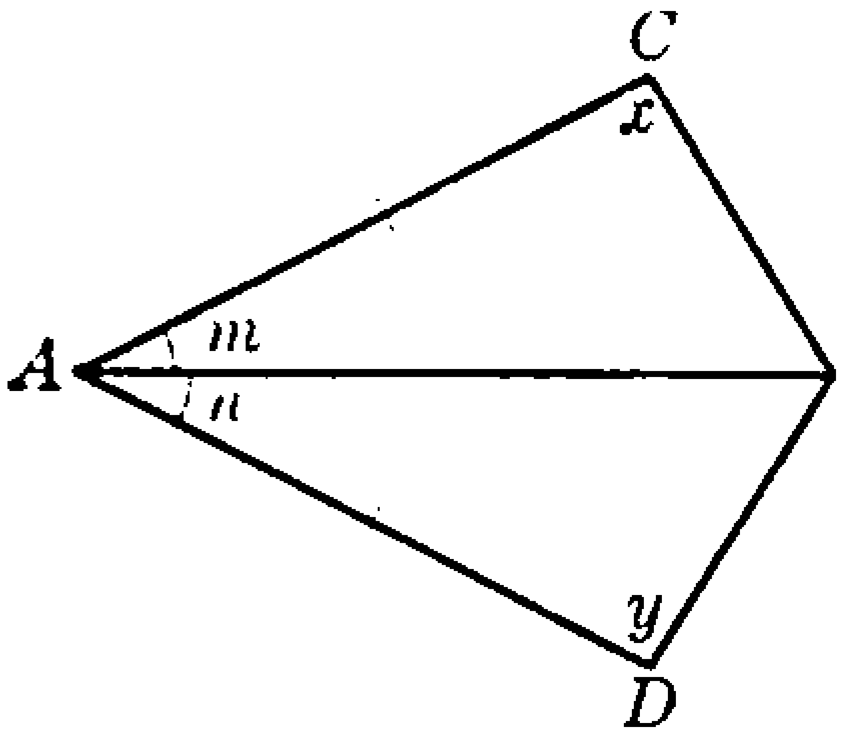


As you have already learned, in the deductive method we argue from a general statement to a particular case. You will now see this method illustrated again.

To prove two line segments or two angles equal, we shall first show that they are corresponding parts of two triangles that agree in three consecutive parts. We have already assumed that such triangles are congruent. Therefore, the parts in which we are interested are also equal—as a particular case.

Since a definition is reversible, our definition of congruent triangles states: (1) If two triangles agree in their corresponding parts, they are congruent; and (2) corresponding parts of congruent triangles are equal. The last step of the example that follows uses the second part of the definition.

Illustrative Example



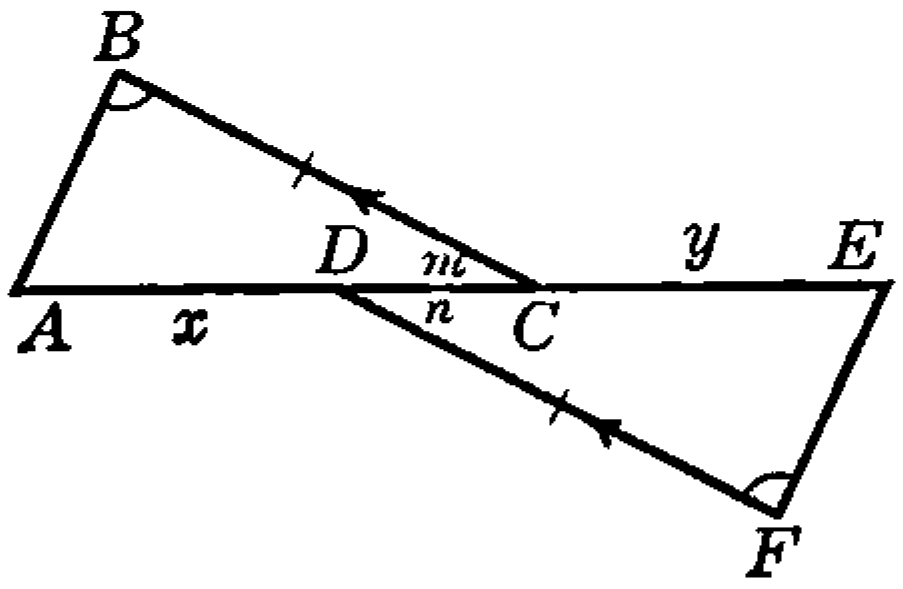
Given:
$AC = AD$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

PROOF

Statements	*Reasons*
1. $AC = AD$	1. Given.
2. $AB = AB$	2. Identity.
3. $\angle m = \angle n$	3. Given.
4. $\triangle ABC \cong \triangle ABD$	4. s.a.s.
5. $\angle x = \angle y$	5. Def. $\cong$ $\triangle$s.

Explain Statement 4 as follows: "If two sides and the included angle" (indicate the sides AC and AB and the included $\angle m$) "of one triangle" (trace $\triangle ABC$) "are equal respectively to two sides and the included angle" (indicate the sides AD and AB and the included $\angle n$) "of another triangle" (trace $\triangle ABD$), "then the triangles" (again trace $\triangle ABC$ and $\triangle ABD$) "are congruent."

Illustrative Example



Given:
$BC = DF$
$BC \parallel DF$
$\angle B = \angle F$
Prove:
$AD = CE$

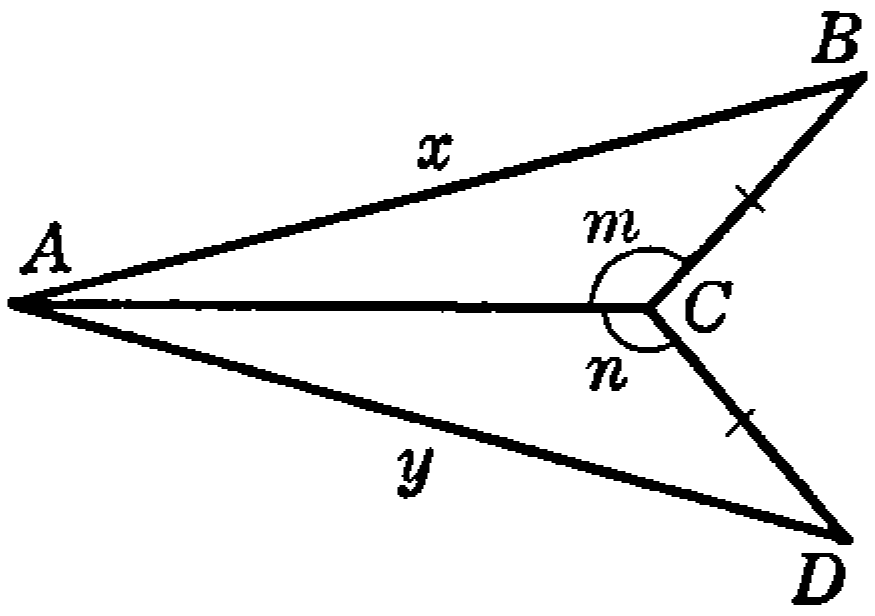
PROOF

Statements	*Reasons*
1. $BC \parallel DF$	1. Given.
2. $\angle m = \angle n$	2. $\parallel$s, alt. int. $\angle$s are =.
3. $\angle B = \angle F$	3. Given.
4. $BC = DF$	4. Given.
5. $\triangle ABC \cong \triangle DEF$	5. a.s.a.
6. $AC = DE$	6. Def. $\cong$ $\triangle$s.
7. $DC = DC$	7. Identity.
8. $AD = CE$	8. Subt. Ax.

Explain Statement 5 as follows: "If two angles and the included side" (point to $\angle B$, $\angle m$, and side BC) "of one triangle" (trace $\triangle ABC$) "are equal respectively to two angles and the included side" (point to $\angle F$, $\angle n$, and side DF) "of another triangle" (trace $\triangle DEF$), "then the triangles" (again trace $\triangle ABC$ and $\triangle DEF$) "are congruent."

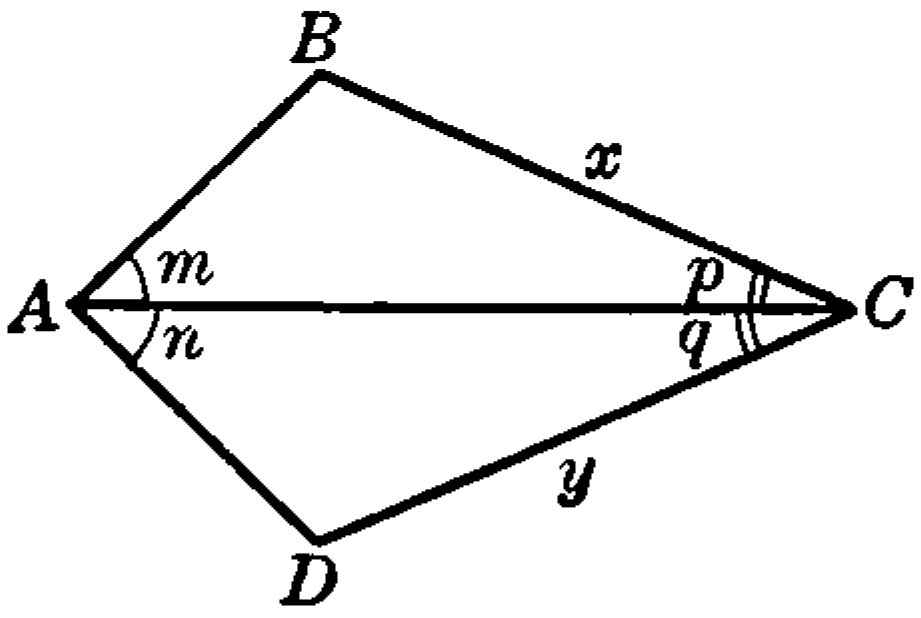
EXERCISES

1.



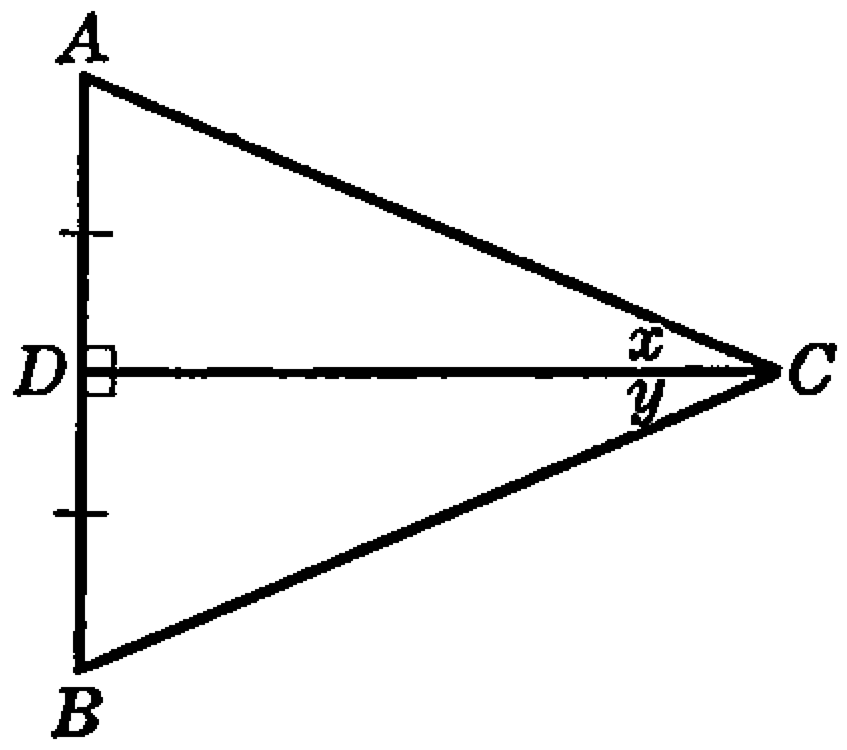
Given:
$BC = CD$
$\angle m = \angle n$
Prove:
$x = y$

2.



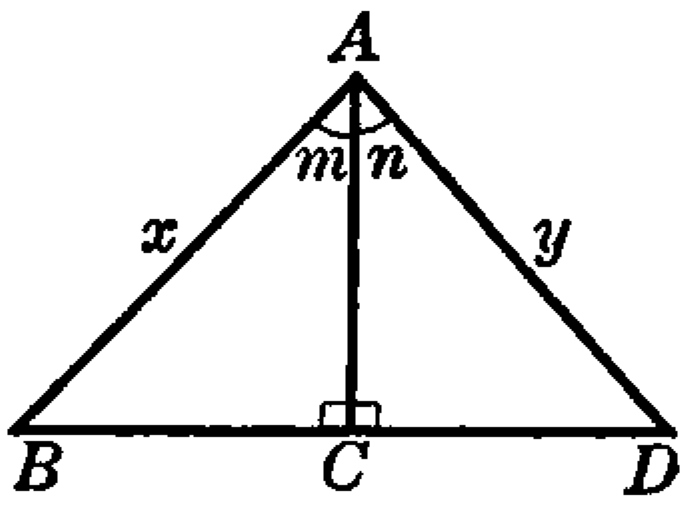
Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x = y$

3.



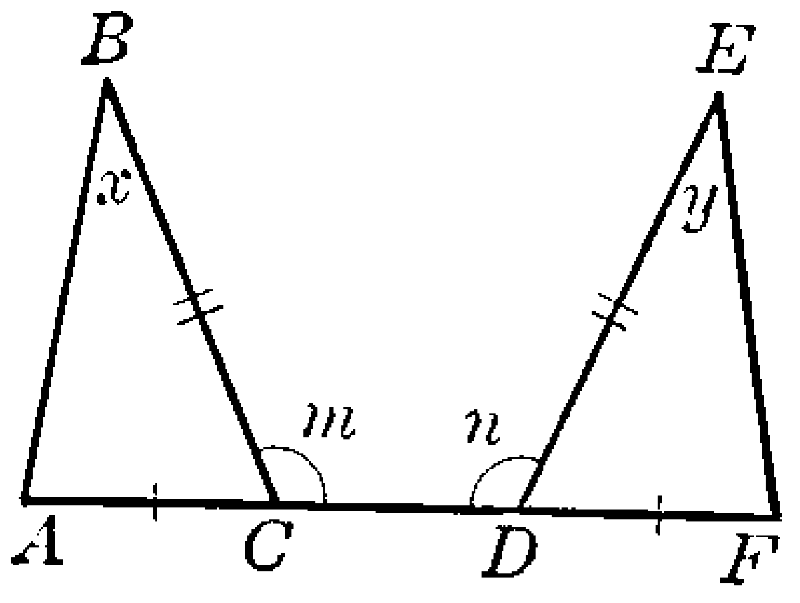
Given:
$AD = DB$
$AB \perp CD$
Prove:
$\angle x = \angle y$

4.



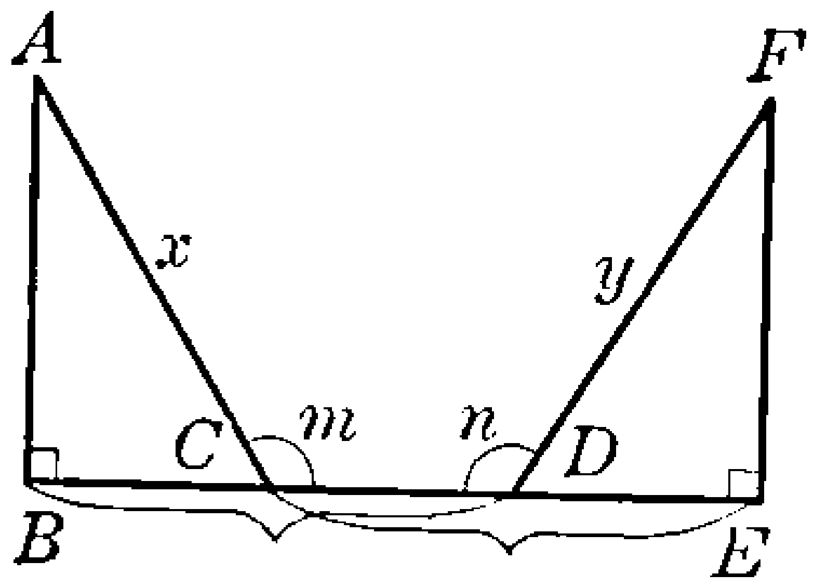
Given:
$\angle m = \angle n$
$AC \perp BD$
Prove:
$x = y$

5.



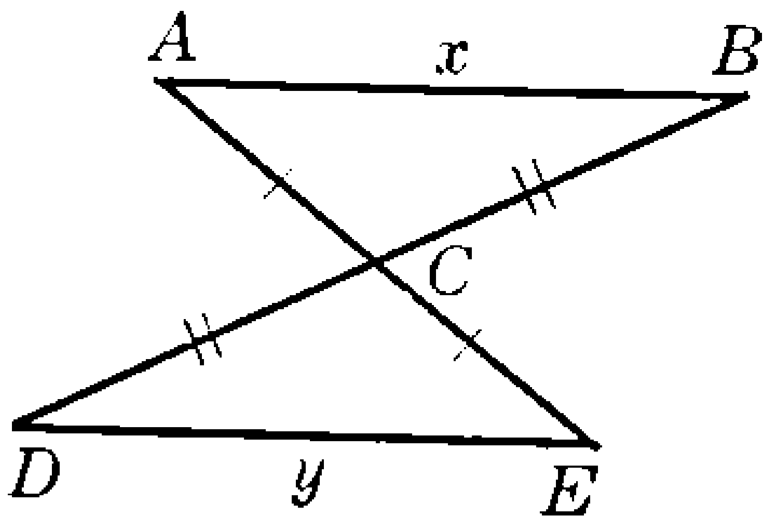
Given:
$AC = DF$
$BC = ED$
$\angle m = \angle n$
AF st. line
Prove:
$\angle x = \angle y$

6.



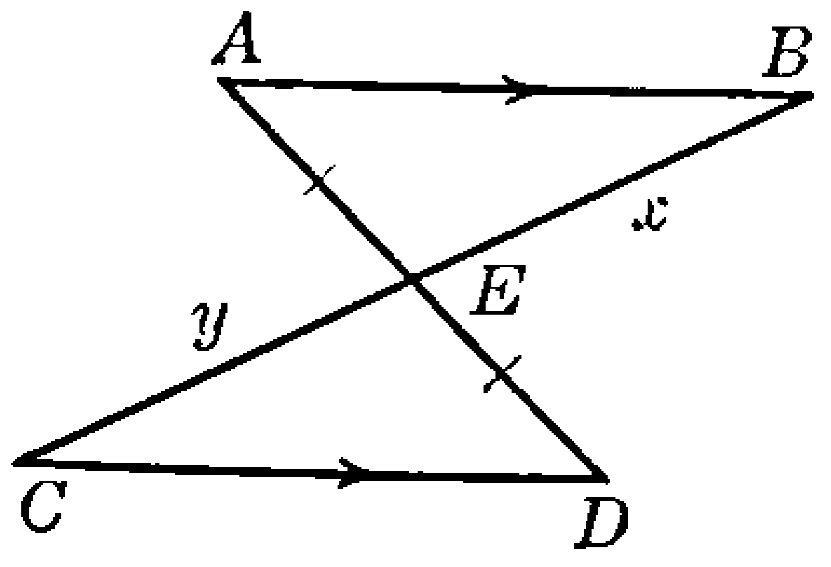
Given:
$AB \perp BE$
$FE \perp BE$
$\angle m = \angle n$
$BD = CE$
BE st. line
Prove:
$x = y$

7.



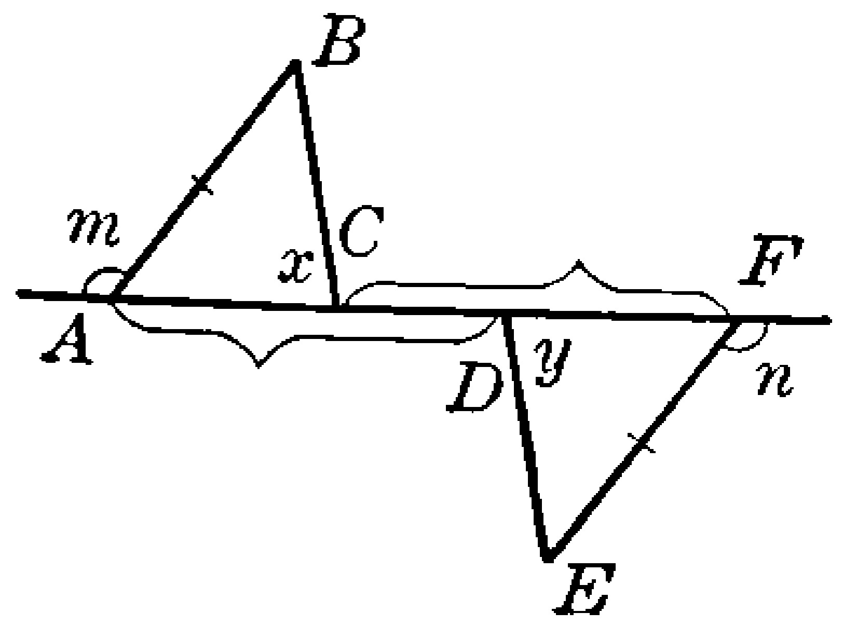
Given:
$AC = CE$
$BC = CD$
Prove:
$x = y$

8.



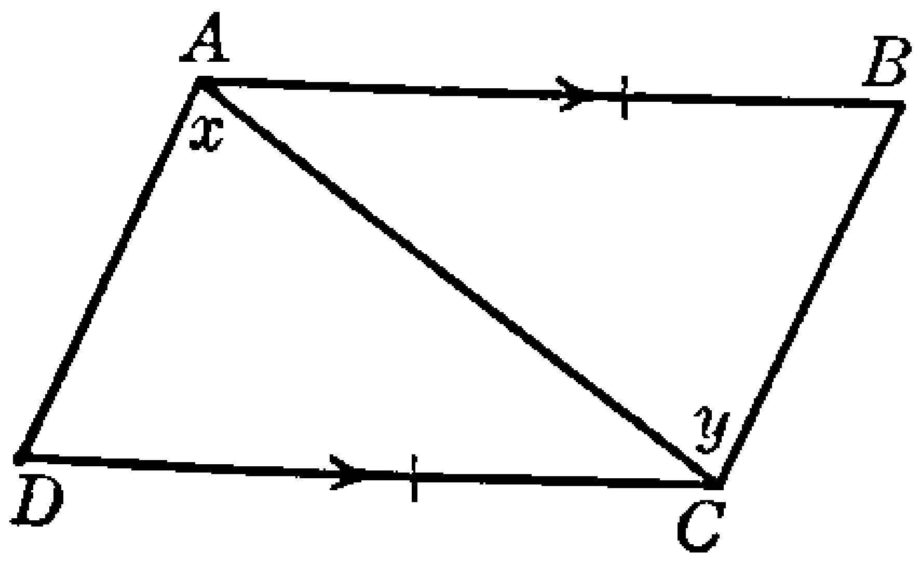
Given:
$AB \parallel CD$
$AE = ED$
Prove:
$x = y$

9.



Given:
$AB = EF$
$AD = CF$
$\angle m = \angle n$
AF st. line
Prove:
$\angle x = \angle y$

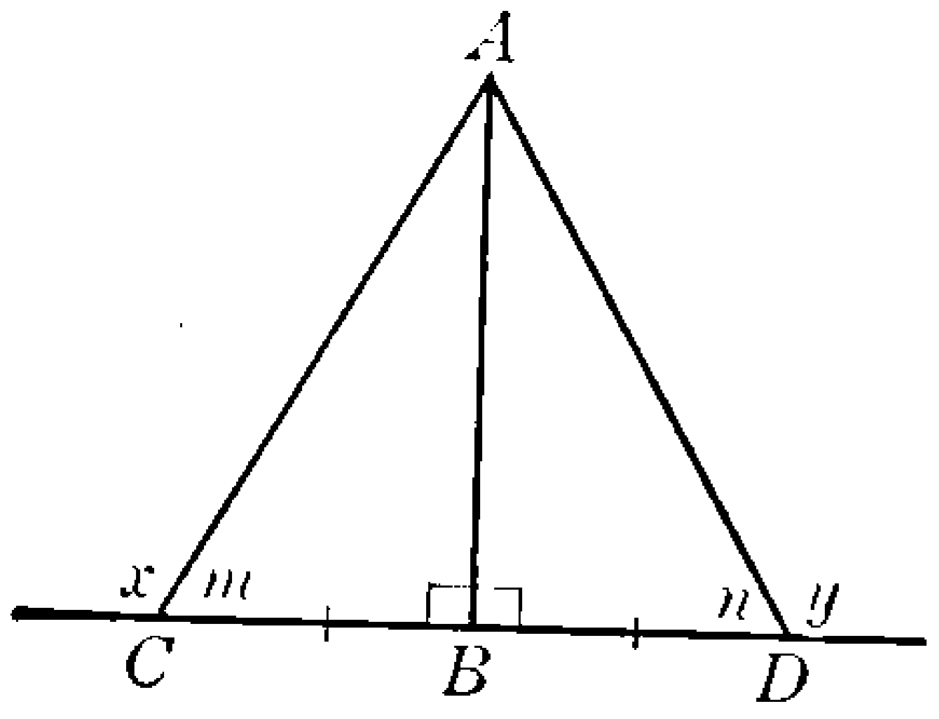
10.



Given:
$AB = DC$
$AB \parallel DC$
Prove:
$\angle x = \angle y$

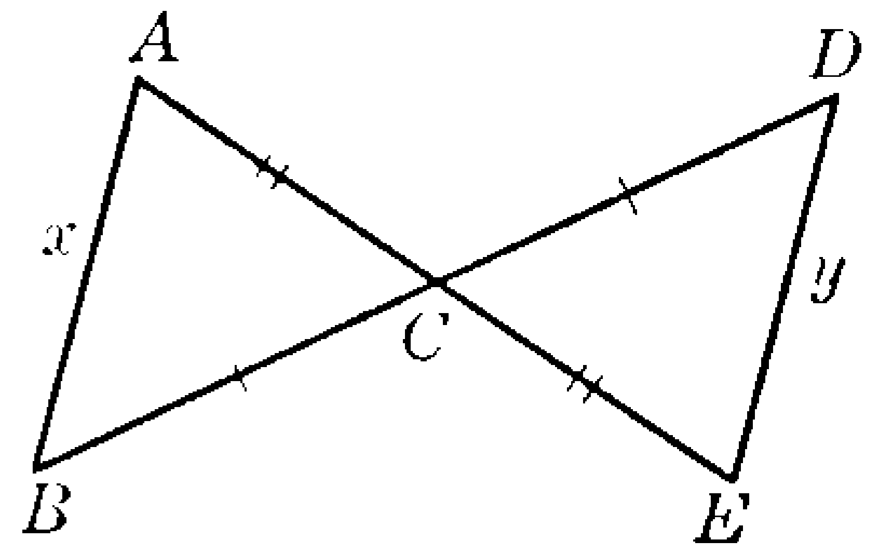
SUPERIOR WORK

S1.



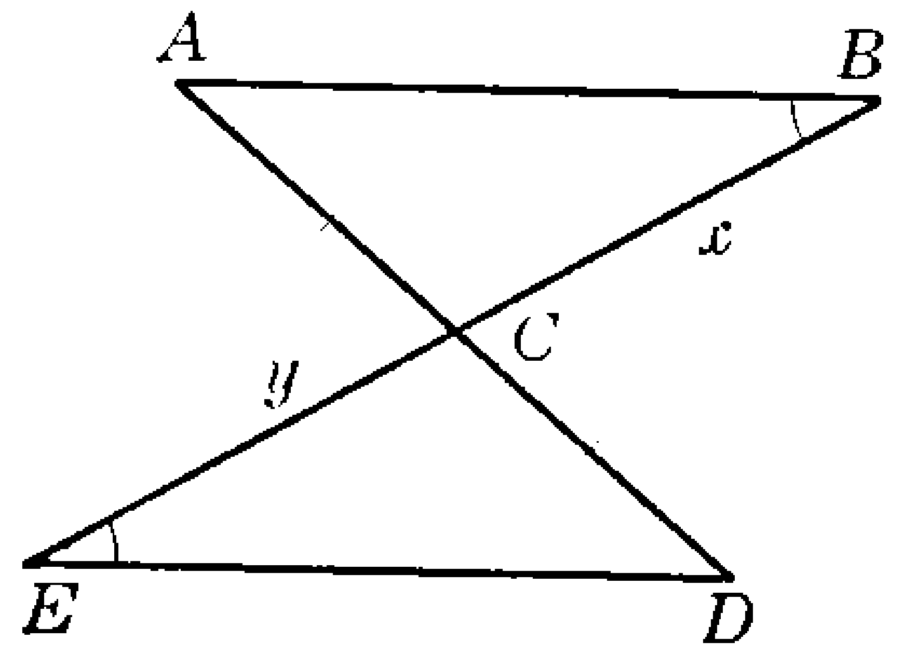
Given:
$AB \perp CD$
$CB = BD$
Prove:
$\angle x = \angle y$
Hint:
First prove
$\angle m = \angle n$.

S2.



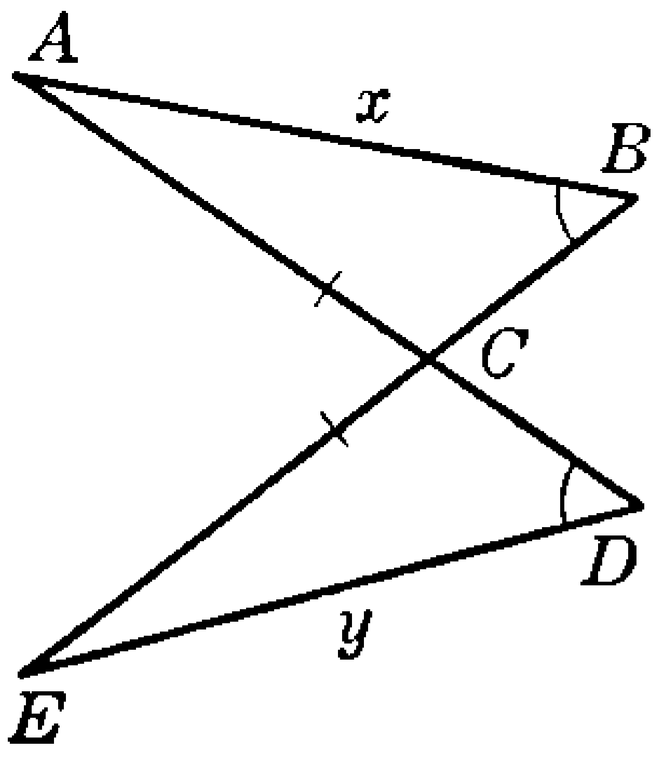
Given:
$BC = CD$
$AC = CE$
Prove:
$x \parallel y$

S3.



Given:
$AC = CD$
$\angle B = \angle E$
Prove:
$x = y$

S4.



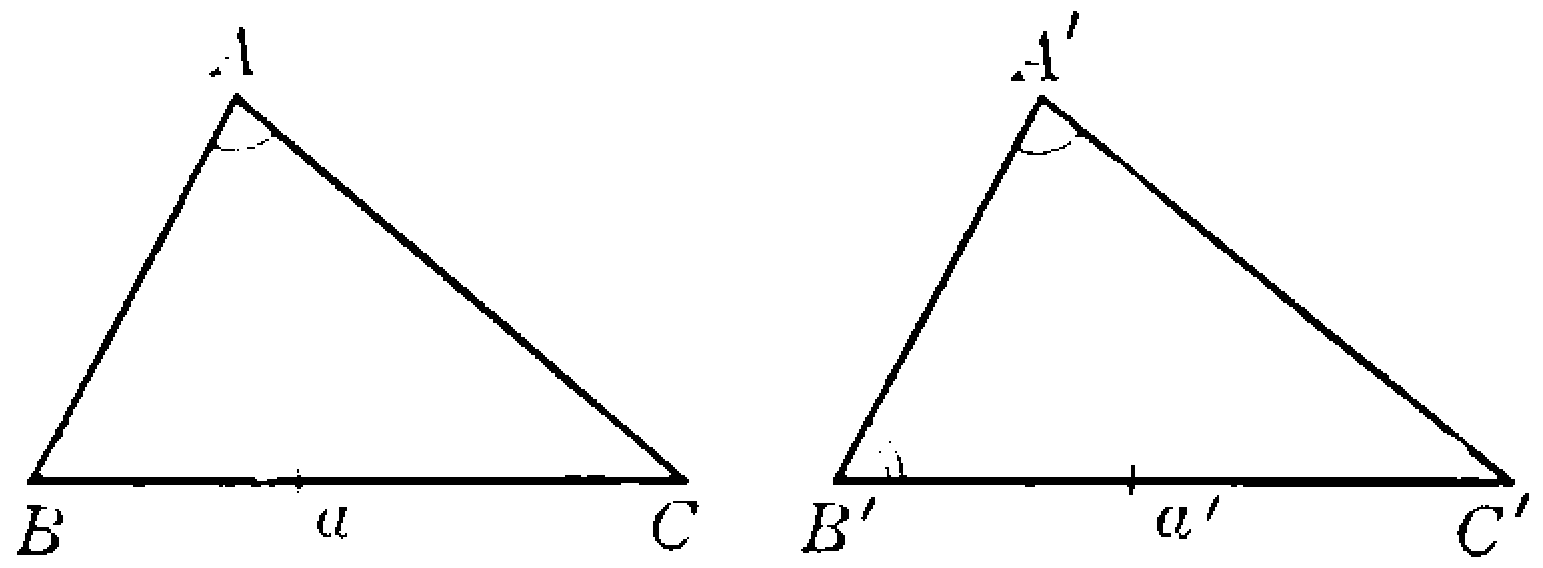
Given:
$AC = CE$
$\angle B = \angle D$
Prove:
$x = y$

Two Angles and a Nonincluded Side

115. If two triangles agree in two angles and a nonincluded side, these parts are not consecutive. But we have proved (see § 65) that if two triangles agree in two angles, they agree in the remaining angle. In this way we can show that the triangles agree in three consecutive parts and are therefore congruent.

---

116. THEOREM. If two angles and a nonincluded side of one triangle are equal respectively to the corresponding parts of another triangle, the triangles are congruent. (This theorem is often abbreviated as s.a.a.)

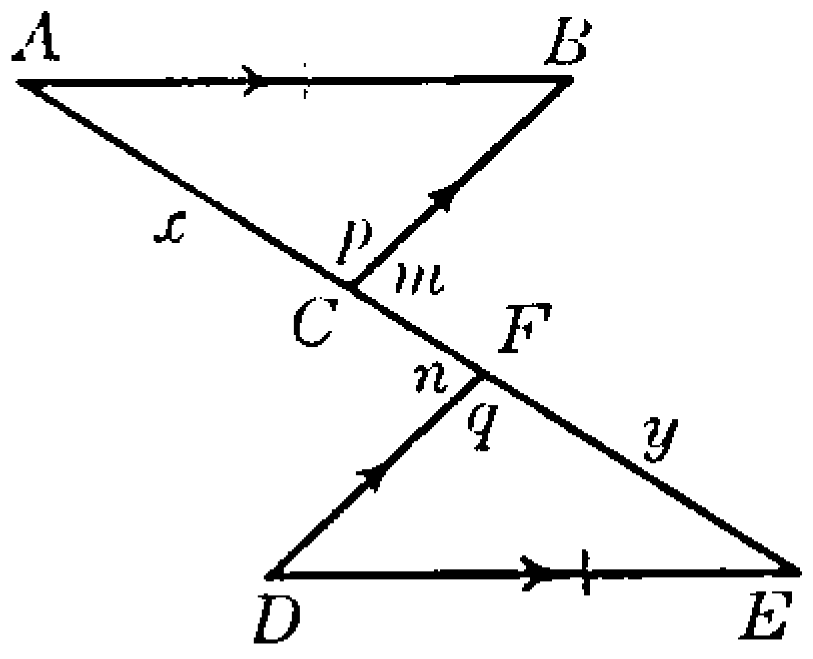


Given: In $\triangle ABC$ and $\triangle A'B'C'$
$\angle A = \angle A'$
$\angle B = \angle B'$
$a = a'$
Prove: $\triangle ABC \cong \triangle A'B'C'$

PROOF

Statements	*Reason*
1. $\angle A = \angle A'$ $\angle B = \angle B'$	1. Given.
2. $\angle C = \angle C'$	2. If 2 ∡ of one △ = respectively 2 ∡ of another △, remaining ∡ are = (§ 65).
3. $a = a'$	3. Given.
4. $\triangle ABC \cong \triangle A'B'C'$	4. a.s.a.

Illustrative Example



Given:
$AB = DE$
$AB \parallel DE$
$BC \parallel FD$
AE st. line
Prove:
$AF = CE$

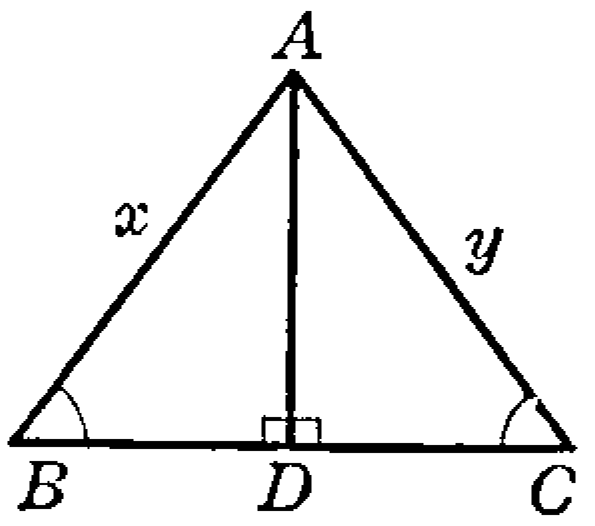
PROOF

Statements	*Reasons*
1. $AB \parallel DE$	1. Given.
2. $\angle A = \angle E$	2. $\parallel_s$, alt. int. ∡ are =.
3. $BC \parallel FD$	3. Given.
4. $\angle m = \angle n$	4. Same as 2.
5. $\angle p = \angle q$	5. Sups. of = ∡ are =.
6. $AB = DE$	6. Given.
7. $\triangle ABC \cong \triangle DEF$	7. s.a.a.
8. $AC = FE$	8. Def. ≅ ∡.
9. $CF = CF$	9. Identity.
10. $AF = CE$	10. Add. Ax.

Explain statement 7 as follows: "If two angles and a nonincluded side" (point to $\angle A$, $\angle p$, and side AB) "of one triangle" (trace $\triangle ABC$) "are equal respectively to the corresponding parts" (point out $\angle E$, $\angle q$, and side DE) "of another triangle" (trace $\triangle DEF$), "then the triangles" (again trace $\triangle ABC$ and $\triangle DEF$) "are congruent."

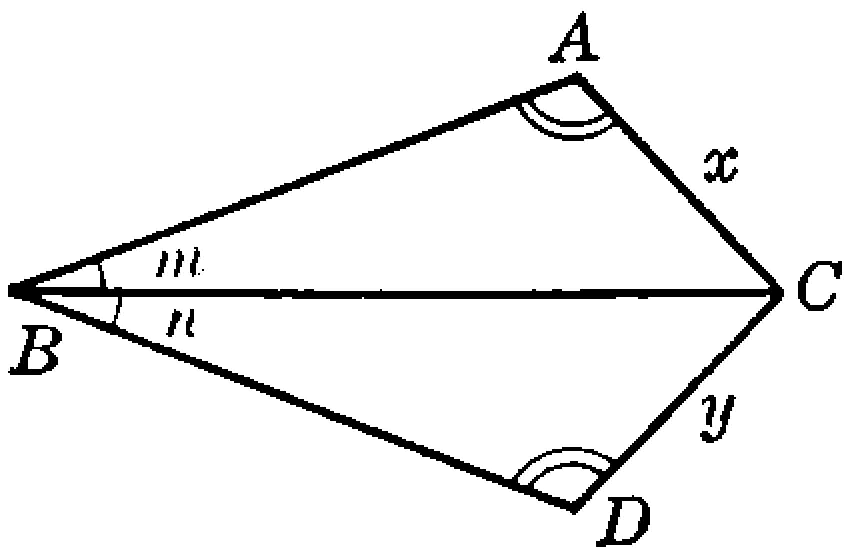
EXERCISES

1.



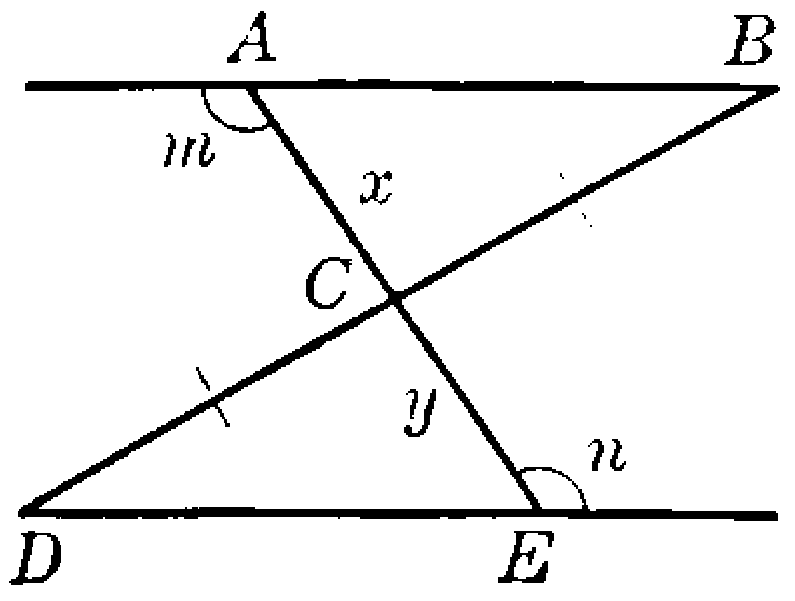
Given:
$AD \perp BC$
$\angle B = \angle C$
Prove:
$x = y$

2.



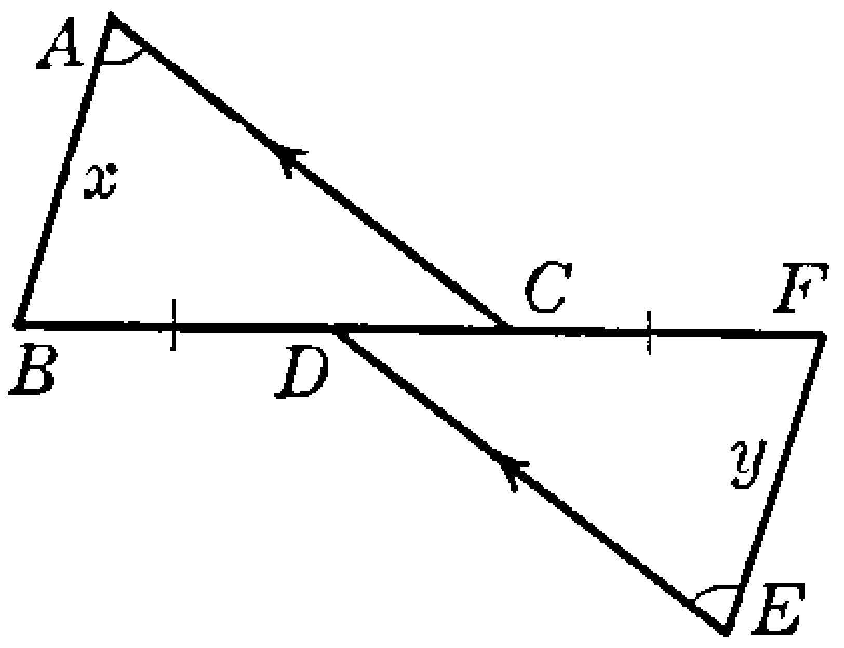
Given:
$\angle A = \angle D$
$\angle m = \angle n$
Prove:
$x = y$

3.



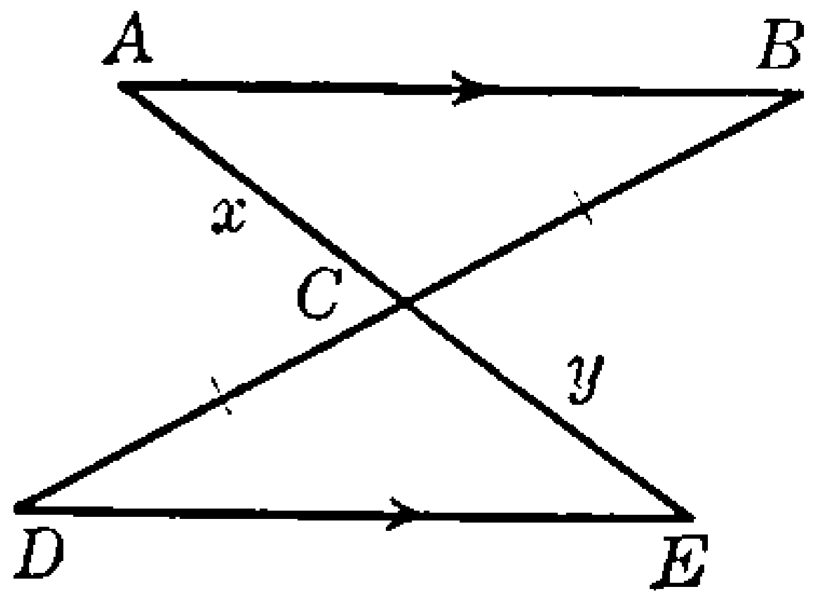
Given:
$BC = CD$
$\angle m = \angle n$
Prove:
$x = y$

4.



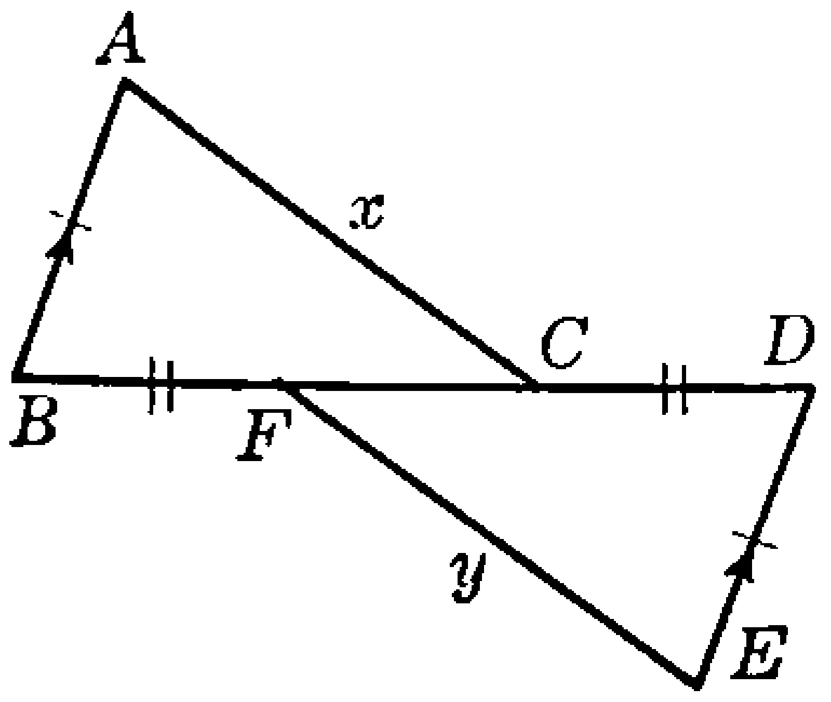
Given:
$BD = CF$
$\angle A = \angle E$
$AC \parallel DE$
BF st. line
Prove:
$x = y$

5.



Given:
$AB \parallel DE$
$BC = CD$
Prove:
$x = y$

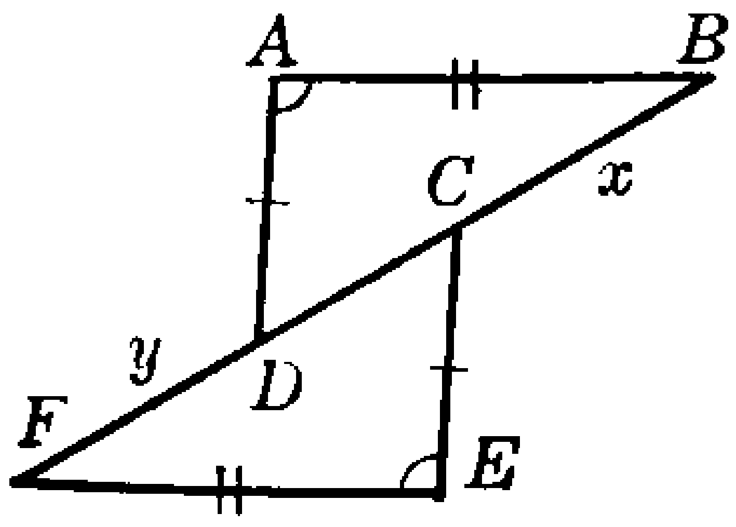
6.



Given:
$AB \parallel DE$
$AB = DE$
$BF = CD$
BD st. line
Prove:
$x \parallel y$

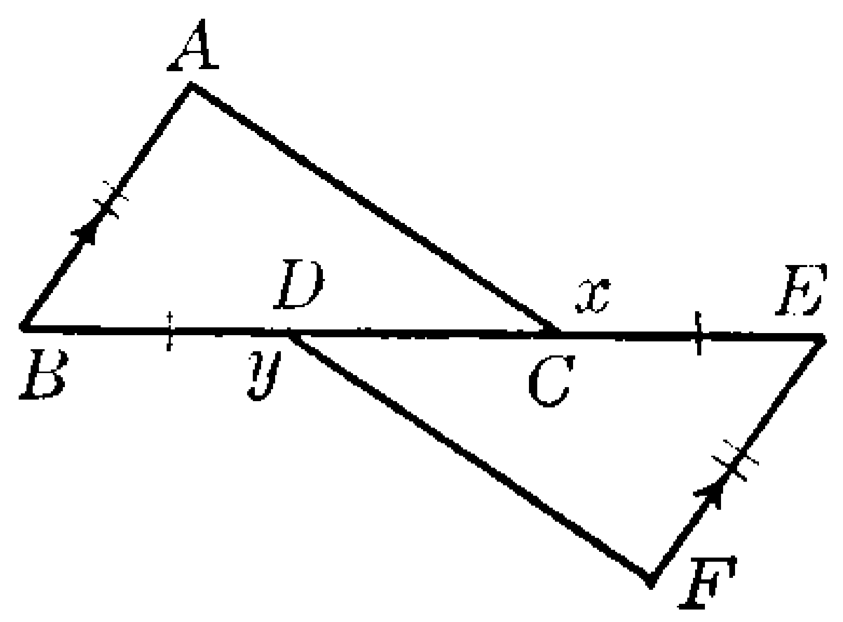
SUPERIOR WORK

S1.



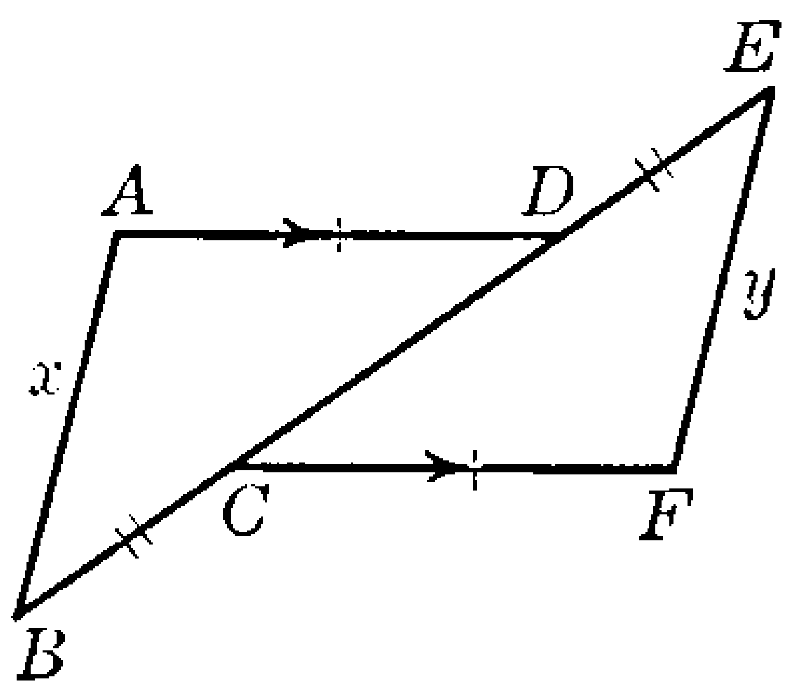
Given:
$AD = CE$
$AB = EF$
$\angle A = \angle E$
Prove:
$FD = CB$

S2.



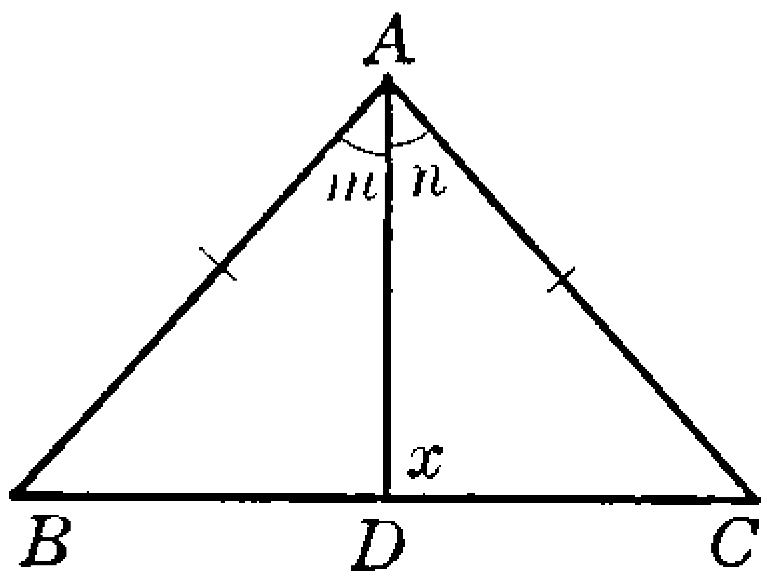
Given:
$BD = CE$
$AB = EF$
$AB \parallel EF$
Prove:
$\angle x = \angle y$

S3.



Given:
$ED = BC$
$AD = CF$
$AD \parallel CF$
Prove:
$x \parallel y$

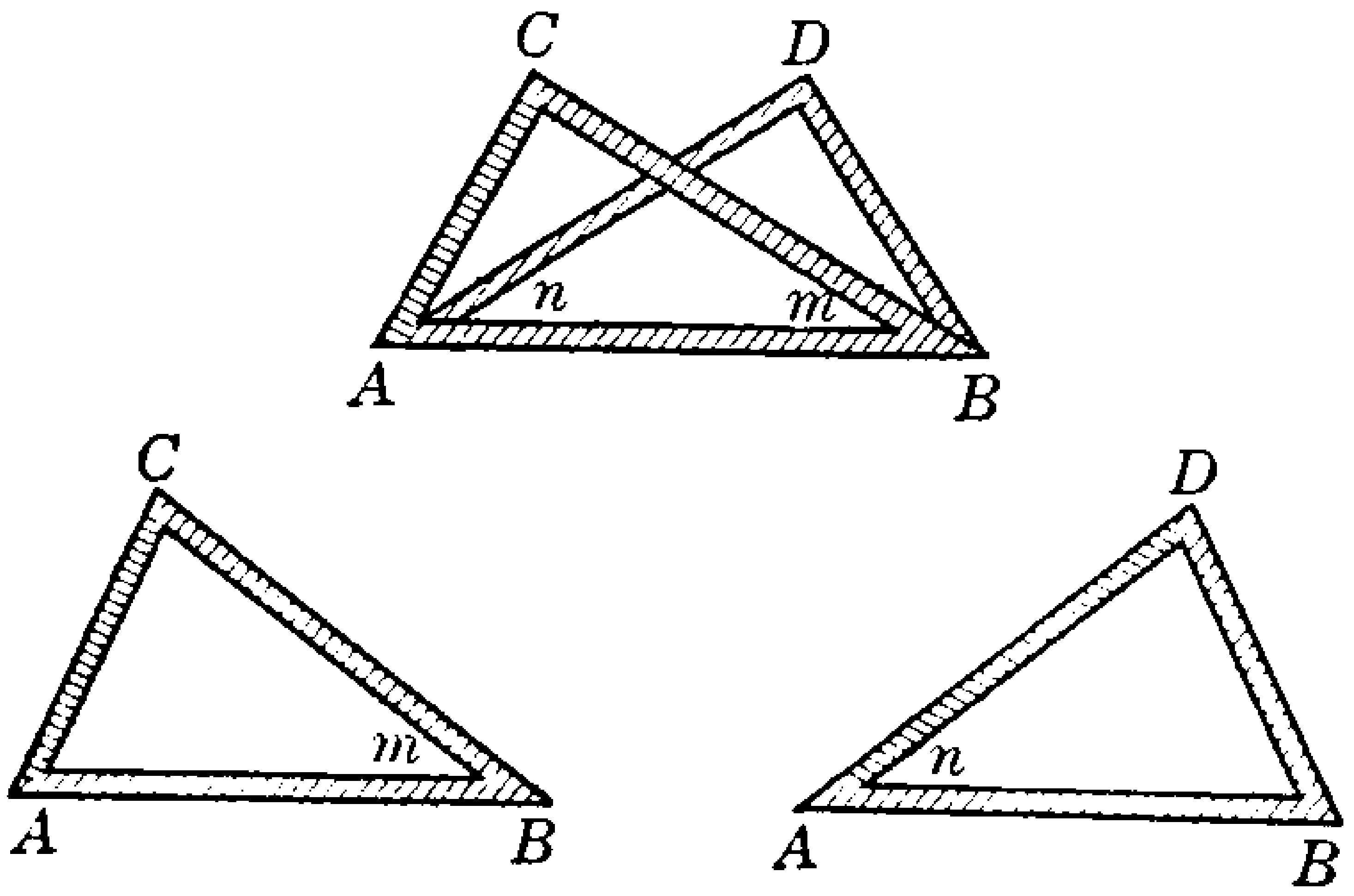
S4.



Given:
$AB = AC$
$\angle m = \angle n$
Prove:
$AD \perp BC$

Overlapping Triangles

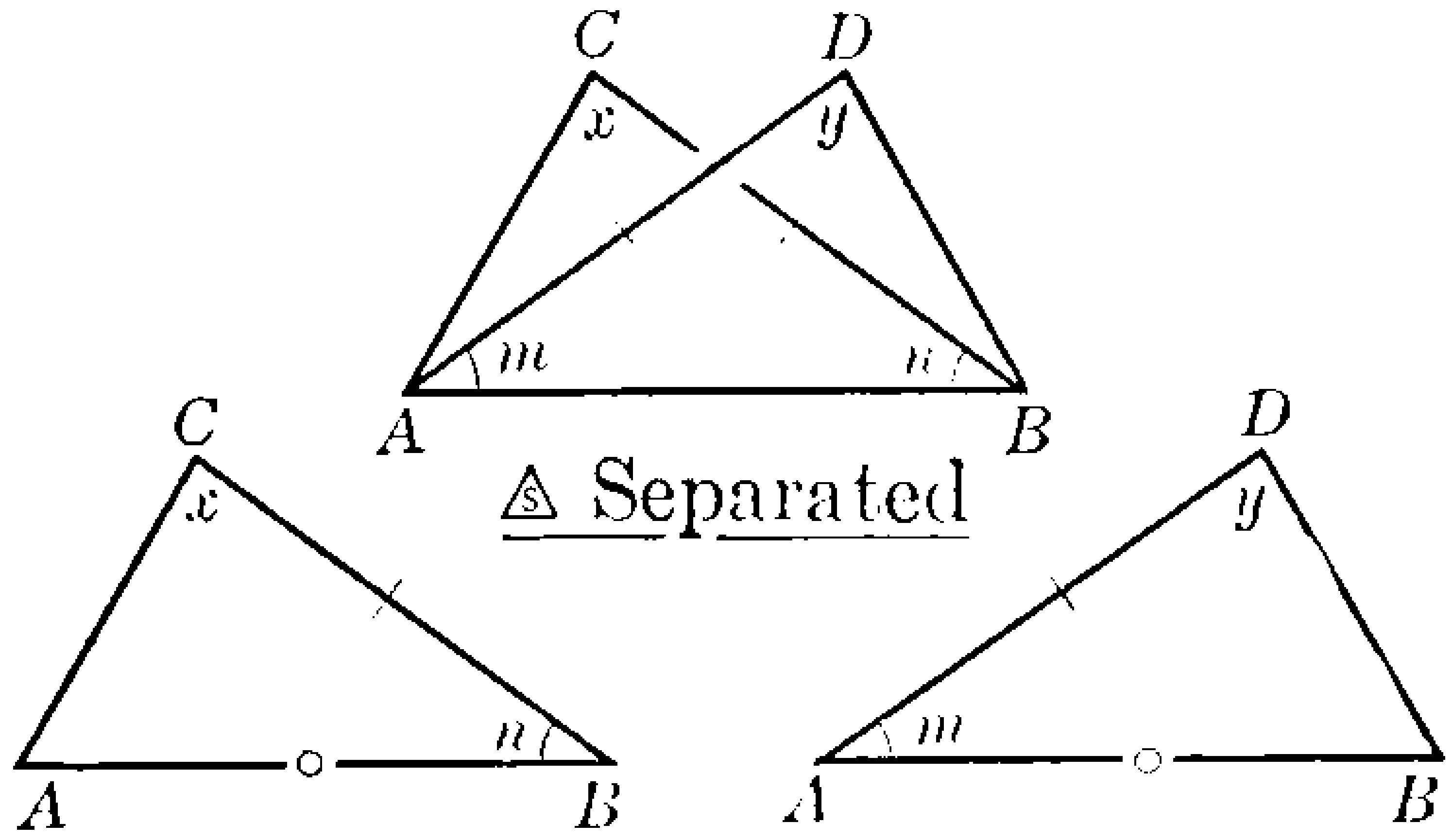
117. In some figures we must work with triangles that overlap. In such cases it is well first to draw the triangles separately. The figure shows how the overlapping triangles ABC and ABD appear when drawn separately. Do you



see that certain angles and line segments in the original figure do not appear after the triangles are separated? Notice that when the triangles are drawn separately, $\angle CAD$ and $\angle CBD$ disappear altogether. Unless you draw the triangles separately, you may mistake these angles for parts of the overlapping triangles. You should therefore separate the triangles until you can imagine them and reason about them without error.

Illustrative Example

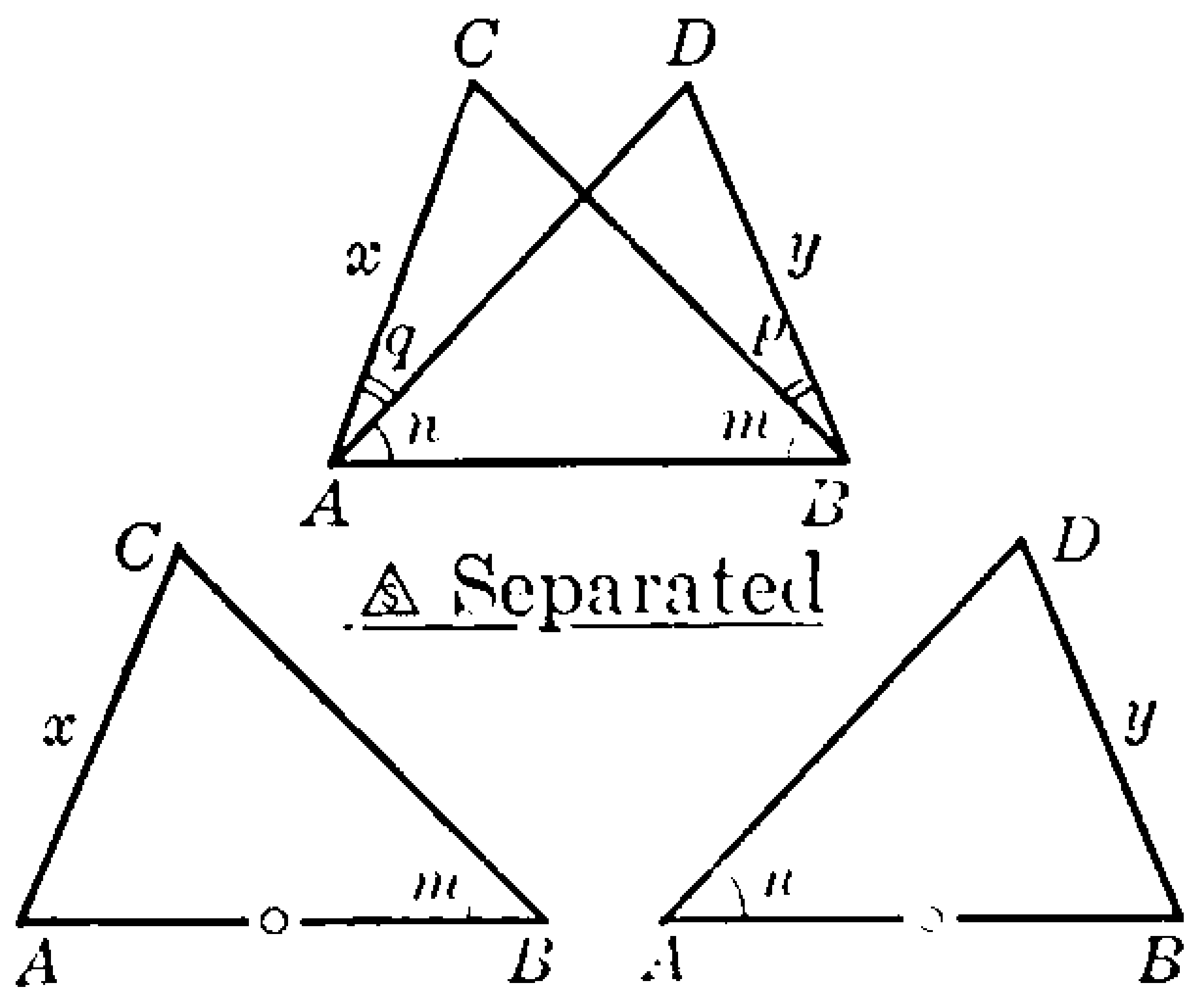
Given: $CB = DA$
$\angle m = \angle n$
Prove: $\angle x = \angle y$



PROOF

Statements	*Reasons*
1. $CB = DA$ $\angle m = \angle n$	1. Given.
2. $AB = AB$	2. Identity.
3. $\triangle ABC \cong \triangle ABD$	3. s.a.s.
4. $\angle x = \angle y$	4. Def. ≅ △.

Illustrative Example



Given: $\angle m = \angle n$
$\angle p = \angle q$
Prove: $x = y$

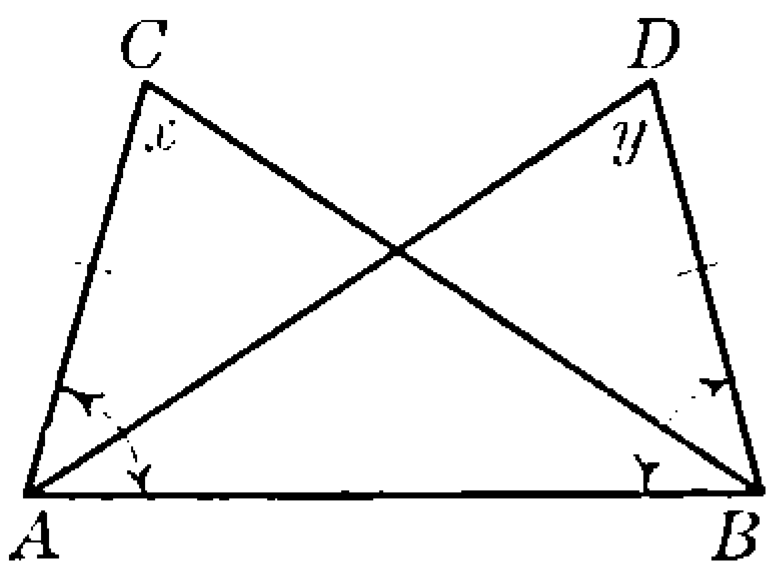
PROOF

Statements	*Reasons*
1. $\angle m = \angle n$ $\angle p = \angle q$	1. Given.
2. $\angle A = \angle B$	2. Add. Ax.
3. $AB = AB$	3. Identity.
4. $\triangle ABC \cong \triangle ABD$	4. a.s.a.
5. $x = y$	5. Def. ≅ △.

Note: Although $\angle p$ and $\angle q$ are not parts of the triangles as they are drawn separately, they did help us to prove $\angle A = \angle B$.

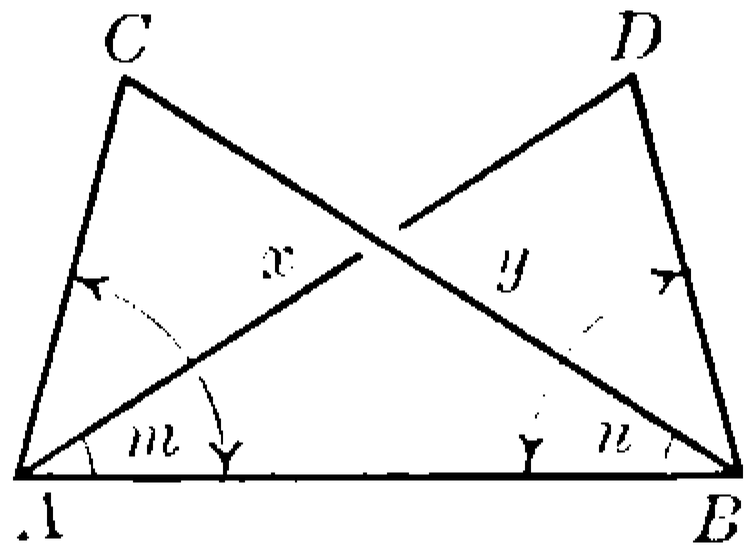
EXERCISES

1.



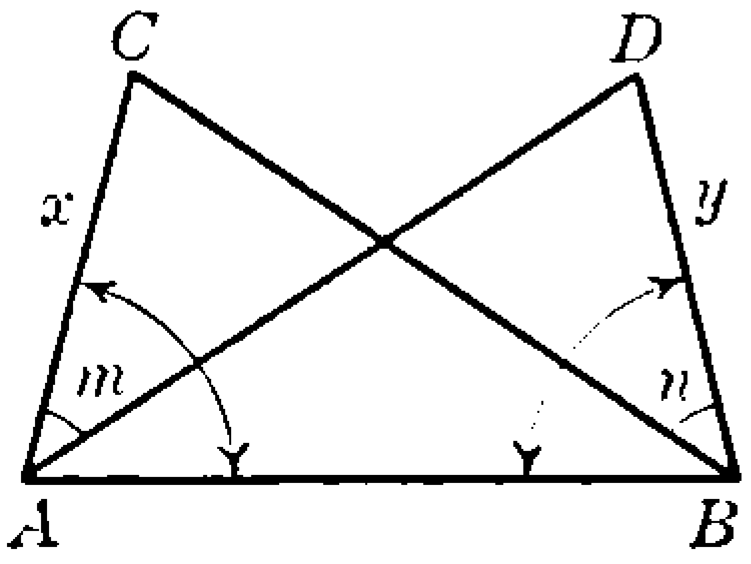
Given:
$CA = DB$
$\angle A = \angle B$
Prove:
$\angle x = \angle y$

2.



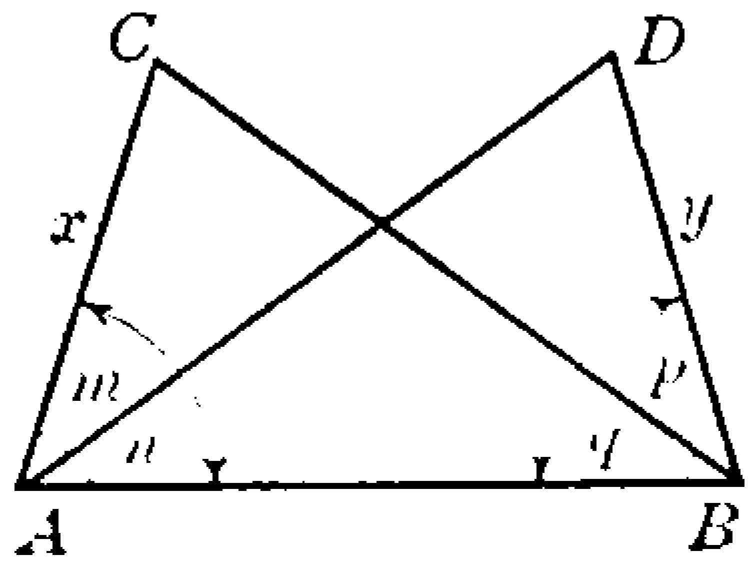
Given:
$\angle A = \angle B$
$\angle m = \angle n$
Prove:
$CB = DA$

3.



Given:
$\angle A = \angle B$
$\angle m = \angle n$
Prove:
$x = y$

4.



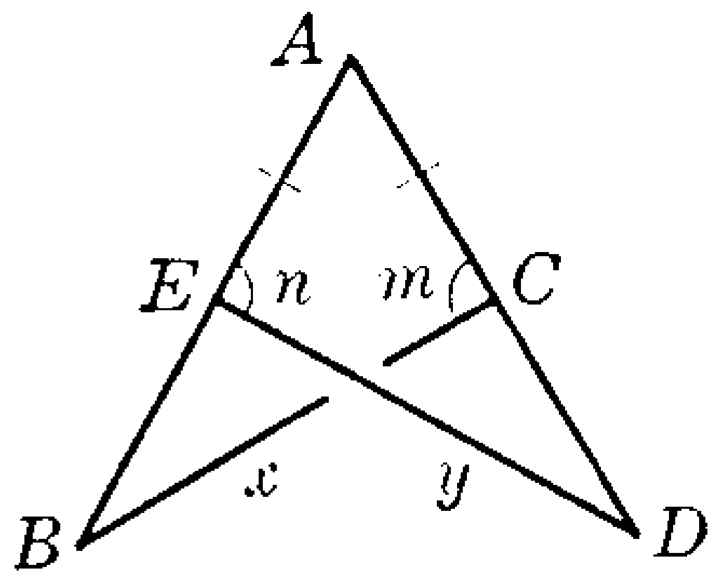
Given:
$\angle A = \angle B$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x = y$

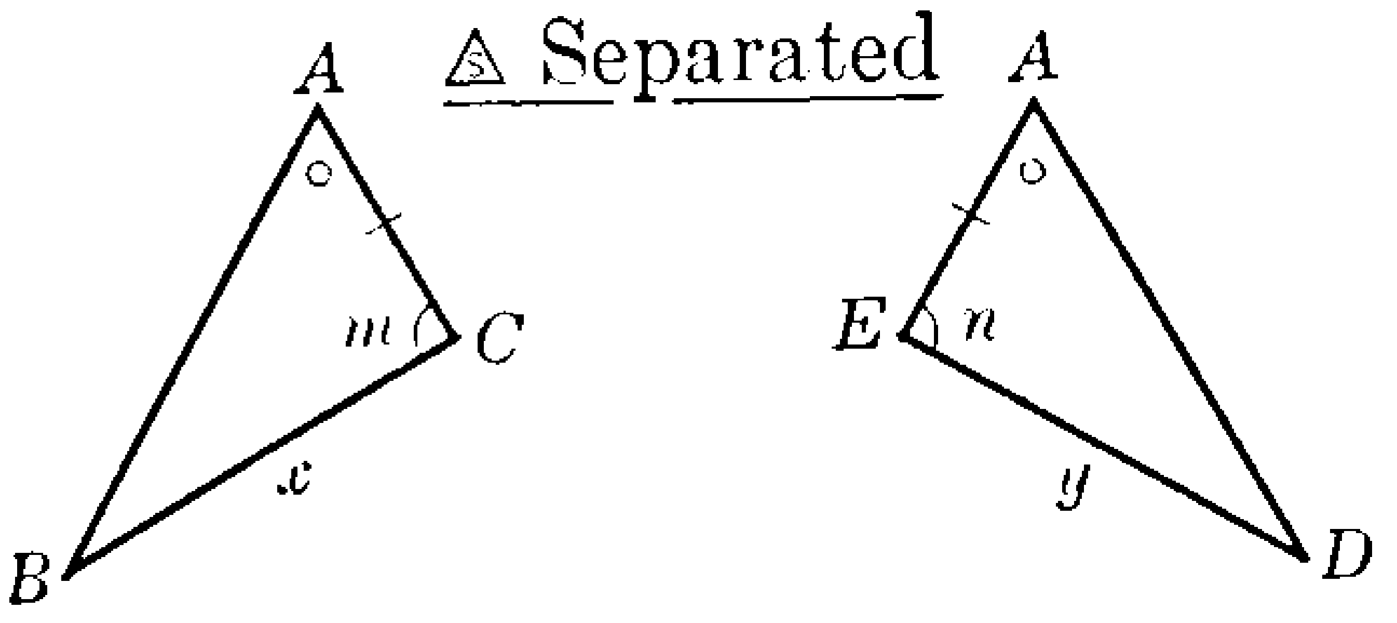
Illustrative Example

Given: $AE = AC$

$\angle m = \angle n$

Prove: $BC = ED$



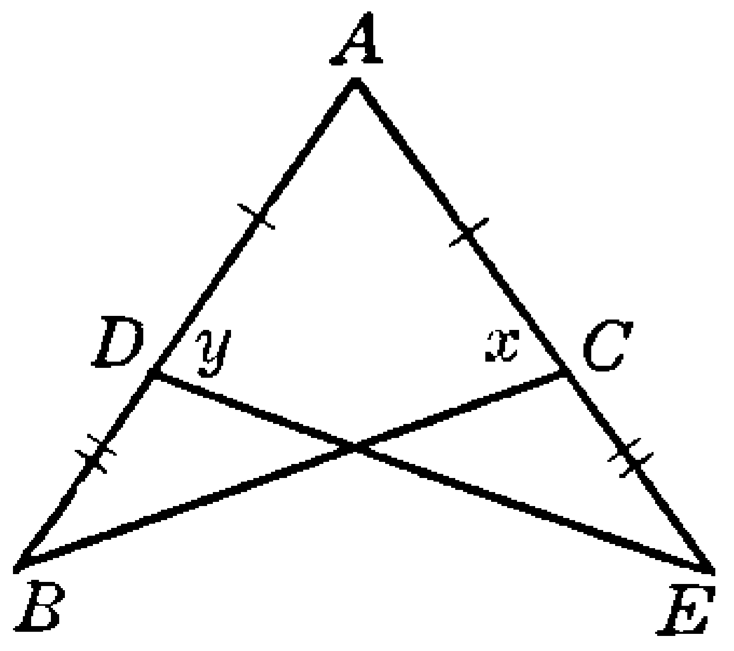


PROOF

Statements	*Reasons*
1. $AE = AC$ $\angle m = \angle n$	1. Given.
2. $\angle A = \angle A$	2. Identity.
3. $\triangle ABC \cong \triangle ADE$	3. a.s.a.
4. $x = y$	4. Def. $\cong$ △s.

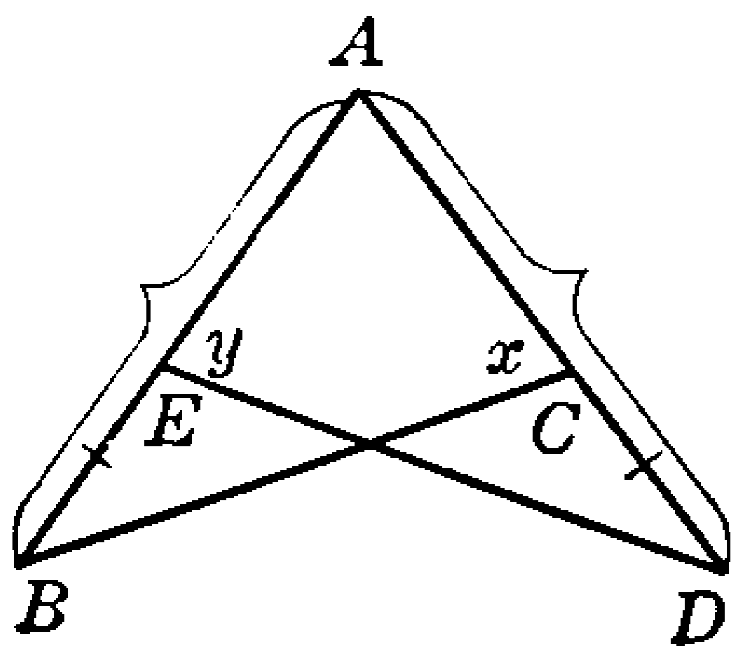
In the exercises that follow, it will not be given directly that the overlapping triangles agree in three parts. You will have to prove this first as you did in Ex. 3 and Ex. 4.

5.



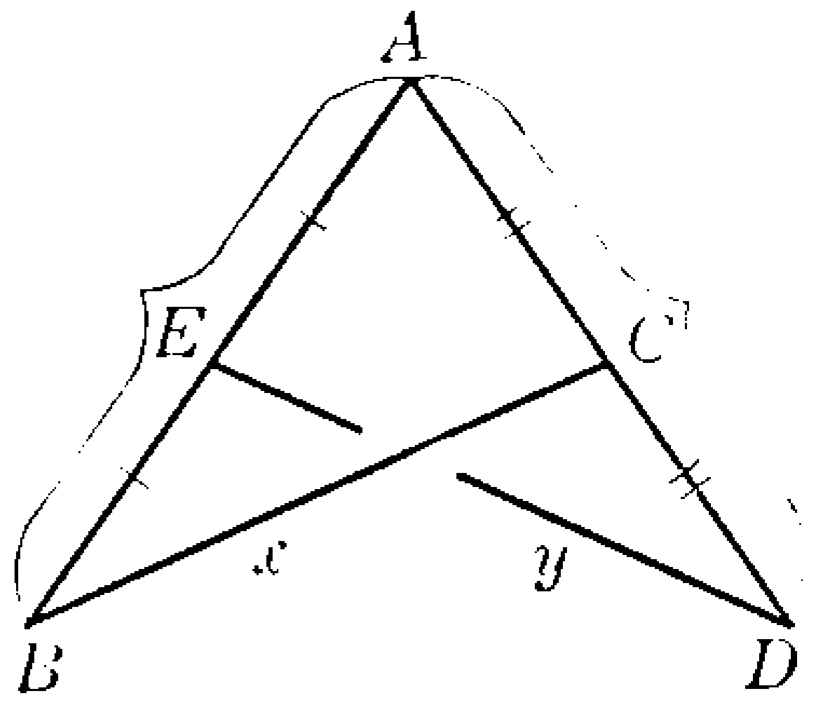
Given:
$AD = AC$
$DB = CE$
Prove:
$\angle x = \angle y$

6.



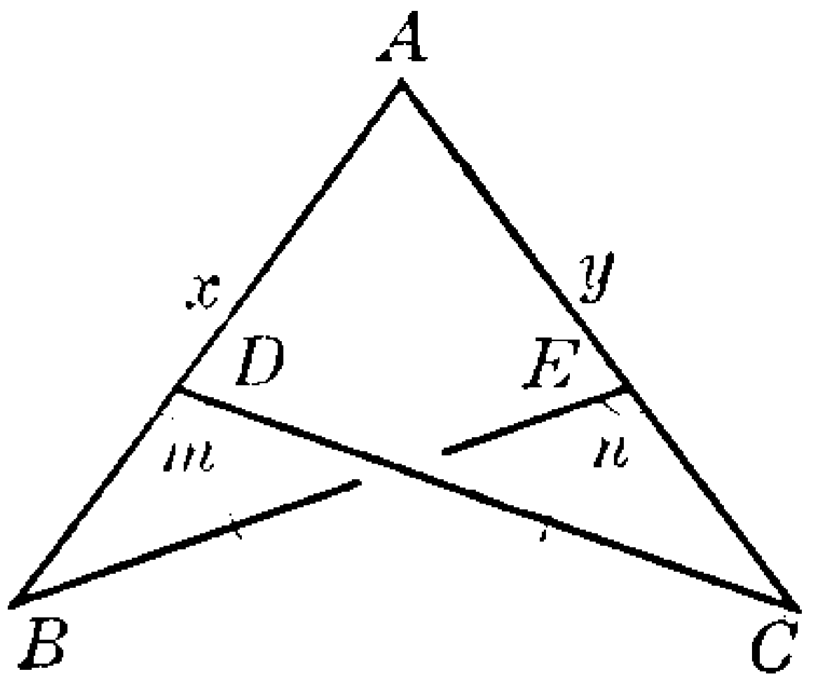
Given:
$AB = AD$
$EB = CD$
Prove:
$\angle x = \angle y$

7.



Given:
$AB = AD$
$AE = EB$
$AC = CD$
Prove:
$BC = DE$

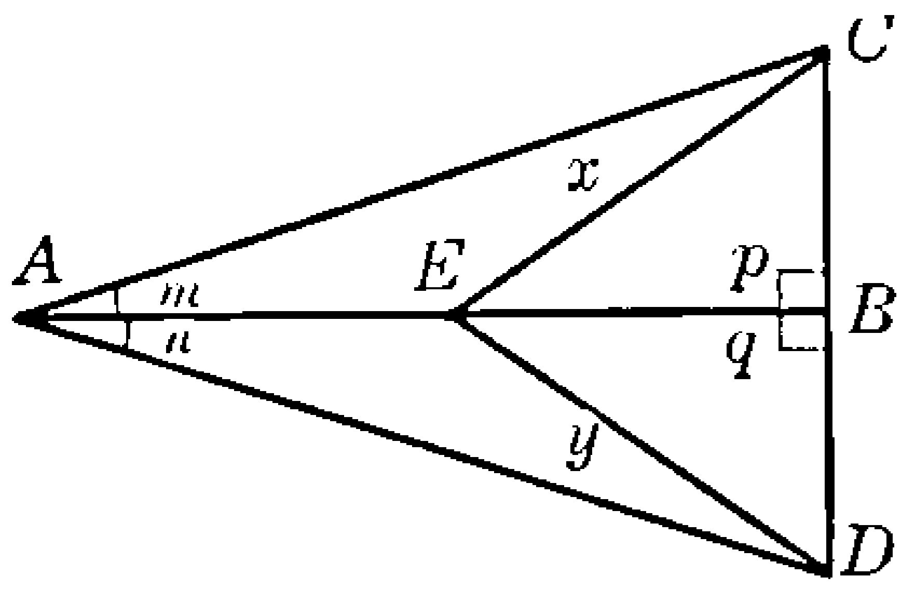
8.



Given:
$\angle m = \angle n$
$DC = BE$
Prove:
$AB = AC$

Sometimes we do not have enough information to prove that particular triangles are congruent. In that case, we try to get this information by first proving other triangles congruent. The problems that follow illustrate this.

Illustrative Example



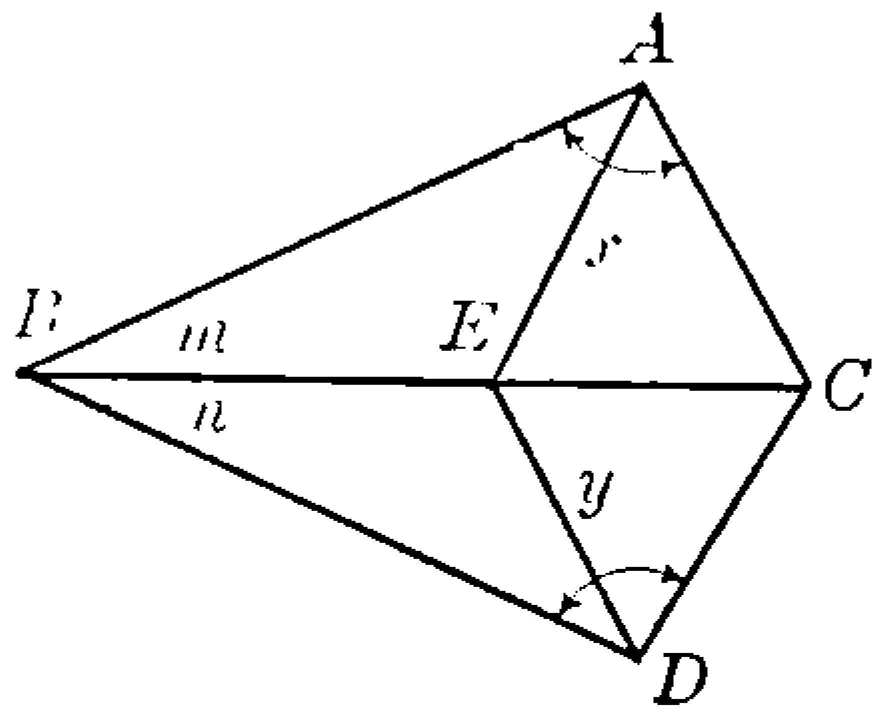
Given:
$CD \perp AB$
$\angle m = \angle n$
Prove:
$x = y$

PROOF

Statements	*Reasons*
1. $\angle m = \angle n$	1. Given.
2. ∠s p and q rt. ∠s	2. Given.
3. $\angle p = \angle q$	3. Rt. ∠s are =.
4. $AB = AB$	4. Identity.
5. $\triangle ABC \cong \triangle ABD$	5. a.s.a.
6. $CB = BD$	6. Def. $\cong$ △s.
7. $EB = EB$	7. Identity.
8. $\triangle BCE \cong \triangle BDE$	8. s.a.s.
9. $x = y$	9. Def. $\cong$ △s.

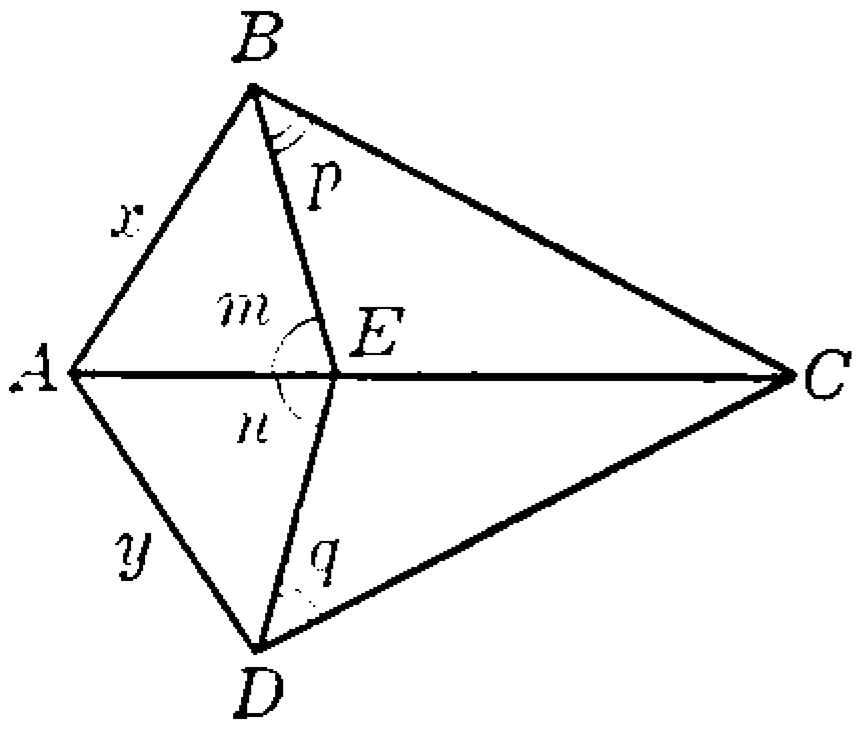
Instead of triangles BCE and BDE, what other triangles might we have proved congruent?

9.



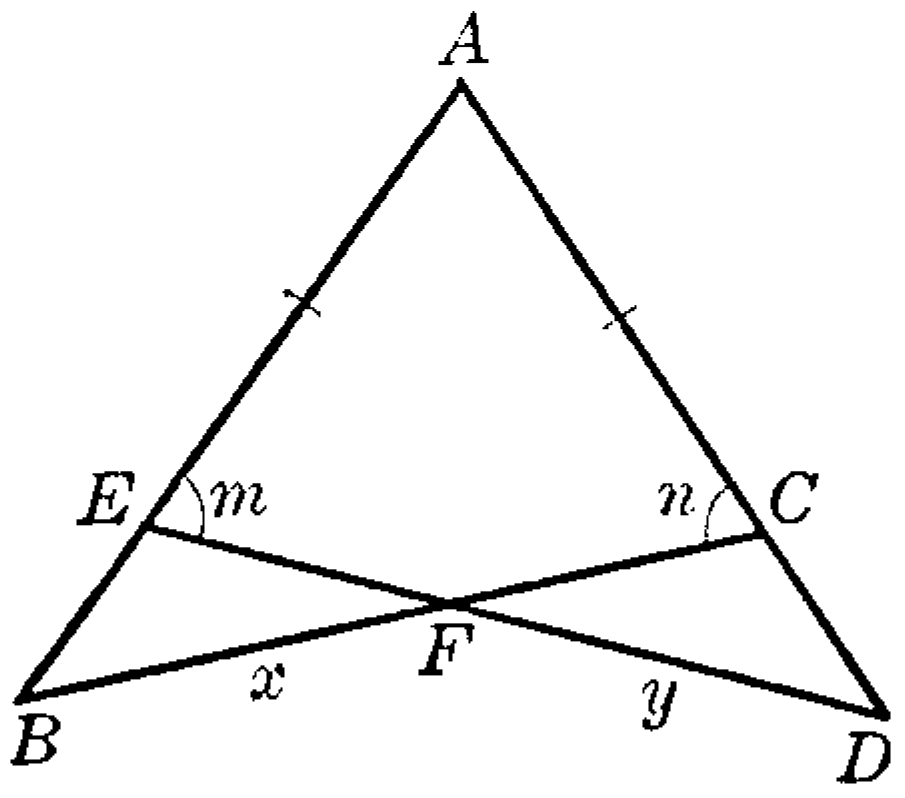
Given:
$\angle m = \angle n$
$\angle A = \angle D$
Prove:
$x = y$

10.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$x = y$

11.

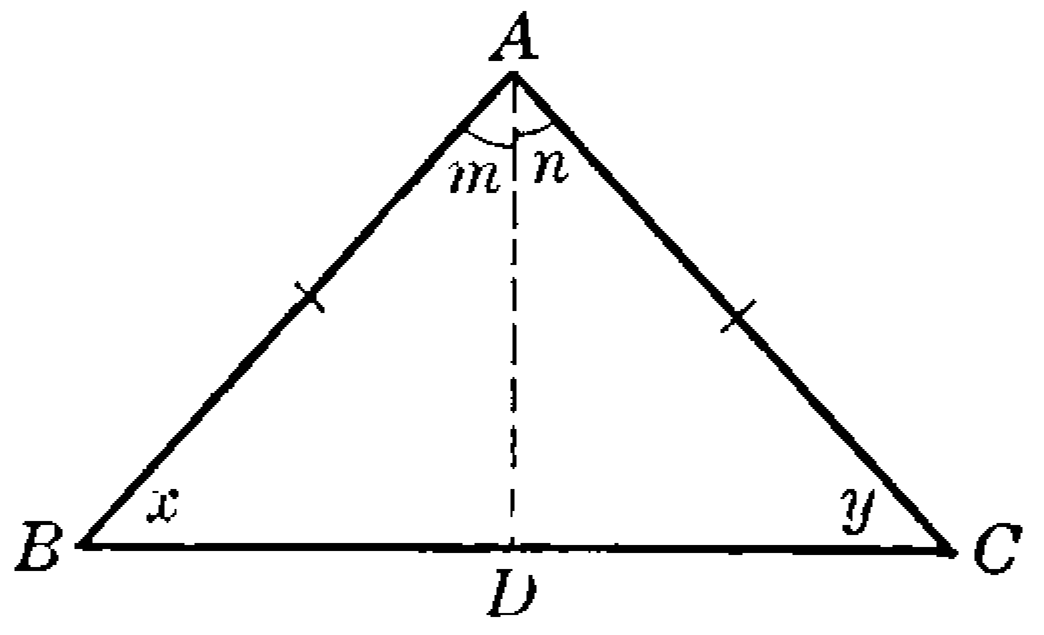


Given:
$AE = AC$
$\angle m = \angle n$
Prove:
$BF = FD$

The Isosceles Triangle

The following shows the deductive proof for: (1) the theorem that the base angles of an isosceles triangle are equal and (2) its converse.

118. THEOREM. If two sides of a triangle are equal, the angles opposite these sides are equal.



Given: In $\triangle ABC$
$AB = AC$
Prove: $\angle B = \angle C$
Cons.: Bisect $\angle A$.

PROOF

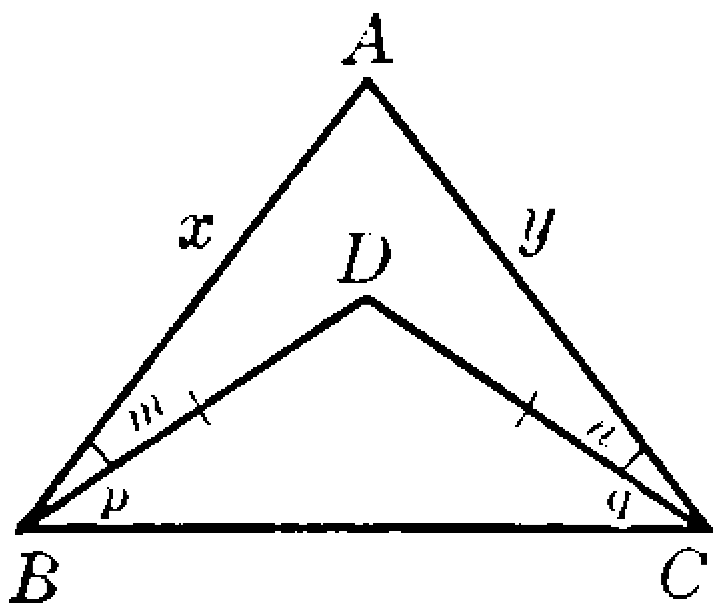
Statements	*Reasons*
1. $\angle m = \angle n$	1. Cons.
2. $AB = AC$	2. Given.
3. $AD = AD$	3. Identity.
4. $\triangle ABD \cong \triangle ACD$	4. s.a.s.
5. $\angle B = \angle C$	5. Def. $\cong$ ⚠.

119. CONVERSE THEOREM. If two angles of a triangle are equal, the sides opposite these angles are equal.

Following the deductive method in the last proof, draw a figure and prove this theorem.

The last two theorems are useful because they enable us to prove two angles equal from the equality of two sides alone, and vice versa.

Illustrative Example



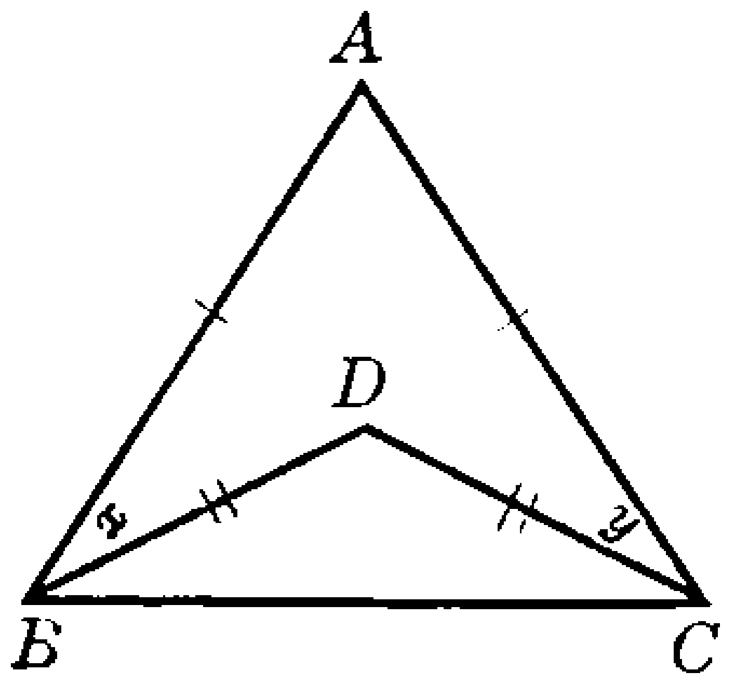
Given:
$DB = DC$
$\angle m = \angle n$
Prove:
$x = y$

PROOF

Statements	*Reasons*
1. $DB = DC$	1. Given.
2. $\angle p = \angle q$	2. Base ∠s of isos. △ are =.
3. $\angle m = \angle n$	3. Given.
4. $\angle B = \angle C$	4. Add. Ax.
5. $x = y$	5. If 2 ∠s of △ are =, sides opposite are =.

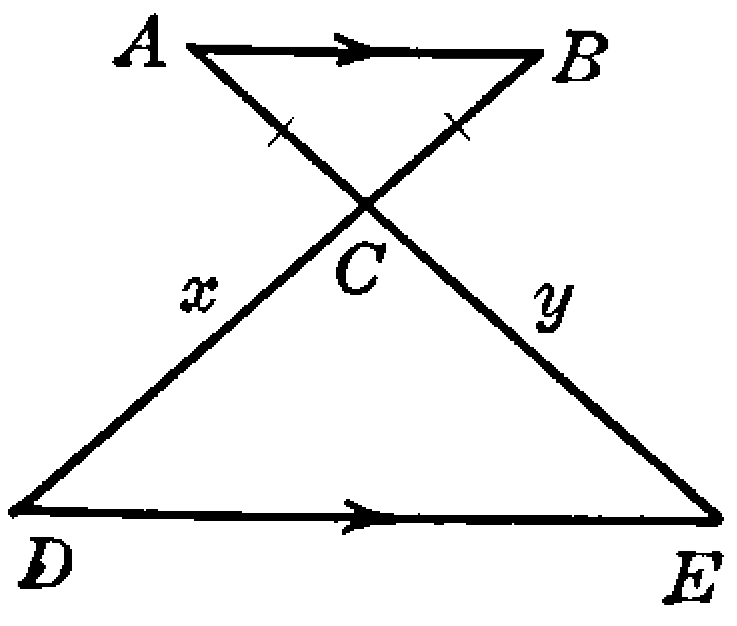
EXERCISES

1.



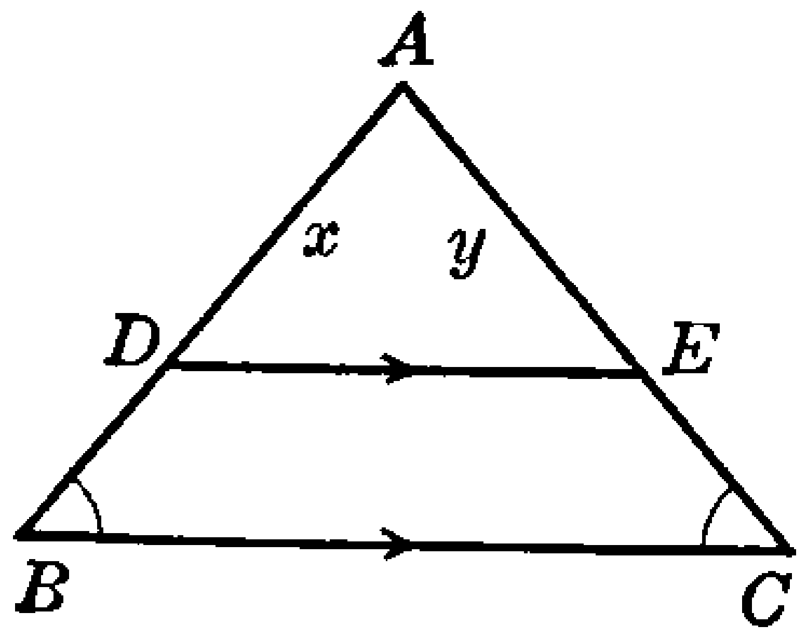
Given:
$AB = AC$
$DB = DC$
Prove:
$\angle x = \angle y$

2.



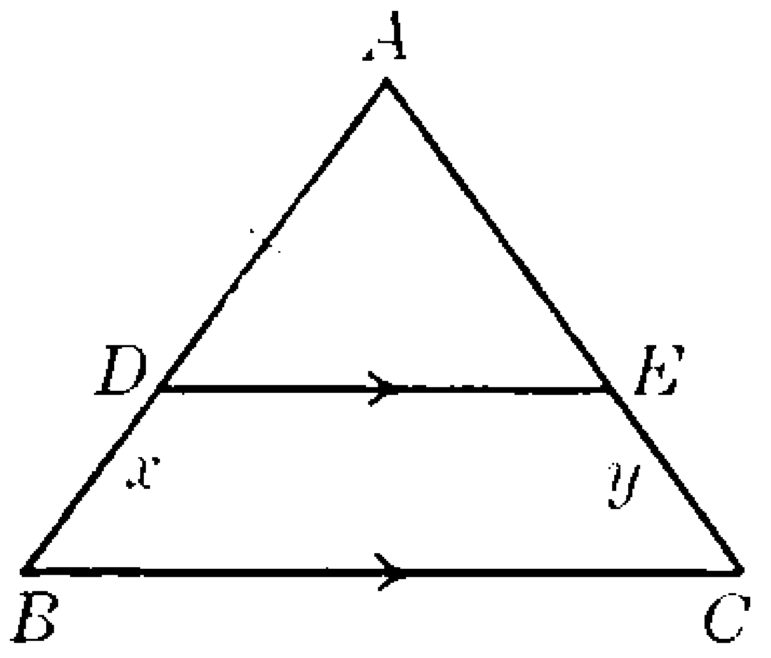
Given:
$AC = CB$
$AB \parallel DE$
Prove:
$x = y$

3.



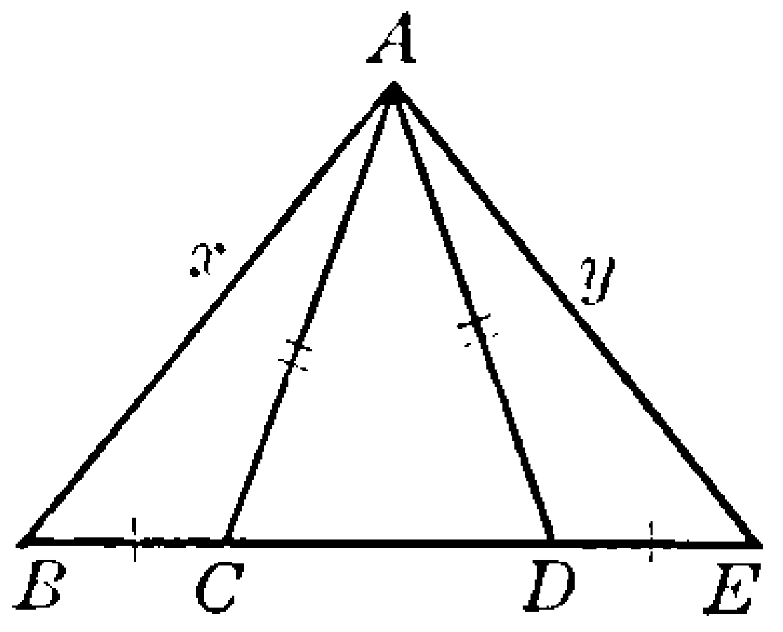
Given:
$\angle B = \angle C$
$DE \parallel BC$
Prove:
$AD = AE$

4.



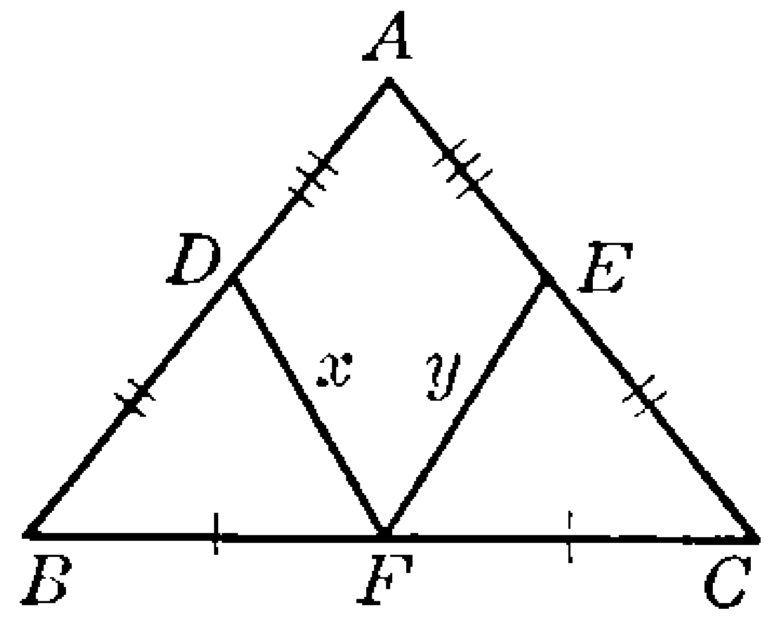
Given:
$AD = AE$
$DE \parallel BC$
Prove:
$DB = EC$

5.



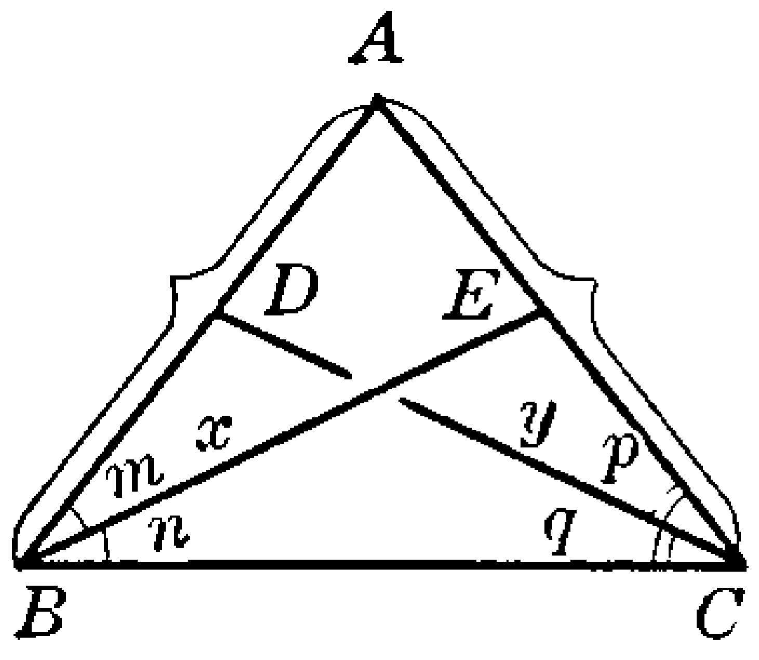
Given:
In $\triangle ABE$
$AC = AD$
$BC = DE$
Prove:
$x = y$

6.



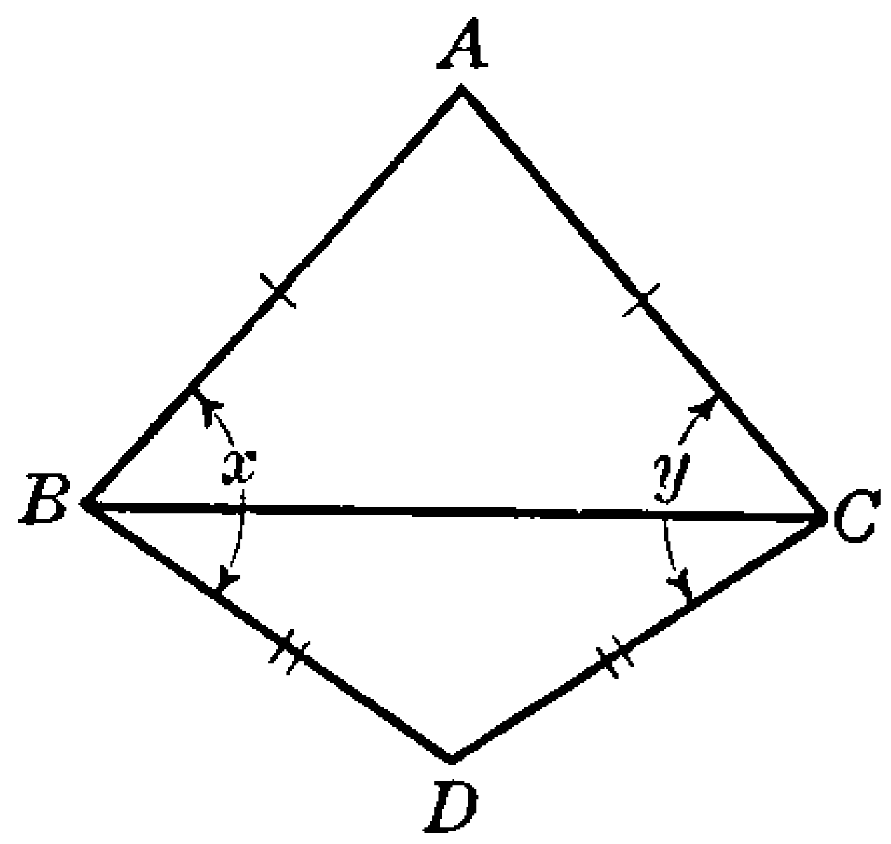
Given:
In $\triangle ABC$
$BF = FC$
$AD = AE$
$DB = EC$
Prove:
$x = y$

7.



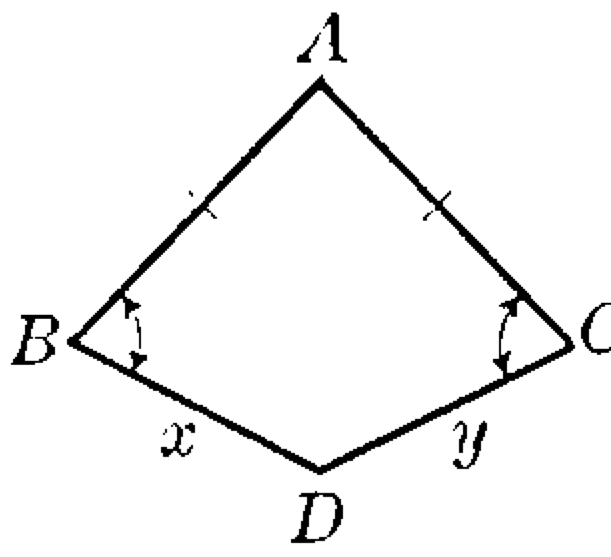
Given:
$AB = AC$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$BE = CD$

8.



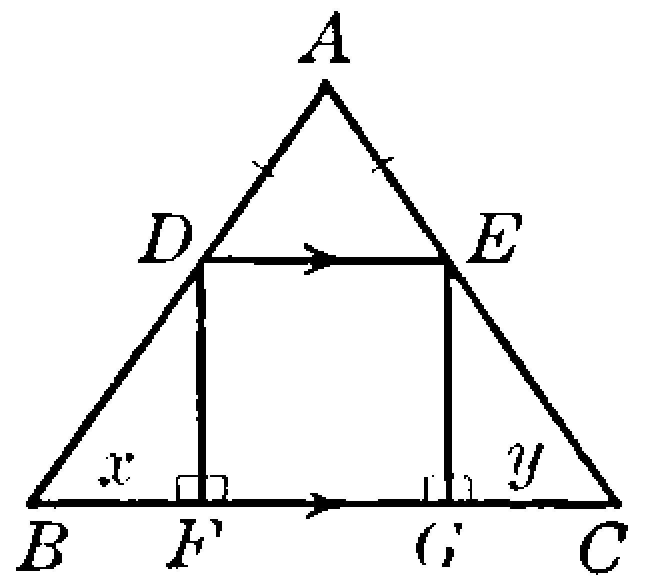
Given:
$AB = AC$
$DB = DC$
Prove:
$\angle B = \angle C$

9.



Given:
$AB = AC$
$\angle B = \angle C$
Prove:
$x = y$

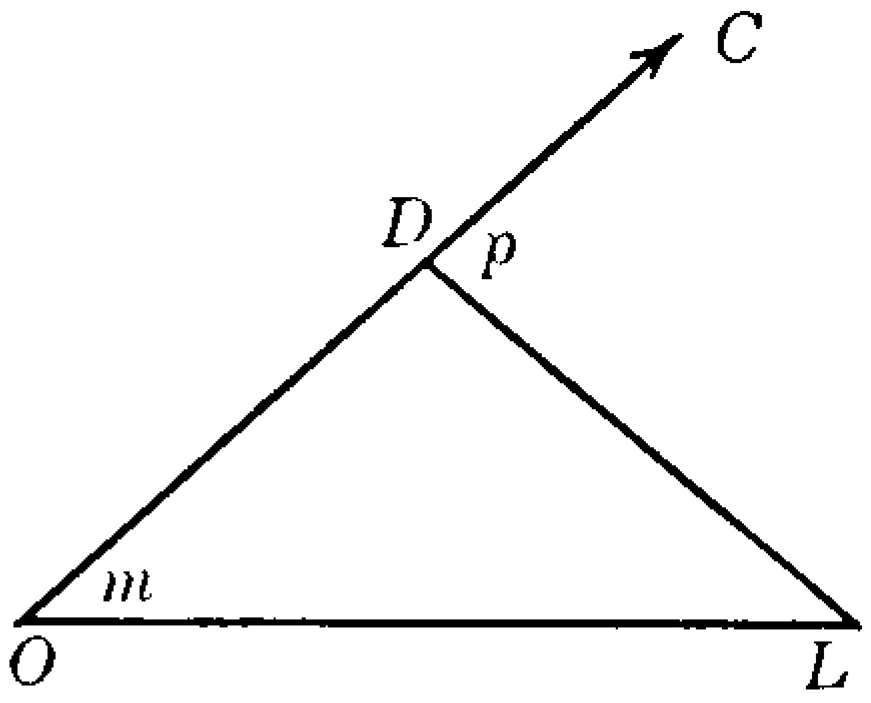
10.



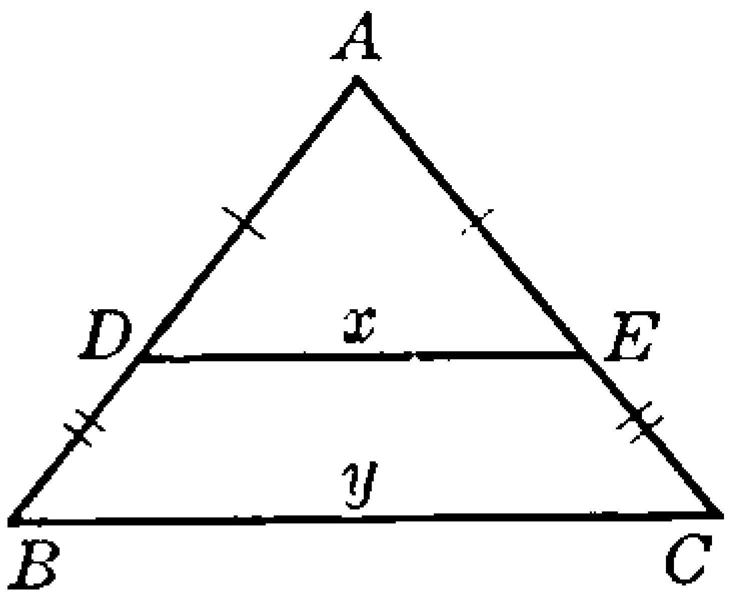
Given:
$AD = AE$
$DE \parallel BC$
DF and $EG \perp BC$
Prove:
$BF = GC$

SUPERIOR WORK

S1. A mariner sailing a course OC, measures the angle m between a lighthouse L and his course OC. This is called "the angle at the bow." He then sails a distance OD until the angle at the bow p is twice the angle m. Prove that the distance to the lighthouse DL equals the distance sailed OD.

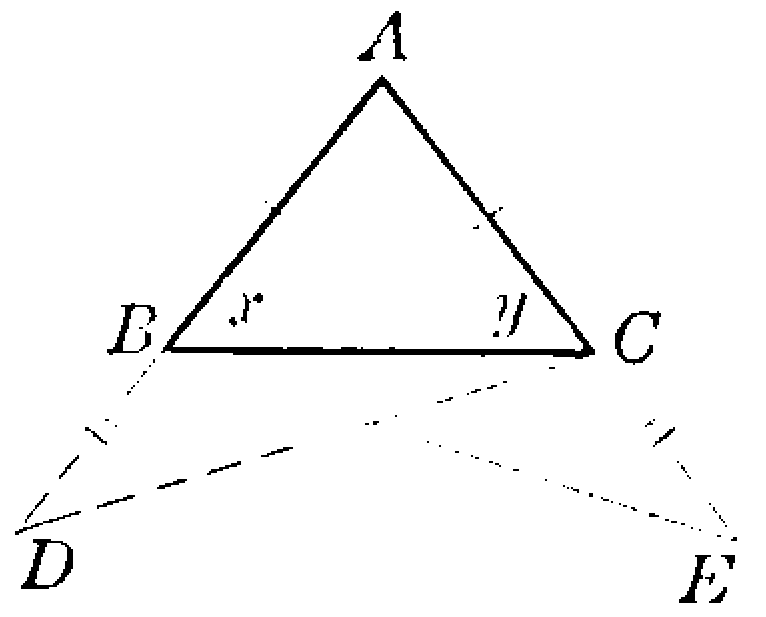


S2. Prove the following converse of Ex. 4 on page 91:



Given:
$AD = AE$
$DB = EC$
Prove:
$DE \parallel BC$

S3. The following diagram suggests the proof that Euclid (see page 72, S5) gave for the theorem that the base angles of an isosceles triangle are equal. Because of its appearance and the difficulty that pupils had with it, the figure was named the *pons asinorum* which means "the bridge of fools".



Given:
$AB = AC$
Prove:
$\angle x = \angle y$

Cons.: Prolong AB and AC so that $BD = CE$; then draw BE and CD.

Write out the proof. *Hint:* First prove $\triangle ACD \cong \triangle ABE$, and then prove $\triangle BCD \cong \triangle BCE$.

Euclid used this long proof because he did not want to bisect the vertex angle (as we have done) before he proved that it can be done with compass and straightedge. As you will see in the next chapter, this proof depends indirectly on the theorem about the base angles of an isosceles triangle.

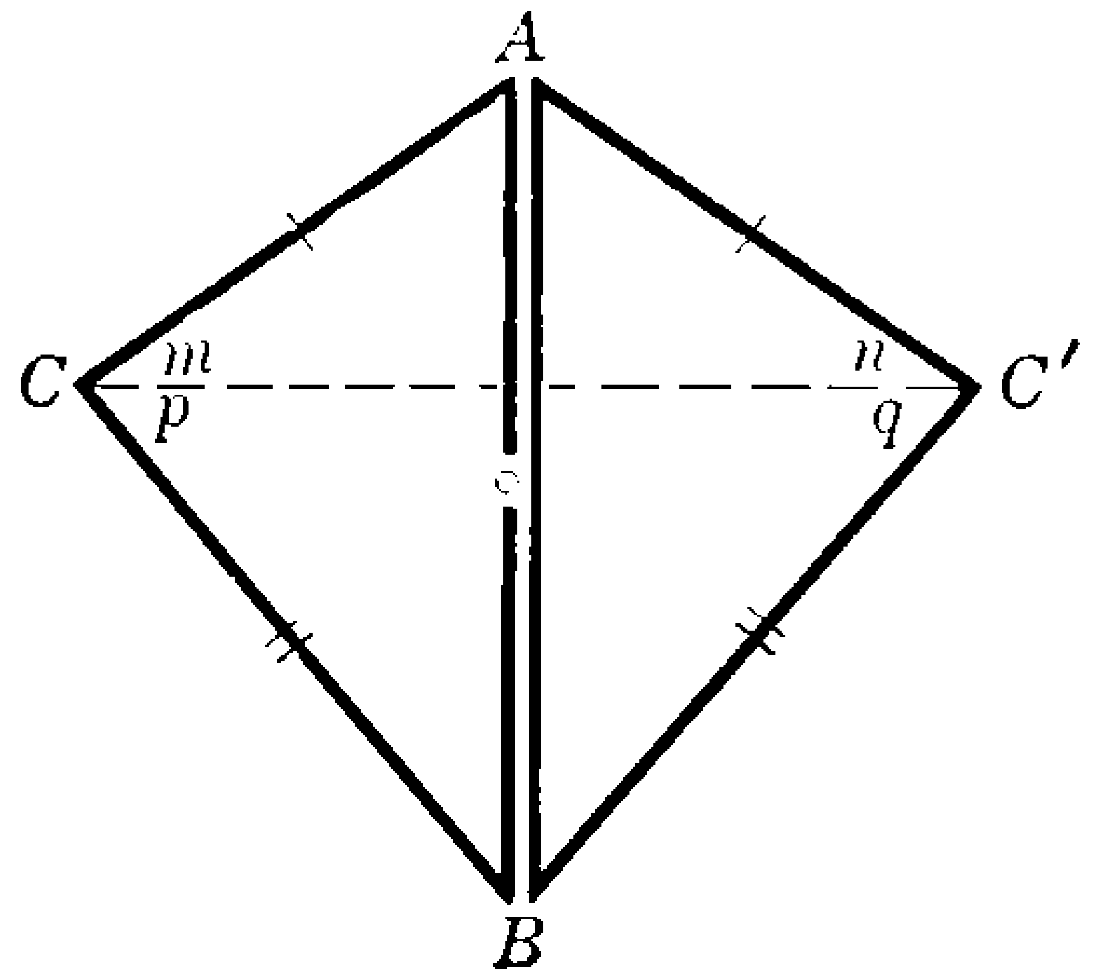
If we used the *construction* of an angle bisector to prove the proposition about an isosceles triangle, and then used this proposition to prove the construction, then we should indeed have a *circular* argument. However, our proof does not depend on the actual construction of the angle bisector, but only on the fact that the angle bisector *exists*. This is not really a new assumption. In the division axiom we have assumed that division is a determinate process (§ 16) which means that half an angle (or any magnitude) exists.

S4. Prove the converse of S3 by Euclid's method.

Triangles with Three Sides Equal

It is possible to prove that two triangles are congruent if they agree in three sides. If we prove one pair of angles equal, the triangles will be congruent by the theorem s.a.s. The theorem about the base angles of an isosceles triangle proves two angles equal from the equality of the sides alone. We therefore draw the two triangles so that they have one side in common. This will enable us to draw a line forming two isosceles triangles.

120. THEOREM. If three sides of one triangle are equal respectively to three sides of another triangle, the triangles are congruent. (This theorem is often abbreviated as s.s.s.)

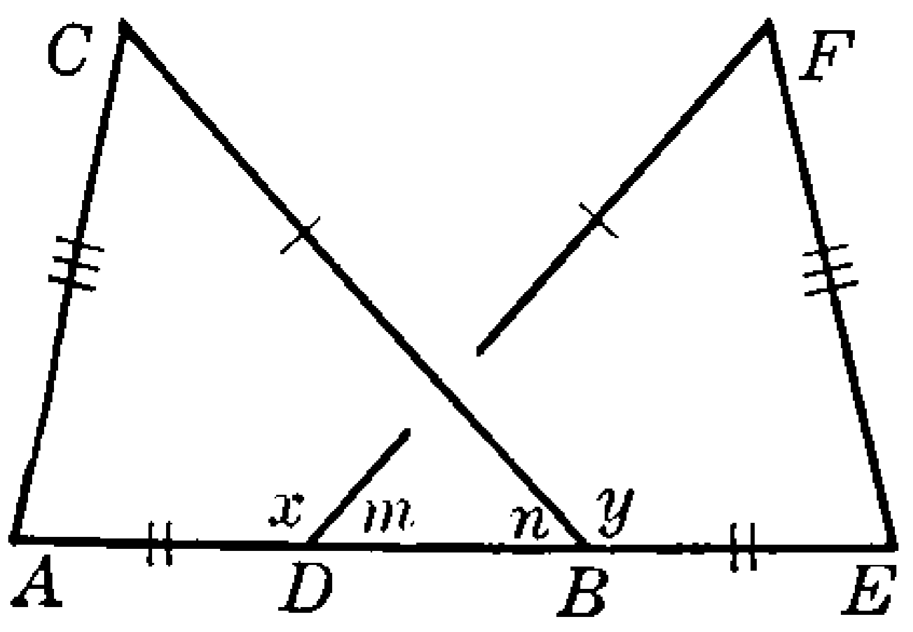


Given: In $\triangle$s ABC and ABC'
$AC = AC'$
$BC = BC'$
$(AB = AB)$
Prove: $\triangle ABC \cong \triangle ABC'$
Cons.: Draw CC'.

PROOF

Statements	*Reasons*
1. $AC = AC'$	1. Given.
2. $\angle m = \angle n$	2. Base $\angle$s of an isos. $\triangle$ are =.
3. $BC = BC'$	3. Given.
4. $\angle p = \angle q$	4. Same as 2.
5. $\angle C = \angle C'$	5. Add. Ax.
6. $\triangle ABC \cong \triangle ABC'$	6. s.a.s.

Illustrative Example



Given:
$CB = FD$
$AD = BE$
$AC = EF$
Prove:
$\angle x = \angle y$

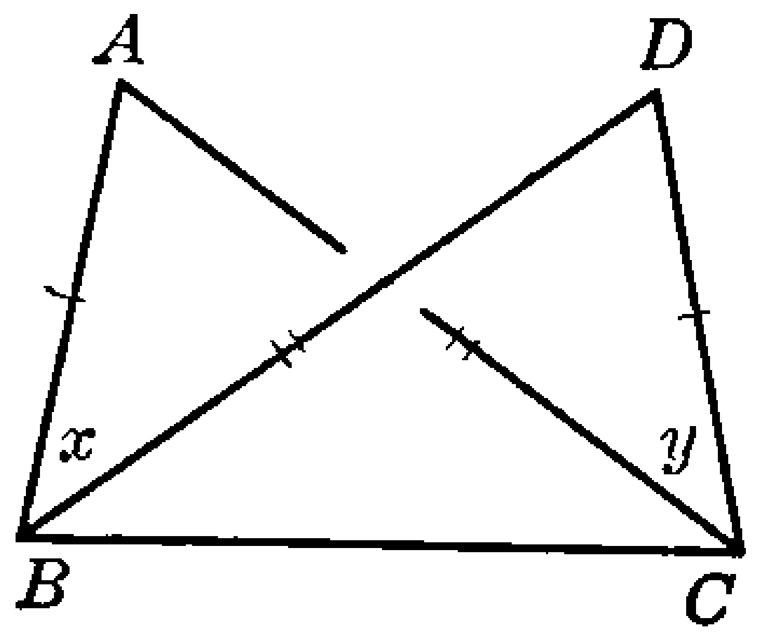
PROOF

Statements	*Reasons*
1. $AD = BE$	1. Given.
2. $DB = DB$	2. Identity.
3. $AB = DE$	3. Add. Ax.
4. $AC = EF$ $CB = FD$	4. Given.
5. $\triangle ABC \cong \triangle DEF$	5. s.s.s.
6. $\angle m = \angle n$	6. Def. $\cong$ $\triangle$s.
7. $\angle x = \angle y$	7. Sups. of = $\angle$s are =.

Explain Statement 5 as follows: "If three sides" (point out sides AC, CB, and BA) "of one triangle" (trace $\triangle ABC$) "are equal respectively to three sides" (point out the sides EF, FD, and DE) "of another triangle" (trace $\triangle DEF$), "then the triangles" (again trace $\triangle ABC$ and $\triangle DEF$) "are congruent."

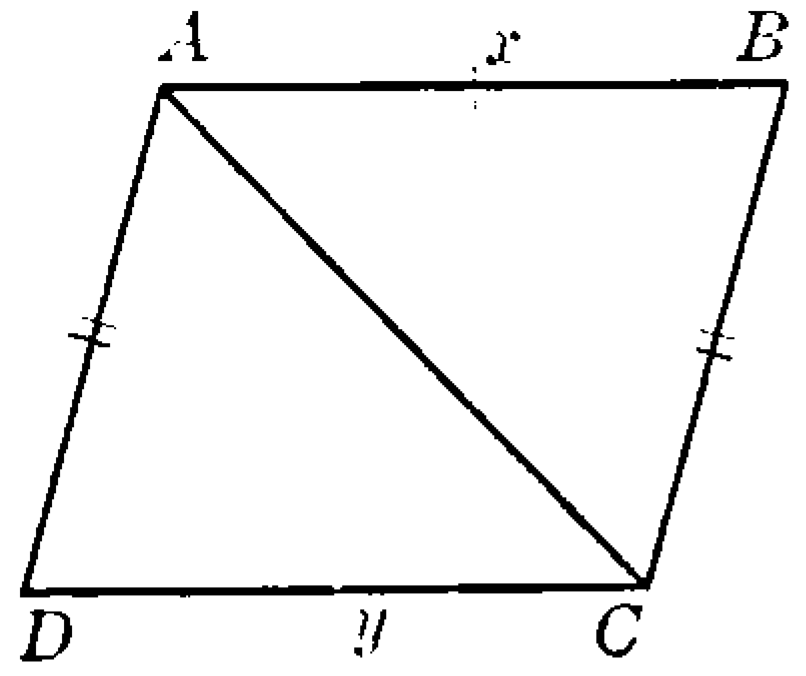
EXERCISES

1.



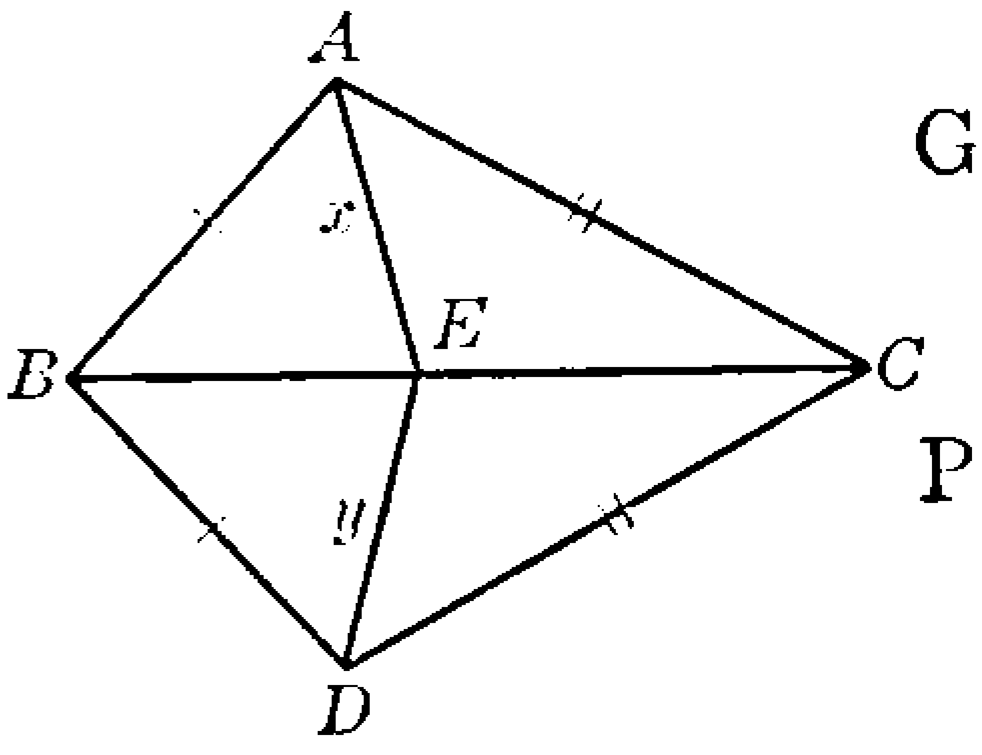
Given:
$AB = DC$
$AC = DB$
Prove:
$\angle x = \angle y$

2.



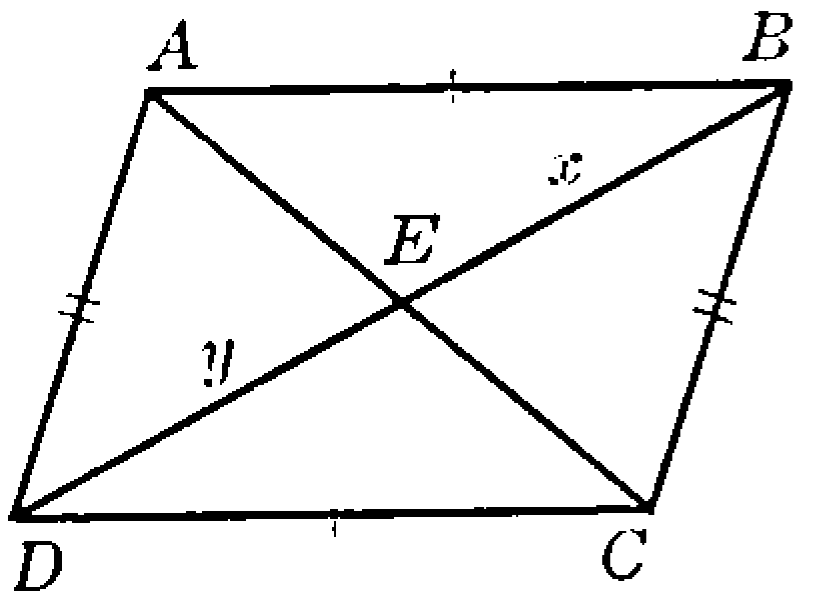
Given:
$AB = DC$
$AD = BC$
Prove:
$AB \parallel DC$

3.



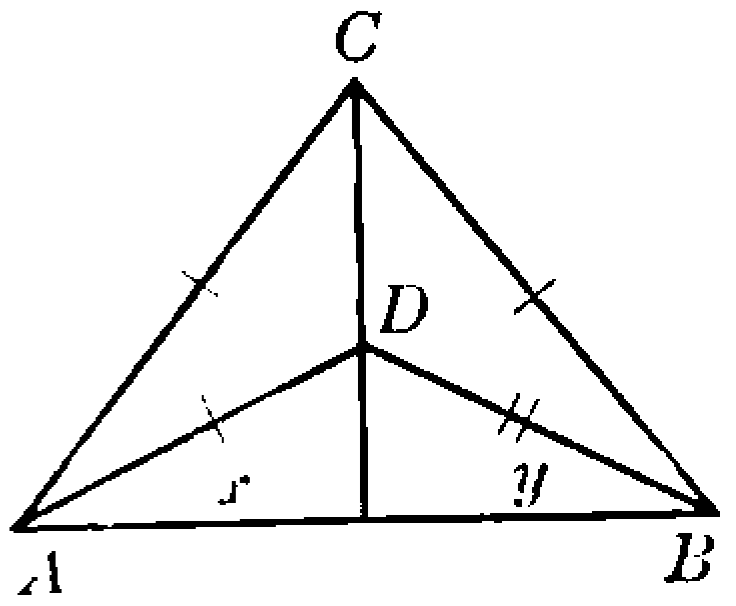
Given:
$AB = BD$
$AC = CD$
Prove:
$AE = ED$

4.



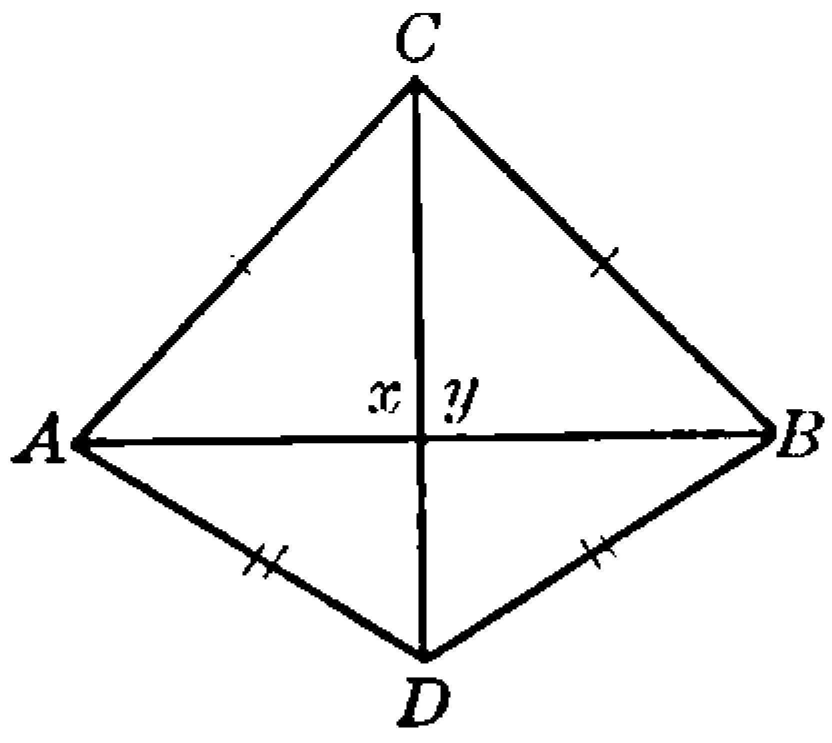
Given:
$AB = DC$
$AD = BC$
Prove:
$x = y$

5.



Given:
$AC = CB$
$AD = DB$
Prove:
$x = y$

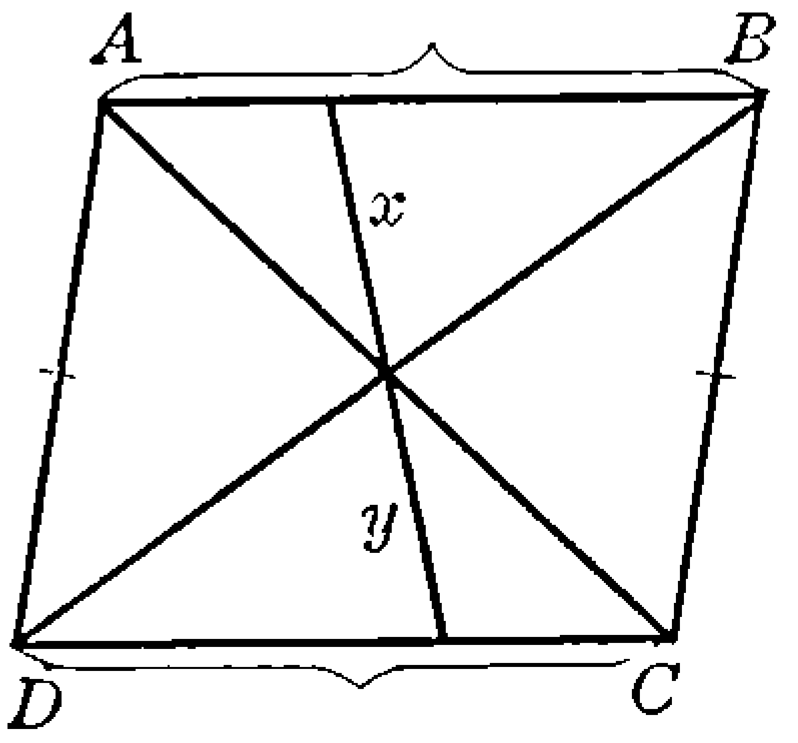
6.



Given:
$AC = CB$
$AD = DB$
Prove:
$\angle x = \angle y$

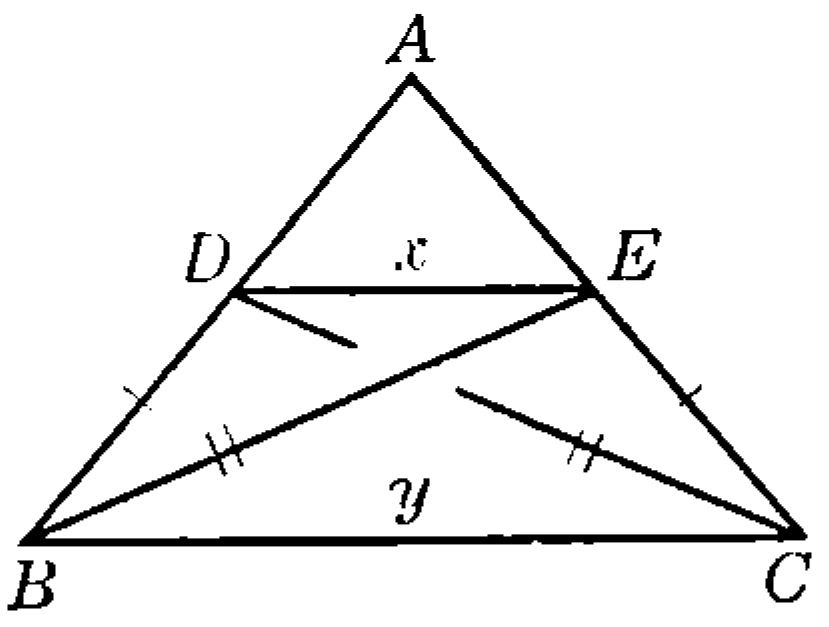
SUPERIOR WORK

S1.



Given:
$AB = DC$
$AD = BC$
Prove:
$x = y$

S2.



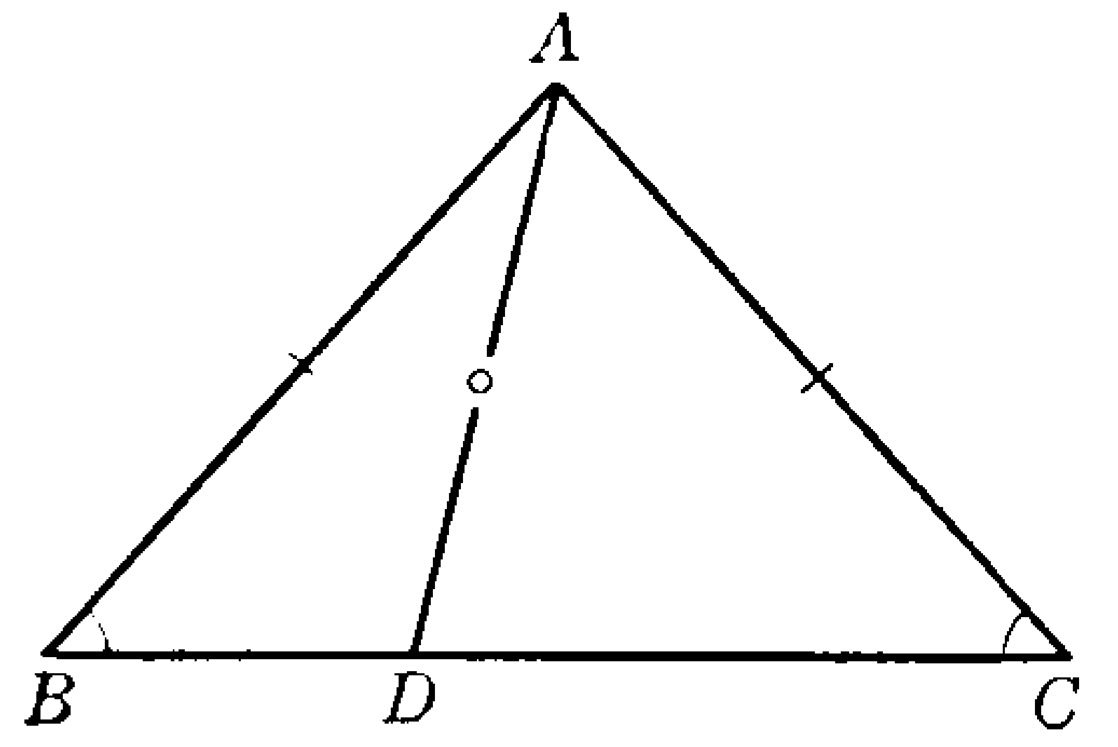
Given:
$DB = EC$
$DC = EB$
Prove:
$x \parallel y$
Hint: First prove $\triangle ABC$ isosceles.

A Nonincluded Angle

121. We have used four ways of proving two triangles congruent when three parts of one are equal respectively to three parts of the other. These parts must be one of the following combinations:

Three sides (s.s.s.)
One side and two angles (a.s.a. or s.a.a.)
Two sides and the included angle (s.a.s.)

Notice that if two sides and one angle are used to prove triangles congruent, the angle must be the included angle. In general, it is not true that two triangles are congruent because they have two sides and a nonincluded angle of one equal respectively to the corresponding parts of the other. Draw a line at random from the vertex of an isosceles triangle to any point in the base. The base angles B and C are equal. Why? The two triangles ABD and ACD have two sides and a nonincluded angle of one

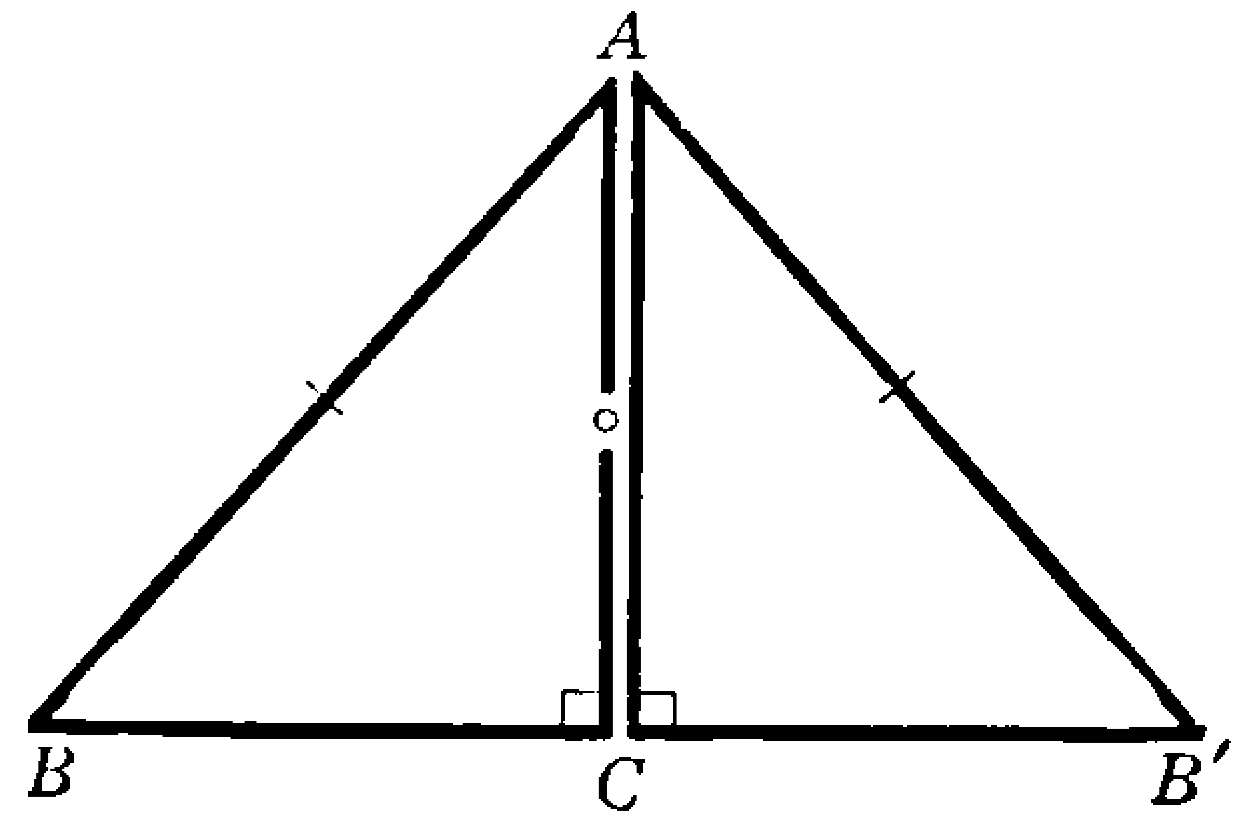


equal respectively to the corresponding parts of the other, but the triangles are not congruent.

Although the proposition about two sides and a nonincluded angle is not generally valid, it does hold in some special cases. One important special case is that in which the nonincluded angle is a right angle. In that case, we have two right triangles in which the hypotenuse and a leg of one are equal respectively to the hypotenuse and a leg of the other. To prove these triangles congruent, we place them so that they have the equal legs in common. We then prove that the whole figure is an isosceles triangle.

---

122. THEOREM. If the hypotenuse and a leg of one right triangle are equal respectively to the hypotenuse and a leg of another right triangle, the triangles are congruent. (This theorem is often abbreviated as h.l., meaning "hypotenuse, leg".)



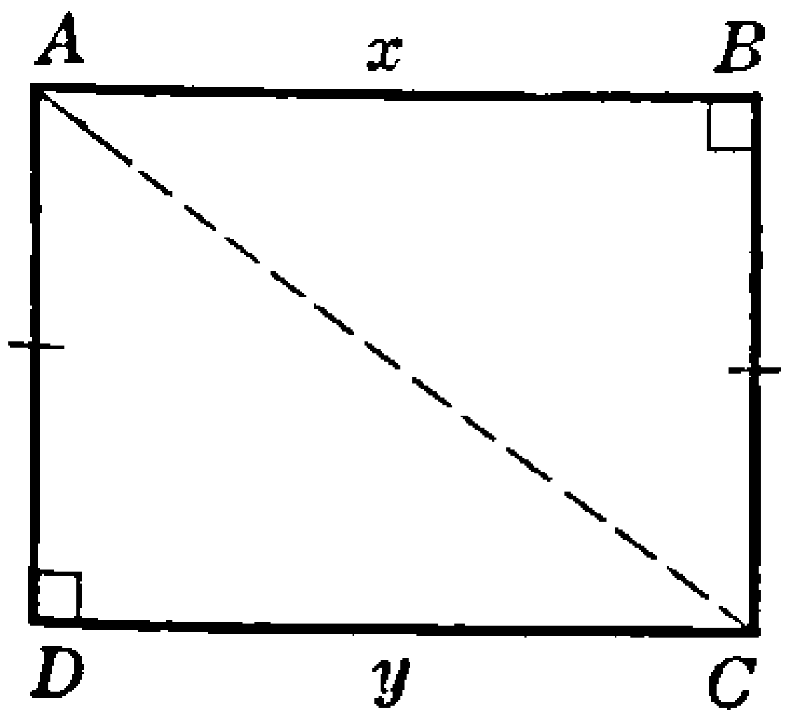
Given: Rt. $\triangle$s ABC and $AB'C$
$AB = AB'$
$(AC = AC)$
Prove: $\triangle ABC \cong \triangle AB'C$

PROOF

Statements	*Reasons*
1. $\angle BCA + \angle ACB' =$ a st. $\angle$	1. Why?
2. BCB' a st. line	2. Why?

Complete this proof.

Illustrative Example



Given:
$\angle B$ and $\angle D$
rt. $\angle$s
$AD = BC$
Prove:
$x = y$

Cons.: Draw AC.

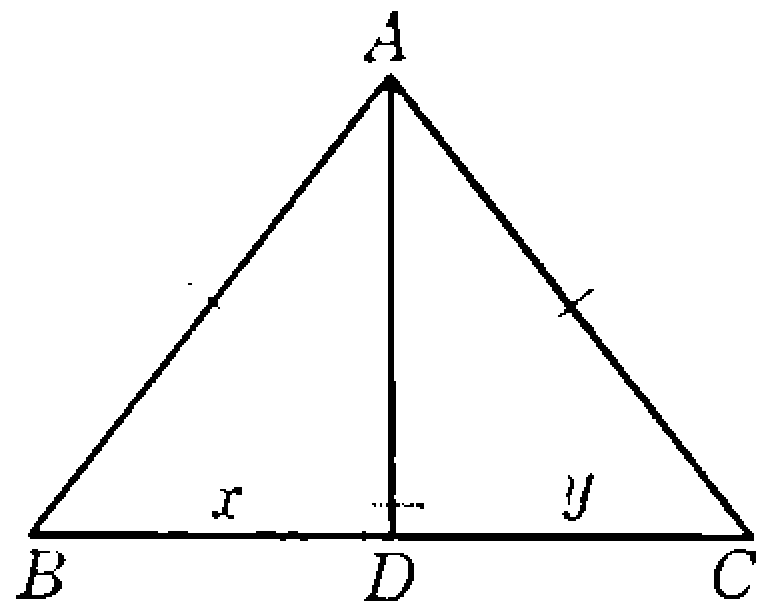
PROOF

Statements	*Reasons*
1. B and D rt. $\angle$s $AD = BC$	1. Given.
2. $AC = AC$	2. Identity.
3. $\triangle ABC \cong \triangle ACD$	3. h.l.
4. $x = y$	4. Def. $\cong$ $\triangle$s.

Explain Statement 3 as follows: "If the hypotenuse" (point to AC) "and a leg" (point to CB) "of one right triangle" (trace $\triangle ABC$, pointing out the right angle B) "are equal respectively to the hypotenuse" (point to AC) "and a leg" (point to AD) "of another right triangle" (trace $\triangle ADC$ with the right angle at D), "then the triangles" (again trace $\triangle ABC$ and $\triangle ADC$) "are congruent."

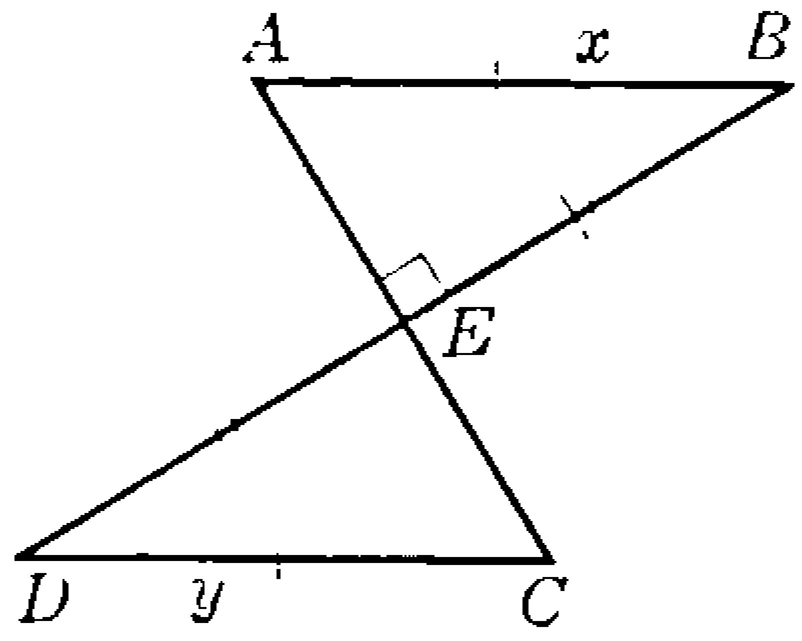
EXERCISES

1.



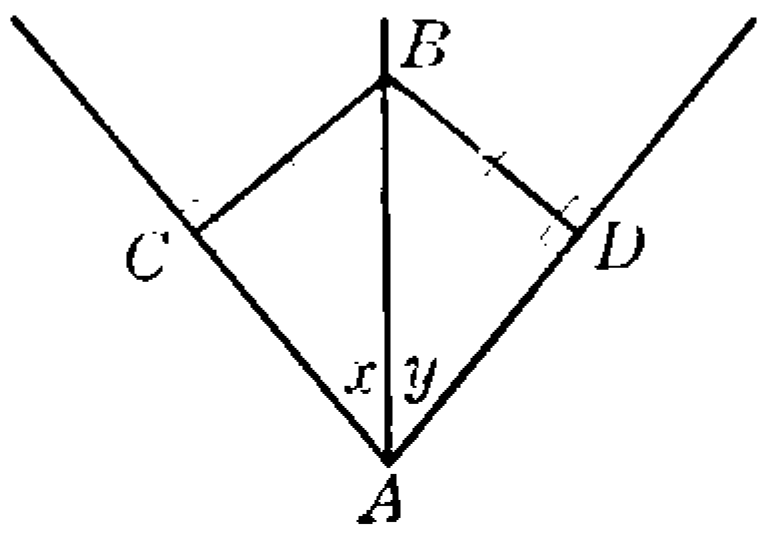
Given:
$AB = AC$
$AD \perp BC$
Prove:
$x = y$

2.



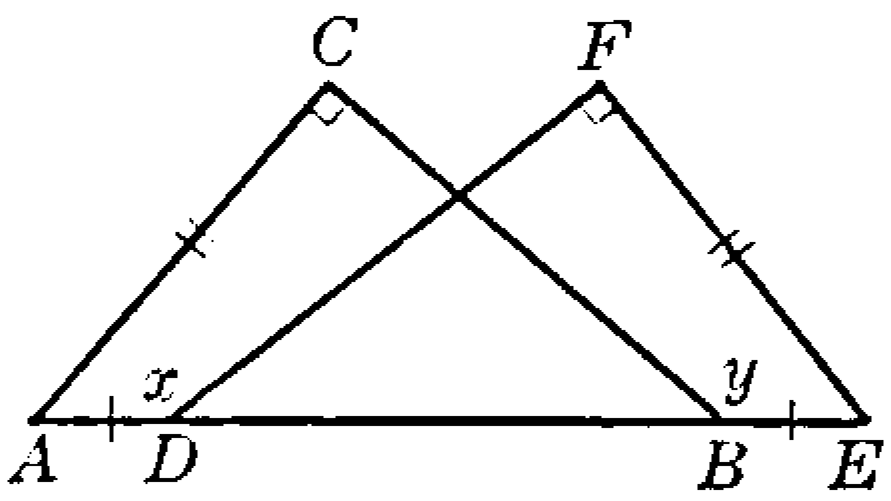
Given:
$AB = DC$
$BE = ED$
$AC \perp BD$
Prove:
$x \parallel y$

3.



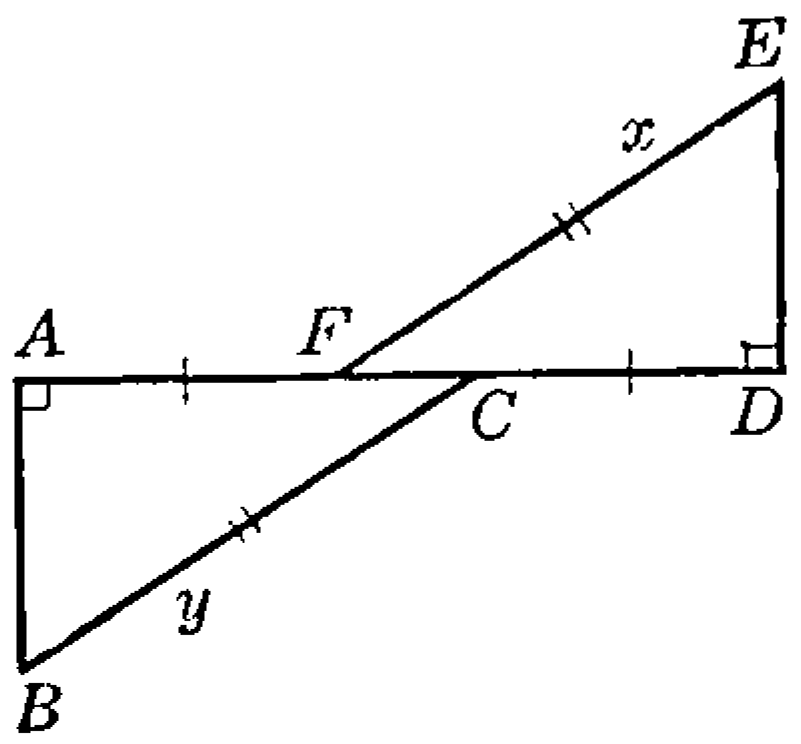
Given:
$BC \perp AC$
$BD \perp AD$
$BC = BD$
Prove:
$\angle x = \angle y$

4.



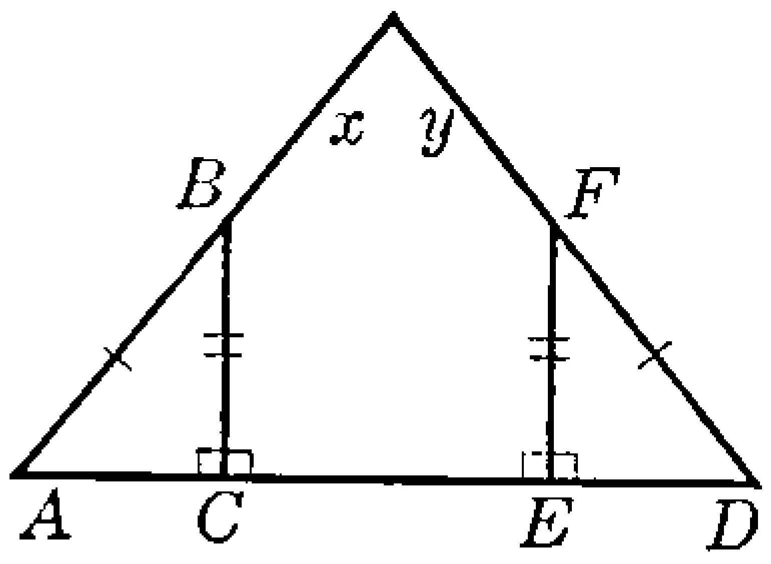
Given:
$\angle C$ and $\angle F$
rt. ∡
$AC = EF$
$AD = BE$
Prove:
$\angle x = \angle y$

5.



Given:
$AB \perp AD$
$ED \perp AD$
$AF = CD$
$BC = EF$
Prove:
$x \parallel y$

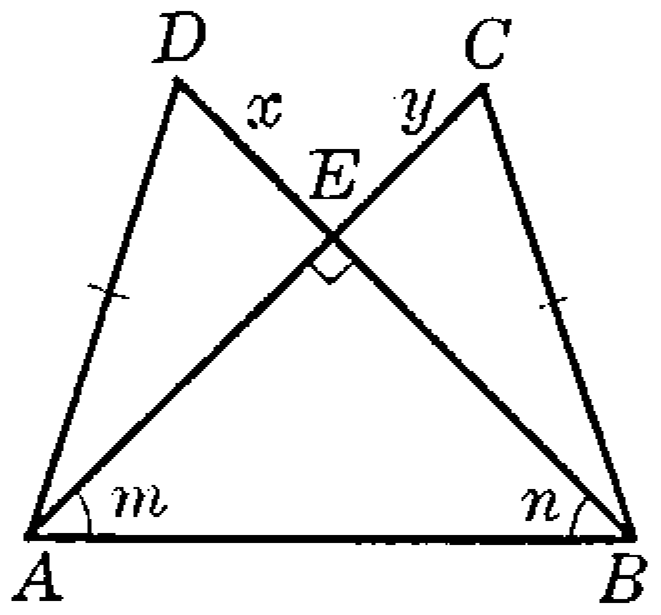
6.



Given:
$AB = DF$
$BC = FE$
$BC \perp AD$
$FE \perp AD$
Prove:
$x = y$

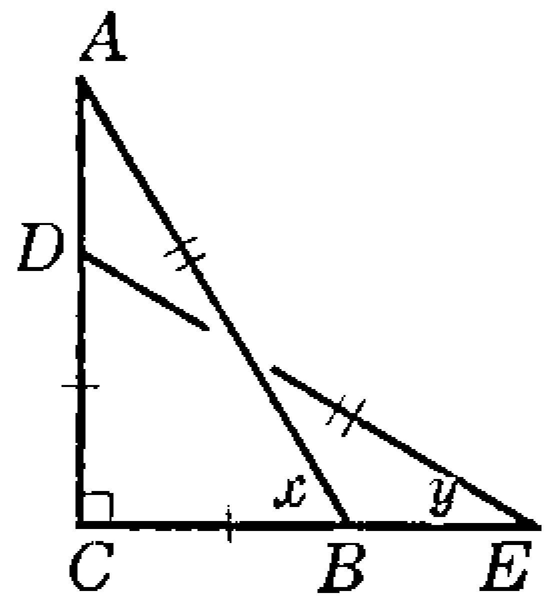
SUPERIOR WORK

S1.



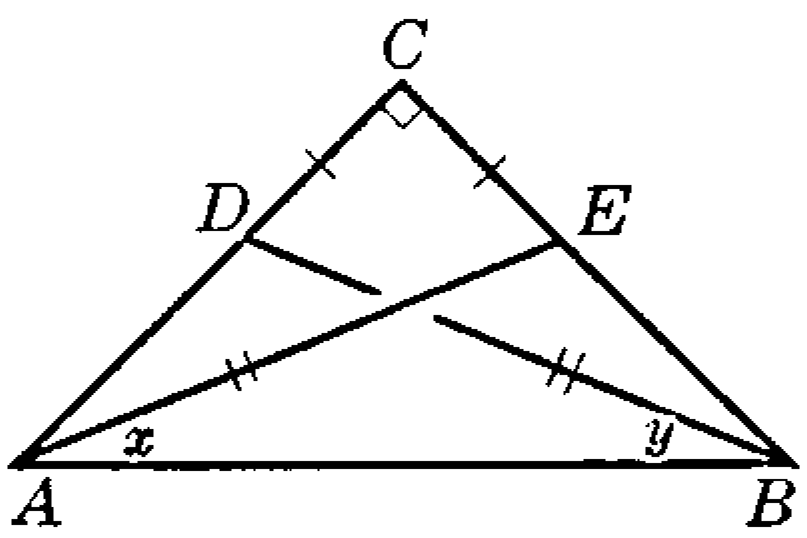
Given:
$AC \perp BD$
$AD = BC$
$\angle m = \angle n$
Prove:
$DE = EC$

S2.



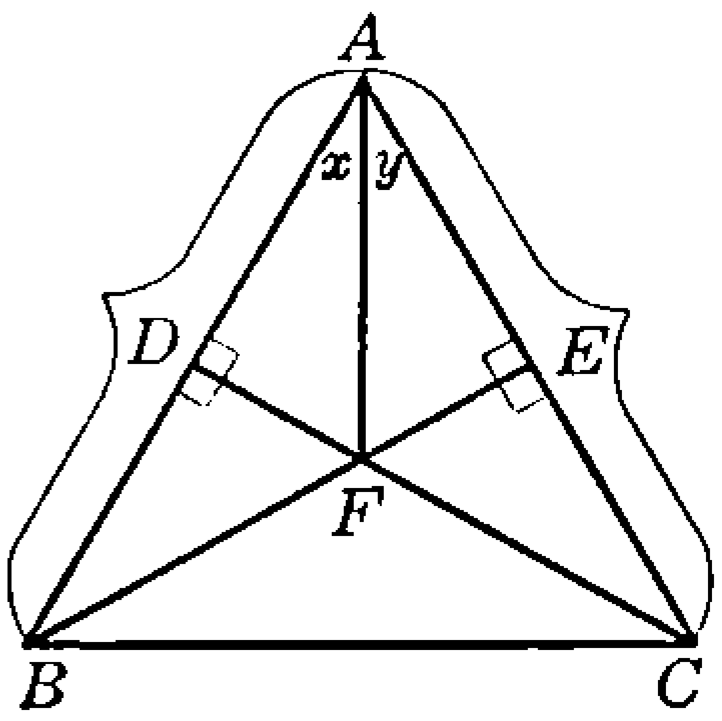
Given:
$DC = CB$
$AB = DE$
$\angle C = 90°$
Prove:
$\angle x + \angle y = 90°$

S3.



Given:
$\angle C = 90°$
$CD = CE$
$AE = BD$
Prove:
$\angle x = \angle y$

S4.



Given:
$AB = AC$
$BE \perp AC$
$CD \perp AB$
Prove:
$\angle x = \angle y$

SUMMARY

123. By the method of induction we argue that a general proposition is true from a number of particular facts. We usually discover these facts by experimenting. By the method of deduction, we argue that a proposition is true because it is a special case of a more general proposition that has been previously accepted.

If we obtain a proposition by induction, we cannot be absolutely sure of it. Much depends on the number of experiments and the skill and accuracy with which they were made. On the other hand, if we reason out a proposition correctly by deduction, we may be just as sure of it as we are of the propositions on which it depends. We often use both methods of thinking in the same problem.

The chief concern in this chapter was to find out which conditions would make two triangles congruent. We discovered by induction that if two triangles agree in three consecutive parts, they are congruent. We assumed this proposition and then proved the remaining propositions about congruent triangles. They may be summarized by saying that two triangles are congruent if they agree in:

(1) Two angles and
  (*a*) the included side (a.s.a.)
  (*b*) a corresponding nonincluded side (s.a.a.)
(2) Two sides and
  (*a*) the included angle (s.a.s.)
  (*b*) a nonincluded right angle (h.l.)
(3) Three sides (s.s.s.)

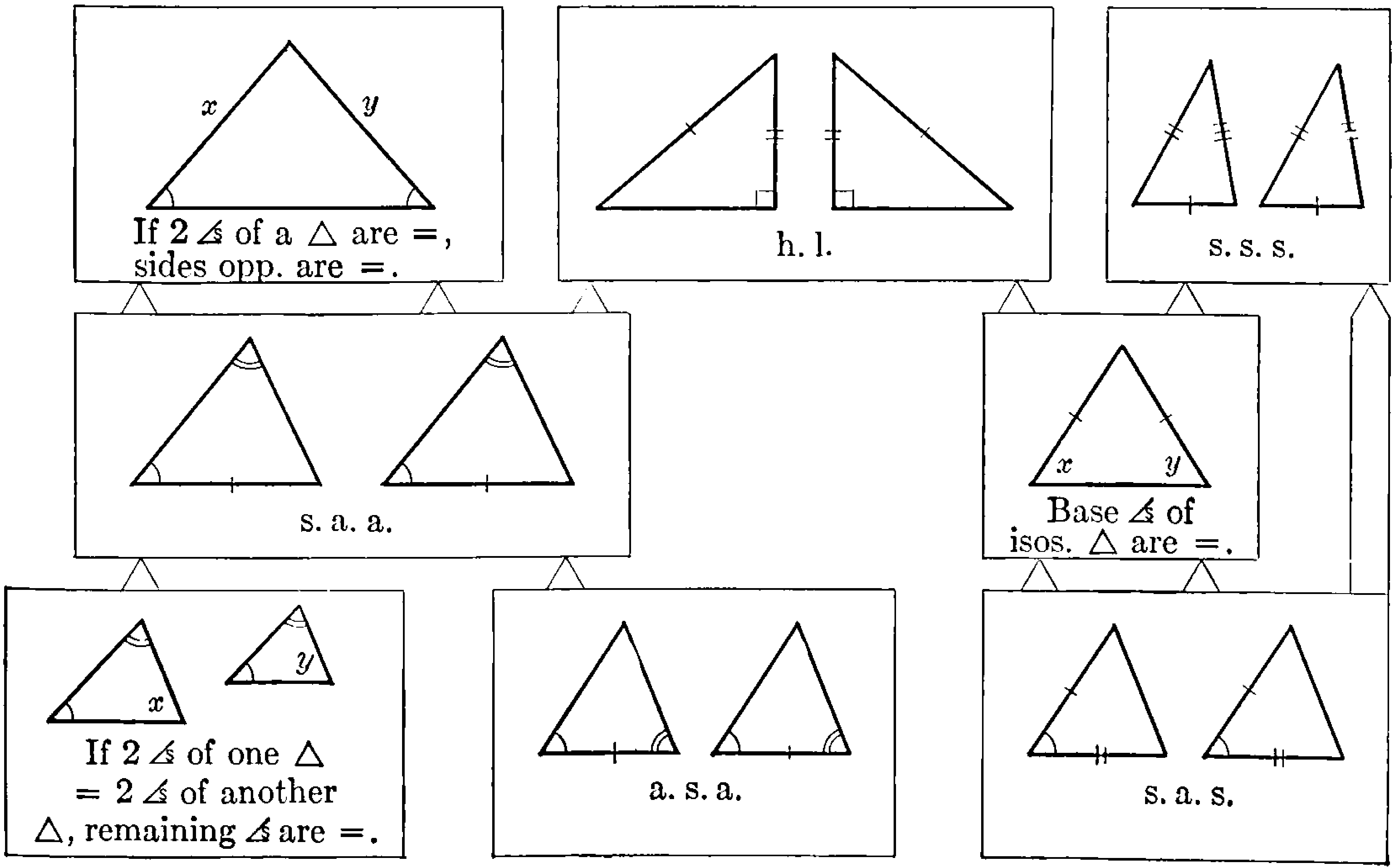
Two triangles may not be congruent even though they agree in two sides and a nonincluded angle, or even though they agree in three angles.

To prove two line segments or angles equal, we often prove that they are corresponding parts of congruent triangles.

We also proved the proposition that the base angles of an isosceles triangle are equal, and its converse. We use these to prove two angles equal when two line segments are given equal, and vice versa.

124. *The following chart shows how the theorems in this chapter depend on one another. With the aid of this chart, answer the questions on page 98.*

DEPENDENCE CHART



1. Which propositions support the theorem s.s.s.?

2. How many propositions support directly or indirectly the theorem h.l.?

3. Which theorems would fall if the proposition a.s.a. were false?

4. Is the theorem about the base angles of an isosceles triangle necessary, or sufficient, or both, for the proof of the theorem s.s.s.?

5. Which theorems would fall if the sum of the angles of a triangle were not constant?

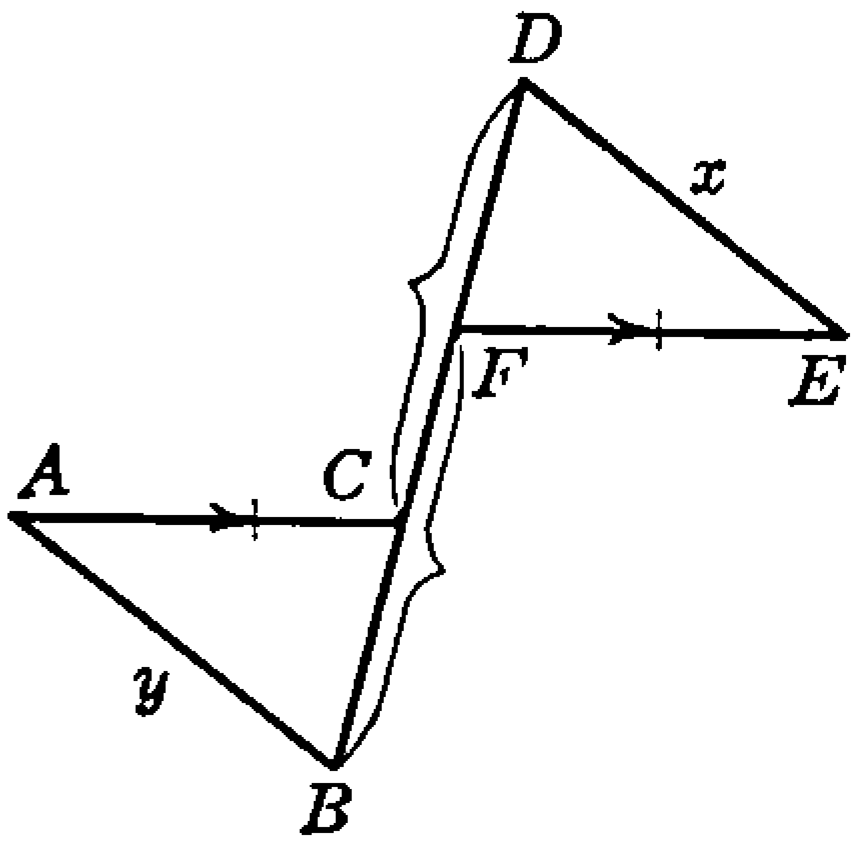
RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

induction	§ 104
deduction	§ 104
congruent triangles ($\cong$ ▲)	§ 111

REVIEW EXERCISES

1.

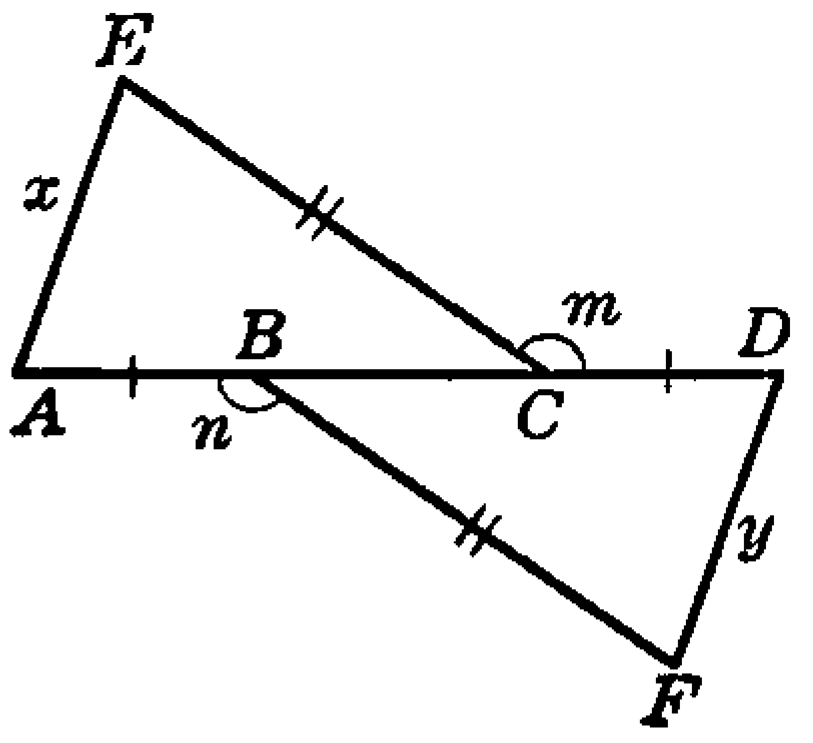


Given:
$FE \parallel AC$
$FE = AC$
$DC = FB$
DB st. line
Prove:
$x \parallel y$

2. Prove that the altitude to the base of an isosceles triangle bisects the base.

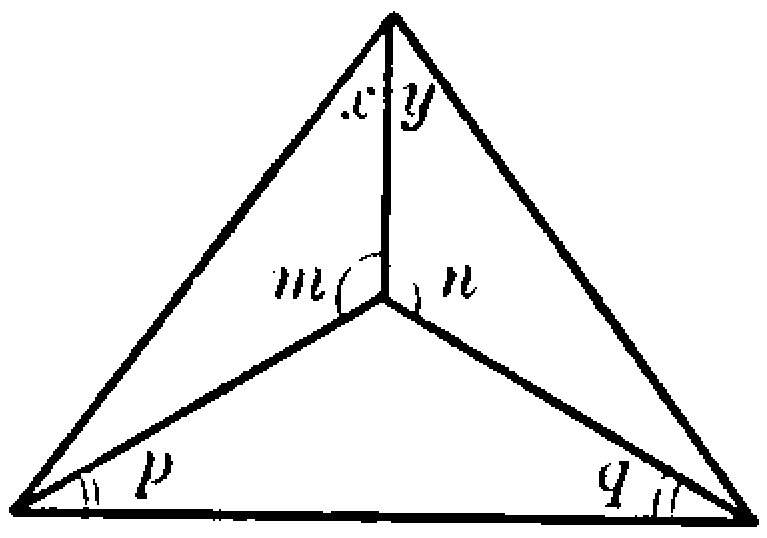
3. Prove the following proposition: Equal line segments drawn from the vertex of an isosceles triangle to the base cut off on the base equal line segments from its ends.

4.



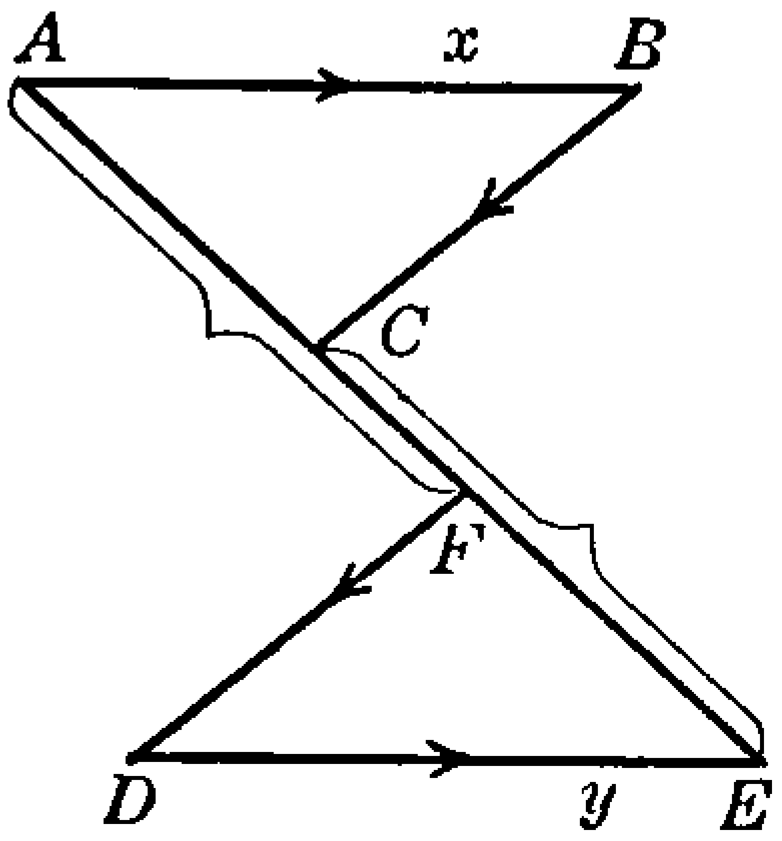
Given:
$AB = CD$
$EC = BF$
$\angle m = \angle n$
AD st. line
Prove:
$x = y$

5.



Given:
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
$\angle x = \angle y$

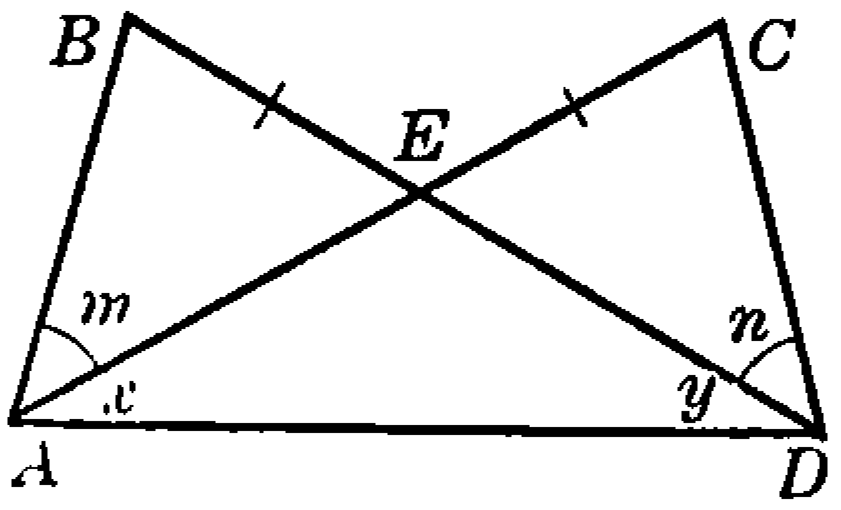
6.



Given:
$AB \parallel DE$
$BC \parallel FD$
$AF = CE$
AE st. line
Prove:
$x = y$

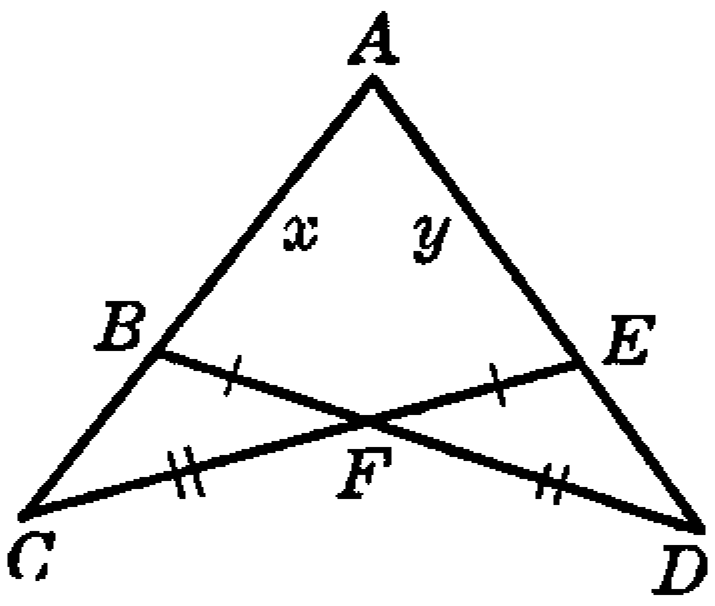
7. Prove that if two altitudes of an acute triangle are equal, the figure has two isosceles triangles.

8.



Given:
$BE = CE$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

9.



Given:
$BF = FE$
$CF = FD$
Prove:
$AB = AE$

10. *Prejudice* means prejudging. A person is prejudiced toward or against something if he judges it before he knows all the facts about it. The opposite of prejudice is called *suspended judgment.* Which form of thinking, inductive or deductive, makes greater use of suspended judgment? Give your reasons.

Useful Corollaries

The following propositions are corollaries that you may use whenever it is convenient. The first three have been proved *informally*. In like manner prove the remaining corollaries.

125. Corollary. A triangle can have but one right angle or obtuse angle.

Informal Proof (indirect). Assuming that a triangle could have two right angles or two obtuse angles, their sum would alone be 180° or more. This would contradict the theorem that the sum of the *three* angles of a triangle is 180°.

126. Corollary. The acute angles of a right triangle are complementary.

Informal Proof. The sum of the angles of a triangle is 180°, and a right triangle has one angle of 90°. Therefore, the sum of the two remaining angles is 90°.

127. Corollary. If one angle of a right triangle is 45°, the triangle is isosceles.

Informal Proof. Since the acute angles of a right triangle are complementary (§ 126) and one of them is 45°, then the other acute angle is 45°. Therefore, the triangle is isosceles because if two angles of a triangle are equal, the sides opposite are equal.

128. Corollary. The acute angles of an isosceles right triangle are each 45°.

129. Corollary. Every triangle has at least two acute angles.

130. Corollary. An equilateral triangle is also equiangular.

131. Corollary. Each angle of an equilateral triangle is 60°.

132. Corollary. An equiangular triangle is also equilateral.

133. Corollary. If one angle of an isosceles triangle is 60°, the triangle is equilateral.

134. Corollary. If two adjacent angles have their noncommon sides in a straight line, they are supplementary.

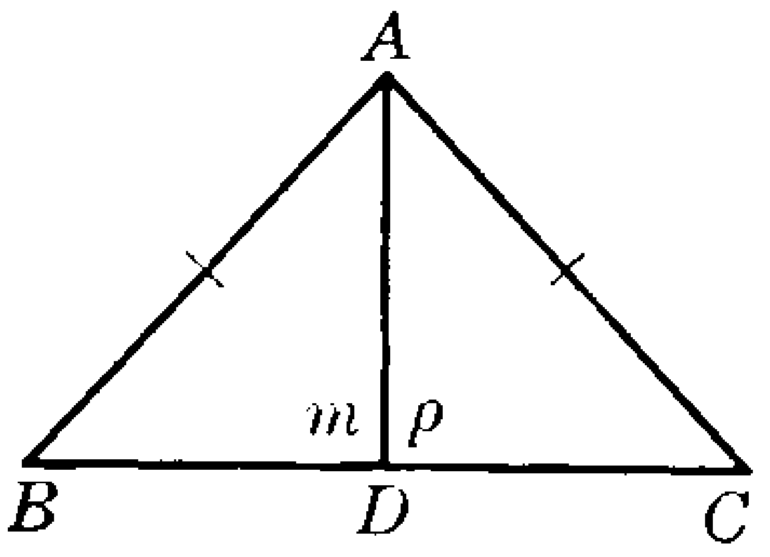
135. Corollary. If two adjacent angles are supplementary, their exterior sides form a straight line.

136. Corollary. The sum of all the angles about a point in a plane equals two straight angles.

137. Corollary. If the adjacent angles formed by two intersecting lines are equal, the lines are perpendicular to each other.

138. Corollary. In an isosceles triangle, the bisector of the vertex angle, the altitude to the base, and the median to the base are one and the same line segment.

Informal Proof. (*a*) If a line segment bisects the vertex angle, it divides the figure into two congruent triangles by s.a.s. Therefore, it bisects the base and is a median. Since it makes equal angles with the base, it is an altitude by the Corollary (§ 137).



(*b*) If the line segment is an altitude to the base, ... Complete the argument.

(*c*) If the line segment is a median to the base, ... Complete the argument.

139. Corollary. In an equilateral triangle, the angle bisectors, the medians, and the altitudes are the same line segments.

140. Corollary. All the angles of a rectangle are right angles.

SUPERIOR WORK

S1. (*a*) Draw a quadrilateral. Find the sum of the four angles by measuring the angles with a protractor, or by cutting out the four angles and placing them alongside one another so that their vertices fall on the same point.

(*b*) Now prove deductively that the sum obtained is correct. Divide the quadrilateral into triangles by drawing a diagonal, or by joining some point within the quadrilateral to all the four vertices.

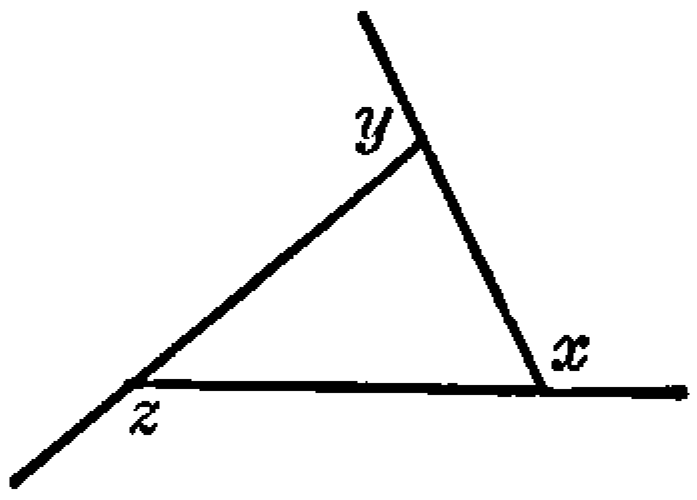
S2. Draw a pentagon (a 5-sided polygon), a hexagon (6-sided), a heptagon (7-sided), and an octagon (8-sided). Find the sum of the angles in each polygon by one of the methods suggested in the preceding exercise. Use the straight angle as a convenient unit for measuring. Copy the table below and enter the results of your experiments under the proper headings. (For the present, do not fill in the last two columns of your copy.) *Do not mark this book.*

The Sum of the Angles of a Polygon

Number of Sides	Name of Polygon	Sum of Int. ∡		Sum of Ext. ∡	
		Degrees	St. ∡	Degrees	St. ∡
3	Triangle	180	1	—?—	—?—
4	Quadrilateral	—?—	—?—	—?—	—?—
5	Pentagon	—?—	—?—	—?—	—?—
6	Hexagon	—?—	—?—	—?—	—?—
7	Heptagon	—?—	—?—	—?—	—?—
8	Octagon	—?—	—?—	—?—	—?—
10	Decagon	—?—	—?—	—?—	—?—
12	Dodecagon	—?—	—?—	—?—	—?—
n	n-gon	—?—	—?—	—?—	—?—

From your experiment try to *predict* the sum of the angles of a decagon and a dodecagon. By the method of induction, find the sum of the angles of an n-gon. Complete the following proposition: The sum of the angles of a polygon equals —?—.

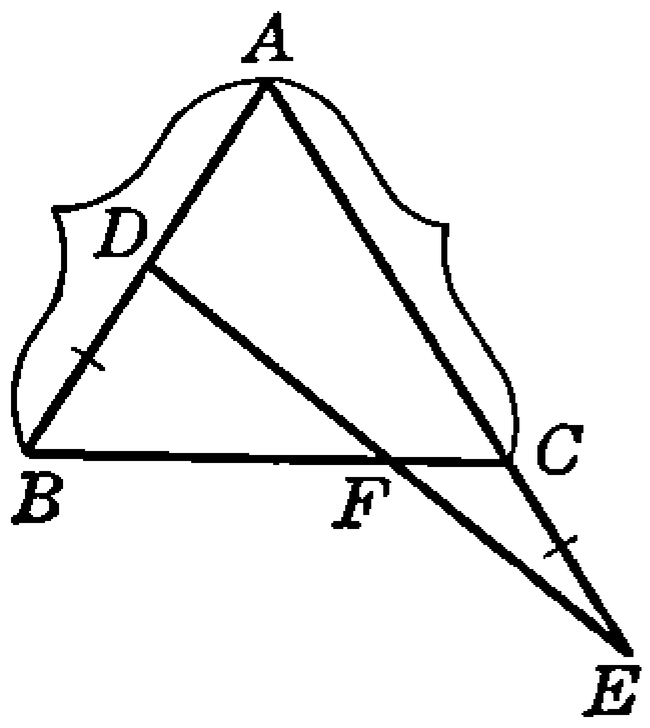
S3. Draw a triangle and find the sum of the exterior angles x, y, and z by one of the methods you used in S1 and S2. Enter this result in the last two columns of the table you made for S2. Repeat the experiment with polygons of four, five, six, seven, and eight sides and enter your results in your table. Now predict the sums of the exterior angles of a decagon and a dodecagon. By induction generalize your results for an n-gon. Complete the following proposition: The sum of the exterior angles of a polygon equals —?—.



S4. In the following problems try to discover some relationships that follow from the given facts. Although you have had enough theorems to prove these relationships, it is not necessary that you prove them all at this time. If, after some effort, you cannot discover or prove any of the relationships, put them down in your notes for continued study. Begin by drawing your figure as accurately as you can.

(*a*) The bisectors of the angles of a parallelogram form with one another a quadrilateral. What can you find out about this quadrilateral?

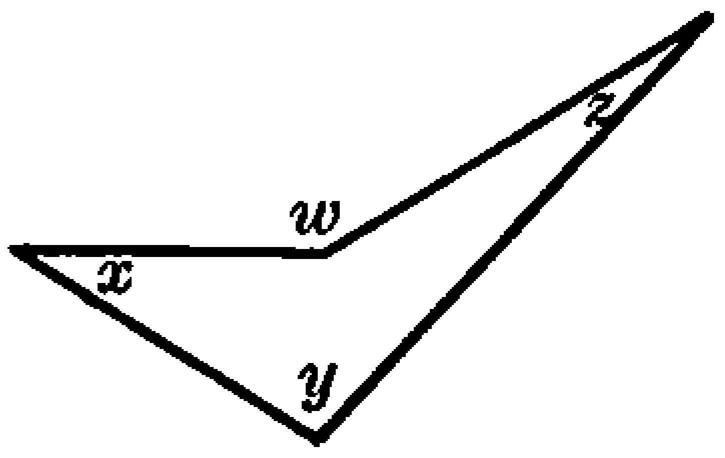
(*b*)



Given:
$AB = AC$
$BD = CE$
Prove:
Something about pt. F

(*c*) The bisector of an exterior angle of a triangle and the bisector of a nonadjacent interior angle form an angle with each other. How is this angle related to the other nonadjacent interior angle?

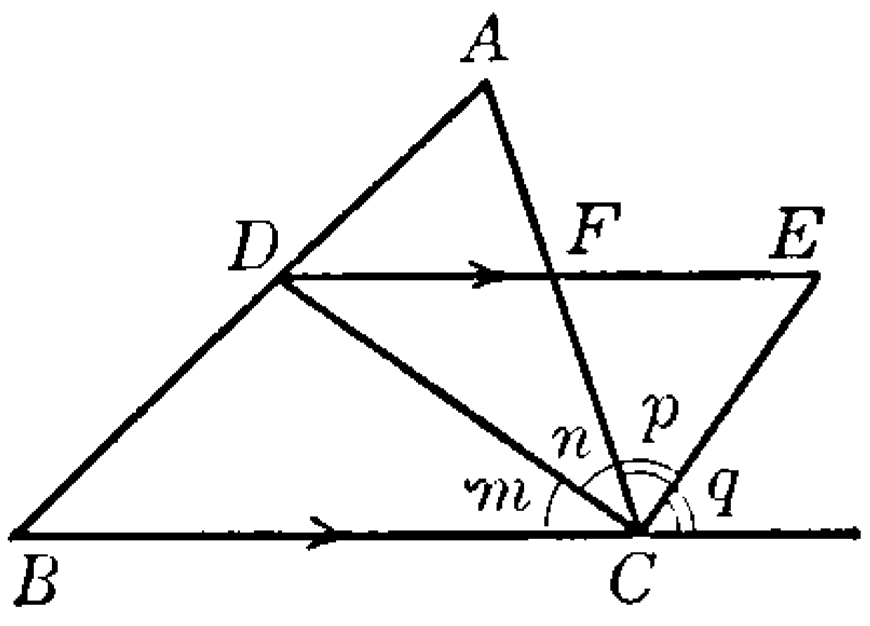
(*d*) The figure shows a *concave* quadrilateral. How are the angles w, x, y, and z related?



(*e*) The bisectors of two exterior angles of a triangle are prolonged until they meet and form an angle. How is this angle related to the third exterior angle of the triangle?

(*f*) What relationship is there between the hypotenuse and the leg of a right triangle that form an angle of 60°?

(*g*)



Given:
$DE \parallel BC$
$\angle m = \angle n$
$\angle p = \angle q$
Prove:
Something about pt. F

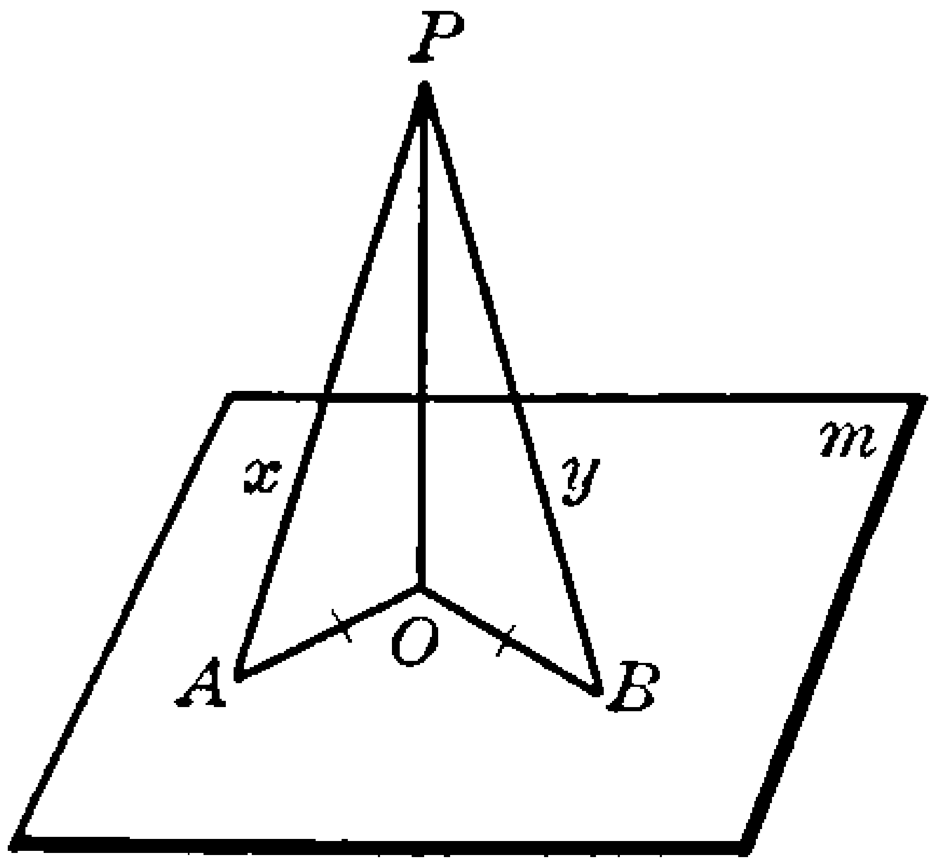
(*h*) What relationship is there between the hypotenuse of a right triangle and the median to it?

S5. (*a*) Change $\frac{1}{17}$ to a decimal, carrying the long division to 15 places. Does the decimal repeat?

(*b*) Reasoning by induction, would you say that the decimal will repeat if carried far enough?

(*c*) Can you reason by deduction that the decimal must repeat after a maximum number of places?

S6. *Theorem.* If, from any point in a line perpendicular to a plane, line segments are drawn ending in the plane at equal distances from the foot of the perpendicular, then these line segments are equal. Prove this theorem (see page 27, S4).



S7. (*a*) Prove the converse of the theorem in S6.

(*b*) In the figure above, if $OA = OB$ and $PA = PB$, does it follow that PO is perpendicular to plane m? Explain.

RECOGNIZING TYPES OF THINKING

141. Repeat each statement below, completing it by using one of the following words: *induction, inductive, deduction,* or *deductive.*

1. The conclusion of —?— reasoning is more general than the propositions from which it is derived.

2. Science makes greater use of —?— thinking.

3. A number of facts each strengthening the same conclusion is an example of —?— thinking.

4. A number of facts arranged in order, each leading to the next, and to a final conclusion is an example of —?— thinking.

5. A proposition properly arrived at by —?— can only be overthrown if we discard the underlying propositions.

6. A proposition properly established by —?— can be overthrown without disturbing the facts that lead to the proposition.

7. A proposition arrived at by —?— is not certain but only probable.

8. The —?— method depends on the accuracy of the experiments.

9. The two-point postulate depends on —?— thinking.

10. The parallel postulate depends on —?— thinking.

11. "John, Henry, and Mary did it. Why can't I?" This is an example of —?— thinking.

12. "Of course, you may think as you do. This is a free country." This is an example of —?— thinking.

13. —?— proof is generally long.

14. An advantage of —?— proof is that the facts appear in a more organized form.

15. —?— thinking makes possible a more orderly arrangement of knowledge.

16. The postulate of knowledge, that nothing happens without a cause, is the basis of —?— proof.

17. When the cause for a certain occurrence becomes known, then the thinking becomes —?—.

18. The underlying assumptions used in —?— proof depend on —?— thinking.

19. Aristotle, an ancient Greek philosopher, reasoned that heavy objects fall to the ground faster than light objects. This is an example of —?— thinking.

20. Galileo, an Italian scientist, studied the speed of falling bodies by dropping them from the Leaning Tower of Pisa. This is an example of —?— thinking.

21. In —?— thinking the conclusion is contained in the propositions from which it is derived.

22. A baseball team leads the league by 10 games with 12 games left to play. The conclusion that the team will win the pennant may be likened to —?— thinking.

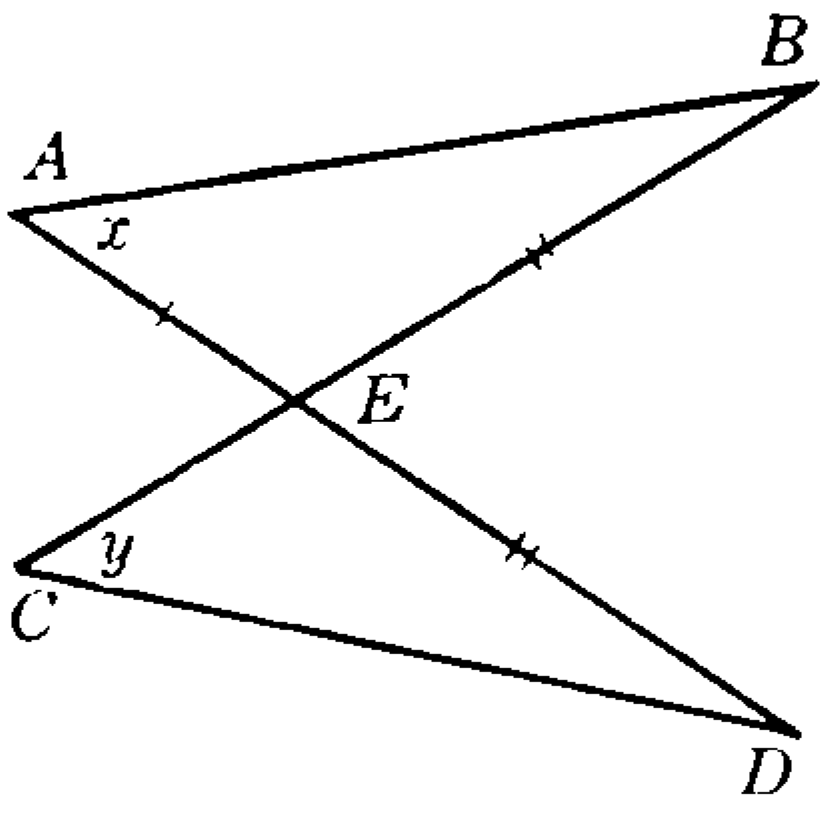
23. If the same team wins 3 more games, the conclusion may be likened to —?— thinking.

24. When a court applies a law to an individual case, it is an example of —?— thinking.

25. The reasoning that led us to adopt the Bill of Rights in our Constitution is an example of —?— thinking.

TEST

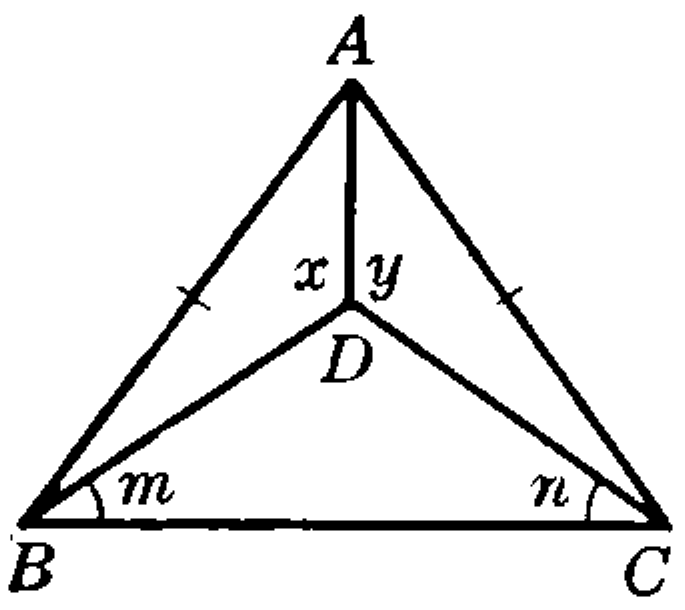
1.



Given:
$AE = EC$
$BE = ED$
Prove:
$\angle x = \angle y$

2. Prove the following proposition: If an altitude of a triangle is also an angle bisector, the triangle is isosceles.

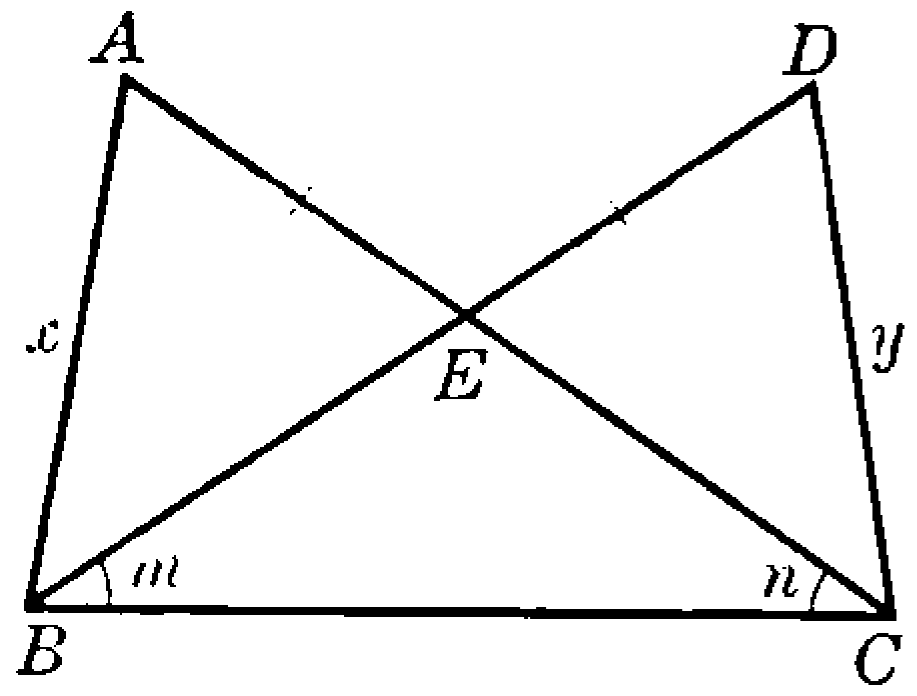
3.



Given:
$AB = AC$
$\angle m = \angle n$
Prove:
$\angle x = \angle y$

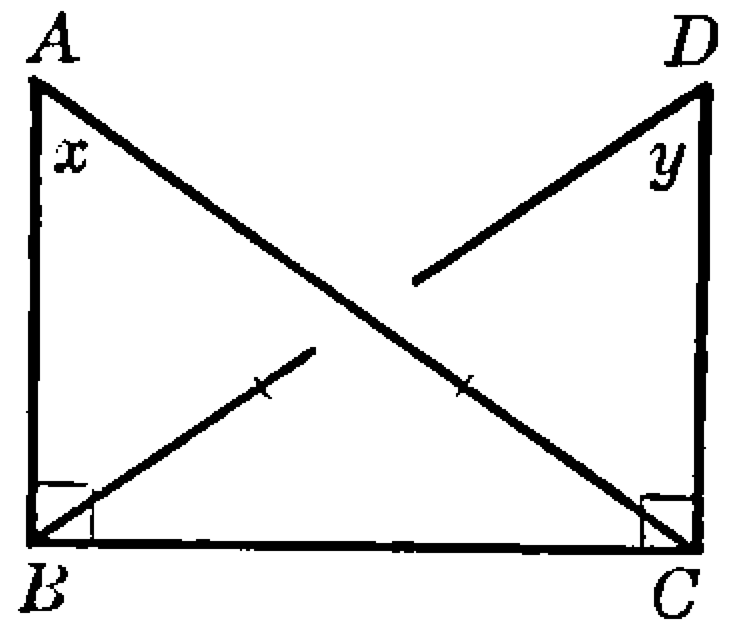
4. Prove the following proposition: Corresponding altitudes of two congruent triangles are equal.

5.



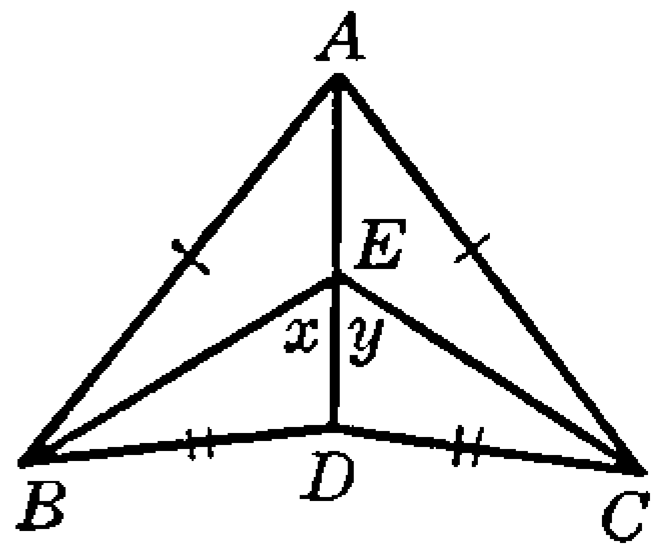
Given:
$AE = ED$
$\angle m = \angle n$
Prove:
$x = y$

6.



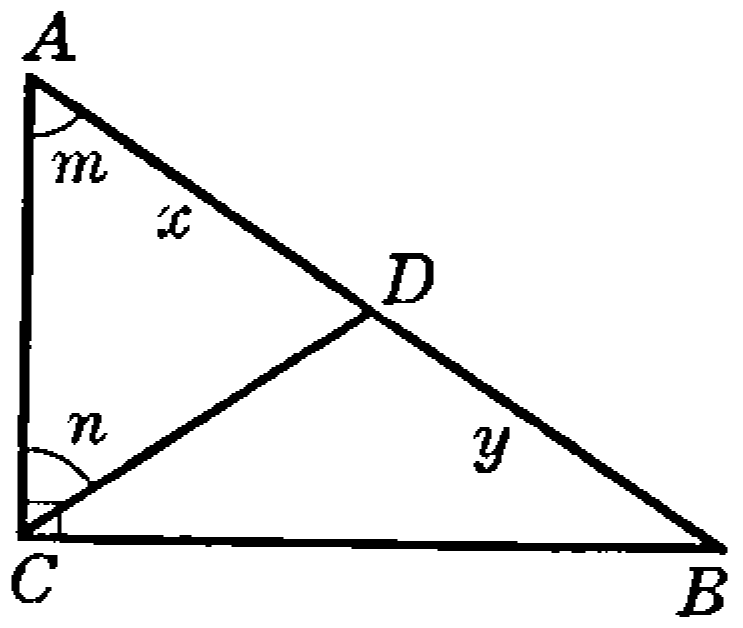
Given:
$AC = BD$
$AB \perp BC$
$DC \perp BC$
Prove:
$\angle x = \angle y$

7.



Given:
$AB = AC$
$BD = DC$
Prove:
$\angle x = \angle y$

8.



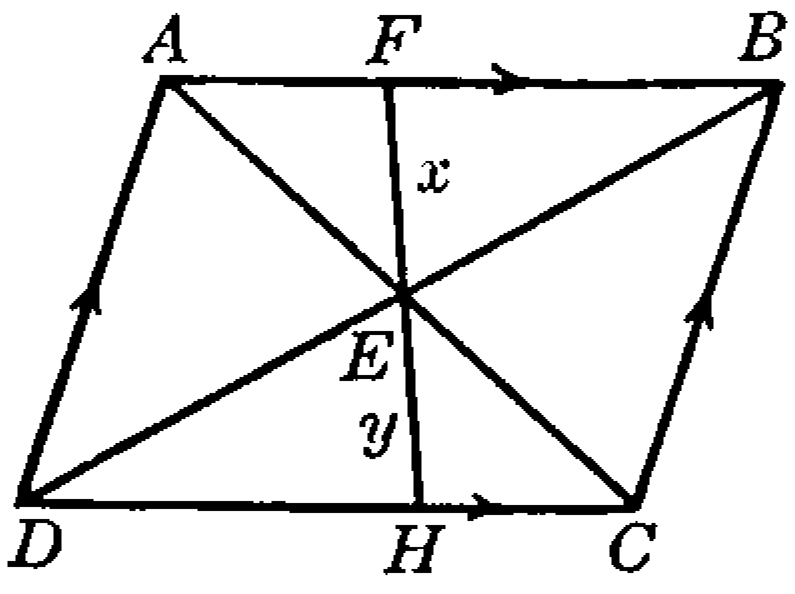
Given:
In $\triangle ABC$
$\angle m = \angle n$
$\angle C = 90°$
Prove:
$AD = DB$

Help: Prove $\triangle$s ACD and BCD isosceles.

9. $ABCD$ is a parallelogram. In proving $FE = EH$, a student proved the following triangles congruent:

(1) $\triangle ABE \cong \triangle CDE$
(2) $\triangle ABD \cong \triangle BCD$
(3) $\triangle BEF \cong \triangle DEH$

Arrange the statements in their logical

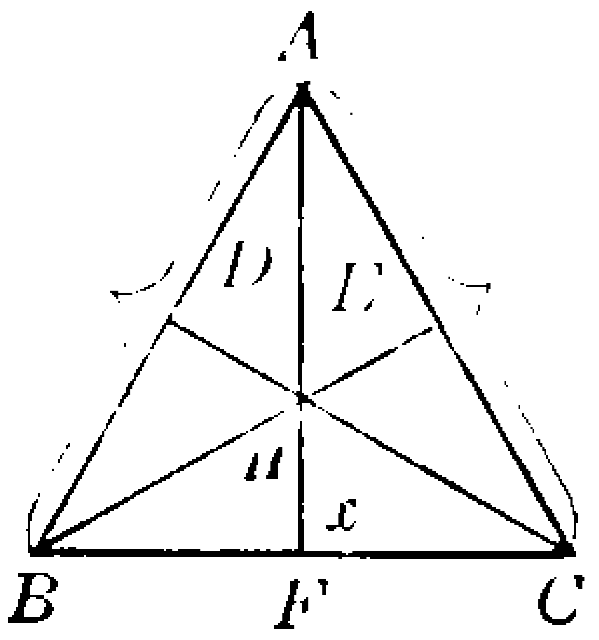


order. Give the reason for each statement.

10. In the figure, the altitudes to the legs of the isosceles triangle ABC intersect at H. To prove that a line from A through H is also an altitude, a student proved the following main statements:

(1) $\triangle BCD \sim \triangle BCE$
(2) $\triangle ABH \sim \triangle ACH$
(3) $\triangle BHC$ isosceles
(4) $\triangle ABF \sim \triangle ACF$

Arrange the statements in their logical order. Give the reason for each statement.



5

The Problem

Constructions and Quadrilaterals

CONSTRUCTIONS

What Is a Problem?

142. You have surely had a *problem* in your own life and you have had to figure out the solution. Most problems arise when we want to do something but do not know how. There are special kinds of problems in which certain facts are given and other facts must be found; problems in arithmetic and in algebra are usually of this kind. Then, in other problems, we are given all the facts and the conclusion, but we need to find the reasoning that leads to the conclusion. Most of our problems in geometry thus far have been of this type.

Locus

143. For some problems in geometry we need to construct certain figures. Such problems are construction problems. In a construction problem we must *locate* points in a certain position. For example, we may wish to locate points that are two miles from school, or points that are one block from Main Street. Usually, we find that such points are located on a certain figure, and we call this figure a *locus*. Locus is a Latin word for *place* or *location*.

144. Definition. The locus of points is a figure that contains all the points that satisfy a certain condition and no others.

The plural of *locus* is *loci*.

For example, the equator is the locus, or location, of the points on the surface of the earth that are equidistant from the North and South poles. This statement has two separate meanings, which are given below:

(1) All the points on the earth that are equidistant from the poles are on the equator.

(2) Conversely, all the points on the equator are equidistant from the poles.

The Circle as a Locus

145. Definitions. A circle is a closed plane curve of which every point is the same distance, called the radius (plural, **radii**)**, from a point within, called the center.**

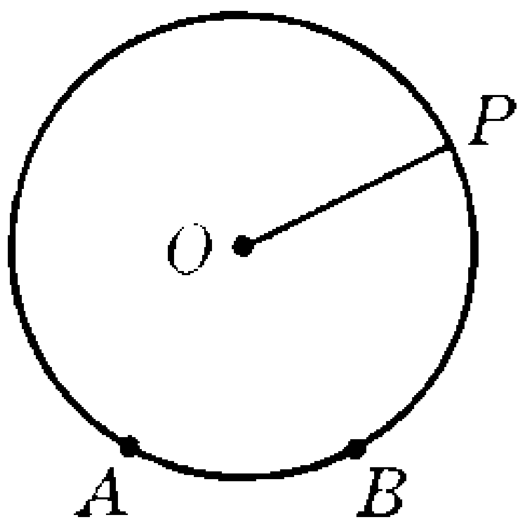
A part of a circle is called an arc.

Concentric circles are circles that have the same center.

From the definitions of a locus and a circle, the following corollary can easily be proved.

146. Corollary. A circle is the locus of points in a plane that are a given distance from a given point.

In the figure, O is the center of the circle and OP is the radius. Every point on the circle is a given distance OP from the given point O, and the circle contains all such points. In other words, the circle is the locus of the points that are a given distance OP from point O.



An arc is denoted as $\overset{\frown}{AB}$. The whole circle is denoted as $\odot O$ and sometimes as $\odot ABPA$.

Locus as the Path of a Moving Point

A locus of points has been described as a figure made up of all the points in a certain position. This means that the figure is completed and that the points are stationary. Such an idea of locus is stationary or *static*.

A locus may also be thought of as a figure that is *being formed* by a *moving point*. The idea of a locus as the path of a point that moves according to some rule is *dynamic*.

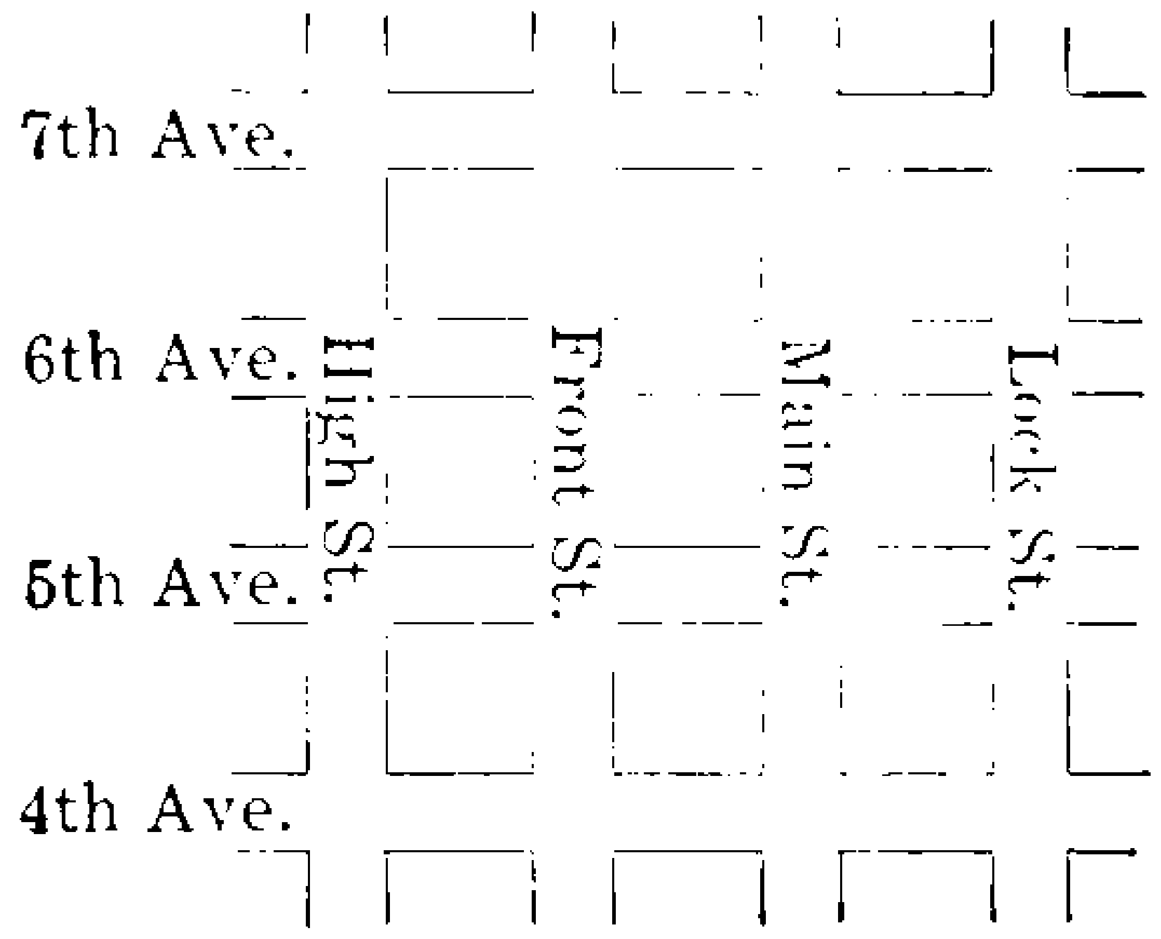
147. Definition. The (dynamic) locus of a point is the path of a point that moves while obeying some law.

Thus we may think of a circle as the path of a point that moves so that it is always the same distance from the center. When we draw a circle, we use this idea of dynamic locus.

148. Definition. Equal circles are circles that have equal radii.

EXERCISES

1. What is the locus of a point on the surface of the water two miles from a life buoy?
2. Is an arc of a circle the locus of a point that is a fixed distance from its center? Why?
3. Answer the following questions by referring to the street map at the top of the next column:
    - (*a*) Is Lock Street the locus of a point one block from Main Street?



    - (*b*) What is the locus of a point one block from the intersection of Main Street and Fifth Avenue?
    - (*c*) What is the locus of a point that is just as far from the corner of Sixth Avenue and Main Street as it is from the corner of Fourth Avenue and Main Street?
    - (*d*) What is the locus of a point that is the same distance from Lock Street as it is from Front Street?
    - (*e*) A police car patrols so that it is midway between Main and High streets. What is the locus of the patrol car?
    - (*f*) It is proposed to cut a new street of which every point would be the same distance from Seventh Avenue as it is from High Street. What would be the location (locus) of this street?

Constructions Based On Postulates

149. In the experiments in Chapter 4, we made certain drawings. We were not limited as to the kind of tools we could use in these drawings. We used a measuring scale, a compass, a protractor, a cardboard triangle, scissors, paper folding, and so on. In making the constructions of this chapter, however, we shall use only two tools: the straightedge (not a graduated ruler) and compass. The reason for this is that we wish to base our constructions on two postulates:

(1) A straight line can be drawn through two given points.

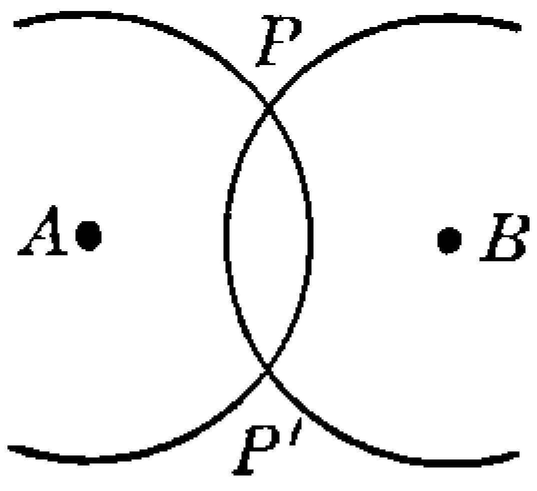
(2) A circle can be drawn if its center and radius are given.

In the two-point postulate we assumed that one and only one straight line can be drawn through two given points. All that we are now assuming is that we have the tool to draw such a straight line. This tool, of course, is the straightedge.

The second postulate means that, given the center and the radius of a circle, we have the instrument for drawing the circle. This instrument is a compass. The compass enables us to locate all the points a given distance from a given point.

When we speak of *geometric construction*, we mean a figure that is made only with a straightedge and compass, and when we are to make such a construction, we refer to it as a problem. Notice in these construction problems that what we are asked to do is written beside the word *Required*, which is often shortened to *Req.* The word *Construction* is often shortened to *Cons.*

150. PROBLEM. Locate a point that is equidistant from two given points.

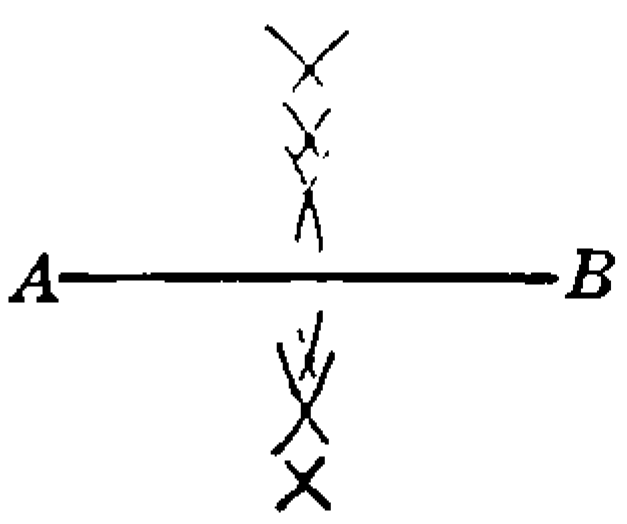


Given: Pts. A and B.
Req.: To locate a pt. equidistant from A and B

Solution

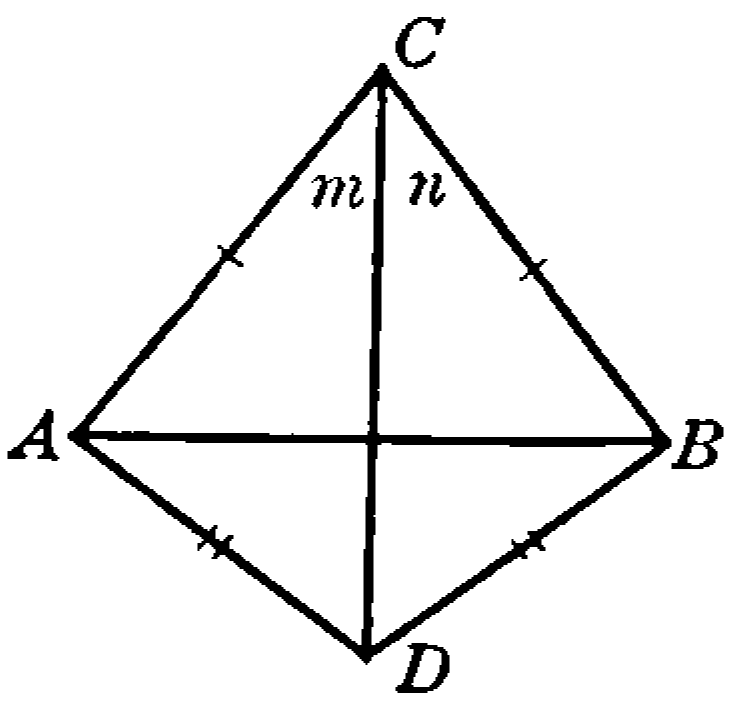
1. With A and B as centers, and the same radius, draw two arcs intersecting each other at P and P'.
2. Then P, as well as P', is equidistant from A and B.

Because we can draw an indefinite number of pairs of equal circles with A and B as centers, we have an indefinite number of points as the solution of this problem. We wish to find the locus of these points. If we construct a number of these points, as shown in the figure below, we may discover the answer to our problem by induction. We may guess that the locus is a straight line.



What can we find out by the deductive method about a straight line that connects two points of this locus, *i.e.*, two points each equidistant from A and B?

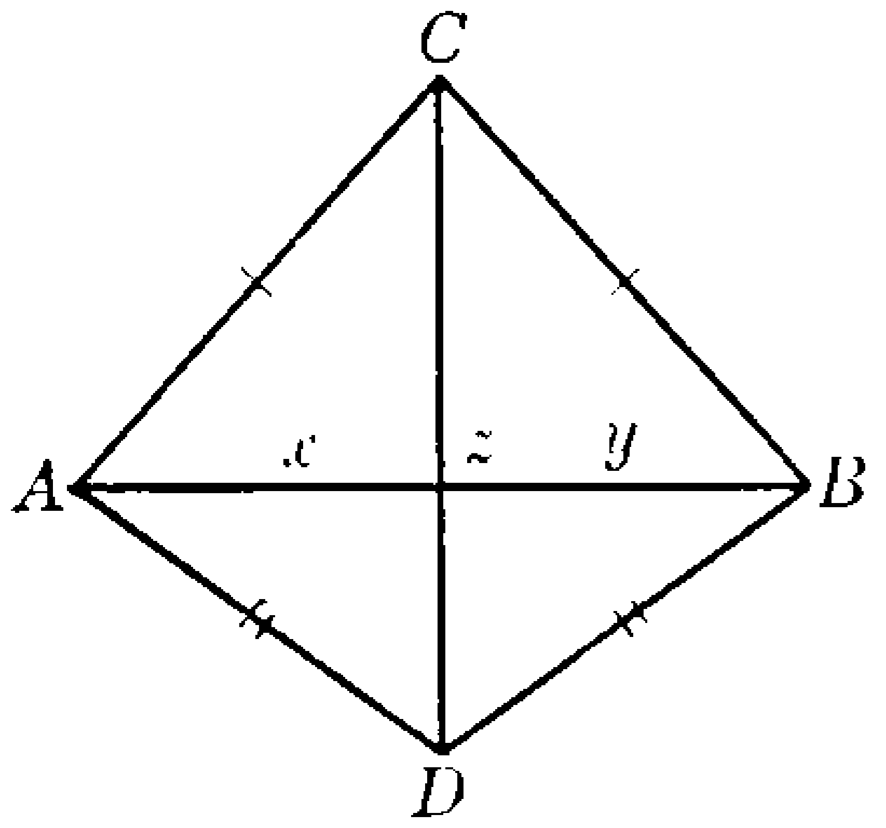
Let C and D be two points each equidistant from A and B. Then $\triangle ACD \cong \triangle BCD$ (s.s.s.) and $\angle m = \angle n$.



The bisector of the vertex angle of an isosceles triangle is the same as the median and the altitude to the base (§ 138). Consequently, CD is the perpendicular bisector of AB.

We arrive at the following important theorem.

151. TWO-POINT THEOREM. Two points each equidistant from the ends of a line segment determine the perpendicular bisector of the line segment.



Given: $AC = CB$
$AD = DB$
Prove: CD the $\perp$ bisector of AB

Write out the proof.

Help: By proving $ACD \cong BCD$, show that $\angle C$ is bisected. Then use Corollary § 138 to prove that CD contains the median and the altitude of the isosceles triangle ABC.

152. Corollary. Any point that is equidistant from the ends of a line segment is on the perpendicular bisector of the line segment.

This corollary tells us that all the points equidistant from the ends of a line segment are located on the perpendicular bisector of the line segment. Before we can say that the perpendicular bisector is the *locus* of such points, we must prove the following converse of this corollary.

153. CONVERSE THEOREM. If a point is on the perpendicular bisector of a line segment, then it is equidistant from the ends of the line segment.

Draw a figure and write out the proof.

154. LOCUS THEOREM. The locus of a point equidistant from two given points is the perpendicular bisector of the line segment connecting the two given points.

Note that this locus theorem contains the Corollary § 152 and its Converse § 153.

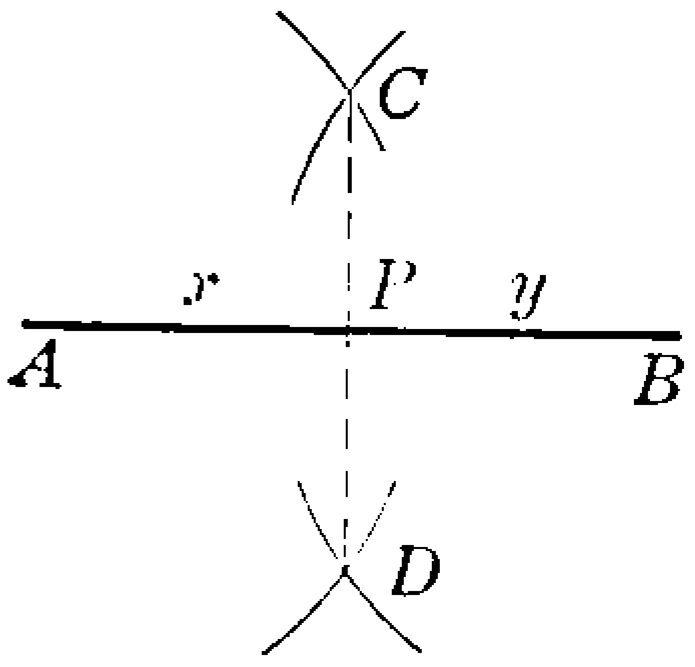
Bisection of a Line Segment

The two-point theorem can be used directly to bisect a given line segment.

In a construction problem, we first draw the given figure and state that it is given. If no figure is given, we merely write, "Given:—." Then we state what is required.

We next make the construction and describe it, making sure to use only the two postulates of construction (§ 149) or constructions that have been proved before. It is a good plan to make dotted all the straight lines that were not given. We then write the conclusion which should agree with what was required. Finally, we prove that our conclusion is correct.

155. PROBLEM. Bisect a given line segment.



Given: Line segment AB
Req.: To bisect AB

Construction

1. Locate two points C and D, each equidistant from A and B.
2. Draw CD cutting AB in P.

Conclusion: $AP = PB$.

PROOF

Statements	*Reasons*
1. C equidistant from A and B D equidistant from A and B	1. Cons.
2. CD the $\perp$ bisector of AB or, $AP = PB$	2. Two-Point Theorem (§ 151).

The point D need not be the same distance from A and B as the point C. However, it is quicker to locate C and D with the same setting of the compass.

If the construction method has been previously used, it is not necessary to describe it in detail again. Thus, in the construction above, it is merely stated that we locate C and D equidistant from A and B. The method for locating such points was described in § 150.

Although at times it is permissible to omit the description of the construction or the proof, always include in your construction three statements telling what is given, what is required, and the conclusion. Never erase the construction lines. On the contrary, make them easy to see. These lines are the evidence that you have solved the problem geometrically: that is, by the use of the two postulates of construction alone.

EXERCISES

1. Divide a given line segment into four equal parts.

2. Construct the three perpendicular bisectors of the sides of a given acute triangle. Make them long enough to meet.

3. Construct the three perpendicular bisectors of the sides of a given obtuse triangle. Make them long enough to meet.

4. Construct the three medians of a given triangle.

5. Construct a quadrilateral whose vertices are the midpoints of the sides of a given quadrilateral.

6. Which of the following five statements have the same meaning?

(*a*) Any point on the perpendicular bisector of a line segment is equidistant from the ends of the line segment.

(*b*) Any point equidistant from the ends of a line segment is on the perpendicular bisector of the line segment.

(*c*) If an altitude of a triangle is also a median, the triangle is isosceles.

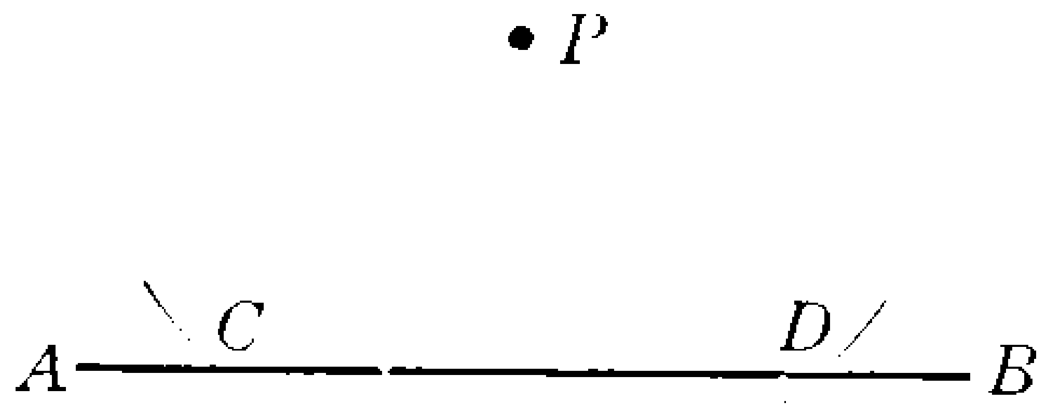
(*d*) The perpendicular bisector of the base of an isosceles triangle passes through the vertex of the triangle.

(*e*) If the perpendicular bisector of a side of a triangle passes through the opposite vertex, the triangle is isosceles.

7. Prove the following propositions:
   (*a*) If two isosceles triangles have a common base, then the line drawn through the vertices of the triangles is the perpendicular bisector of the common base.
   (*b*) The diagonals of an equilateral quadrilateral bisect each other at right angles.
8. (*a*) Prove that the point where two perpendicular bisectors of the sides of a triangle meet is equidistant from the three vertices of the triangle.
   (*b*) Construct a circle that passes through the three vertices of a given triangle.

Construction of a Perpendicular

Given a line AB and a point P, we wish to construct a perpendicular to AB that will pass through P.



The perpendicular to AB through P will be the perpendicular bisector of many line segments on AB. The first step, therefore, is to locate a line segment CD on the line AB whose ends are equidistant from P. This can be done by simply drawing an arc with P as its center, cutting AB in C and D. We then construct the perpendicular bisector of CD. Since P is already equidistant from C and D, we need locate only one more point equidistant from C and D.

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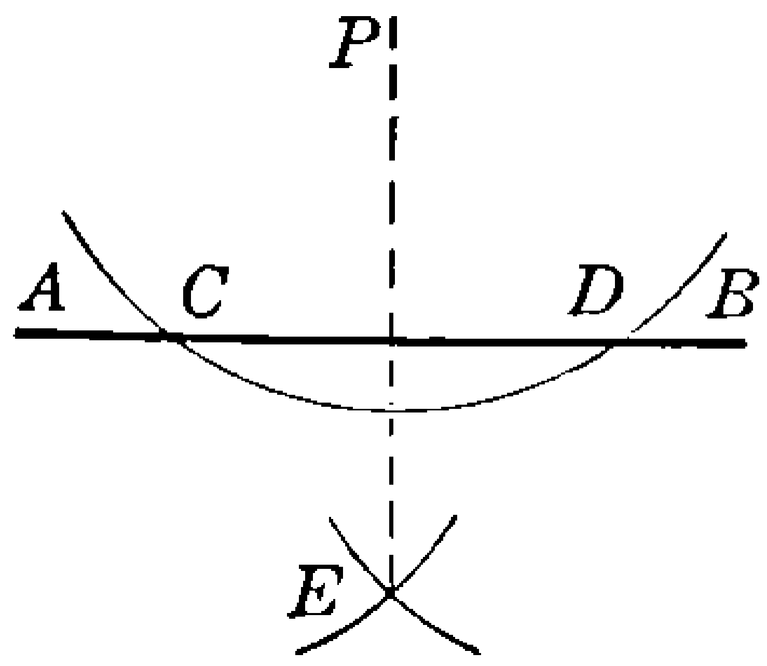
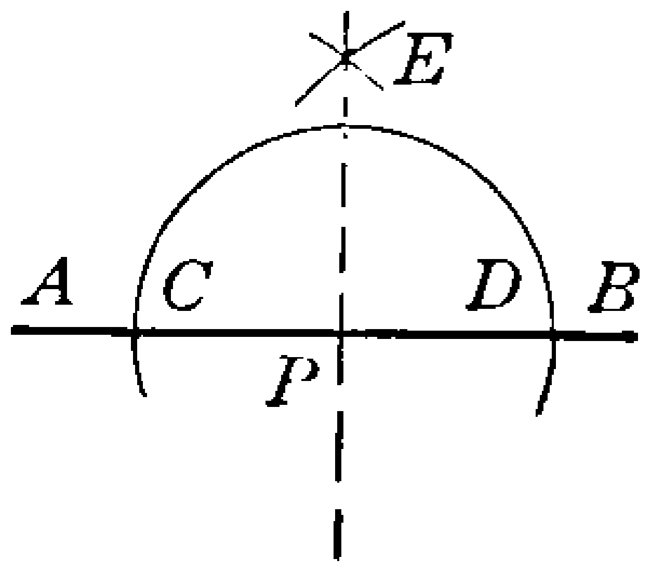
156. PROBLEM. Construct a perpendicular to a given line through a given point.





Given: Line AB and pt. P

Req.: To cons. $PE \perp AB$

The construction is the same when P is on AB or when it is outside of AB.

Construction

1. With P as a center and a radius large enough, draw an arc cutting AB in C and D.
2. Locate E equidistant from C and D.
3. Draw PE.

Conclusion: $PE \perp AB$.

PROOF

Statements	*Reasons*
1. P equidistant from C and D	1. Cons. 1.
2. E equidistant from C and D	2. Cons. 2.
3. PE the perpendicular bisector of CD or $PE \perp AB$	3. Two-Point Theorem (§ 151).

Note: Sometimes it may be necessary to prolong the given line.

EXERCISES

1. Construct the three altitudes of a given acute triangle.
2. Construct the three altitudes of a given obtuse triangle. Prolong the altitudes until they meet.
3. Construct the complement of a given angle.
4. Construct a quadrilateral, making three of the angles 90° each.
5. From the midpoints of two sides of a given triangle, construct perpendiculars upon the third side.
6. Construct an isosceles right triangle, having given one leg.
7. Construct a scalene right triangle and then construct a circle passing through its three vertices.
8. You are given an angle and a point anywhere in the same plane. With the given point as the vertex, construct another angle whose sides are perpendicular to the sides of the given angle.

SUPERIOR WORK

S1. Construct a right triangle, having given one leg and the hypotenuse.

S2. From two vertices of a given triangle, construct perpendiculars upon the median from the third vertex.

S3. Construct a triangle whose sides are perpendicular to the sides of a given triangle.

S4. Construct a quadrilateral in which two opposite angles are right angles. Then construct the perpendicular bisectors of the four sides and prolong them to meet.

S5. Through a given point not on a given line, construct a parallel to the given line. *Help:* First construct a transversal through the given point perpendicular to the given line.

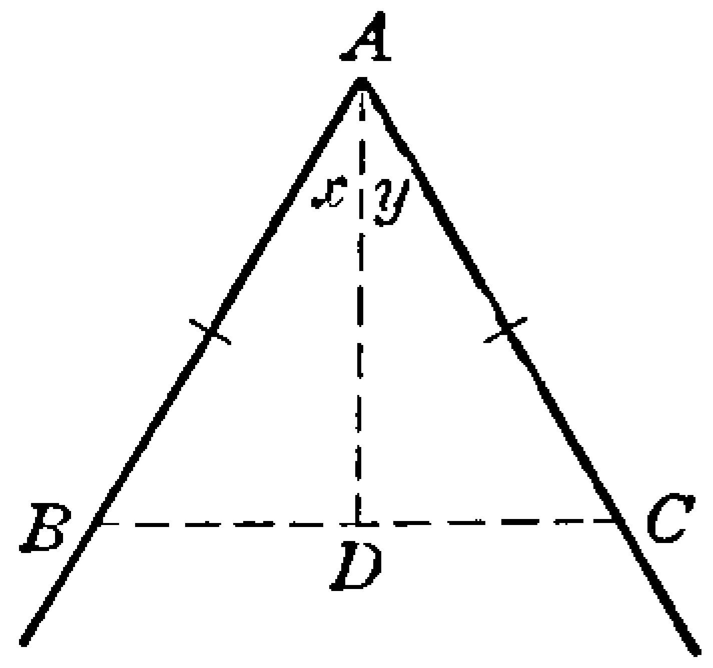
Bisection of an Angle

157. A device that is often useful in discovering the method for solving a problem is to *assume* that the problem has been solved and to examine or analyze the completed solution. This way of attacking a problem is called *analysis*. We take the solution apart so that we may study it further in detail. From the completed problem, we think back step by step and discover the method of solution. We then put the parts together again in their usual and logical order. The process of putting the solution together is called *synthesis* or a *synthetic* solution.

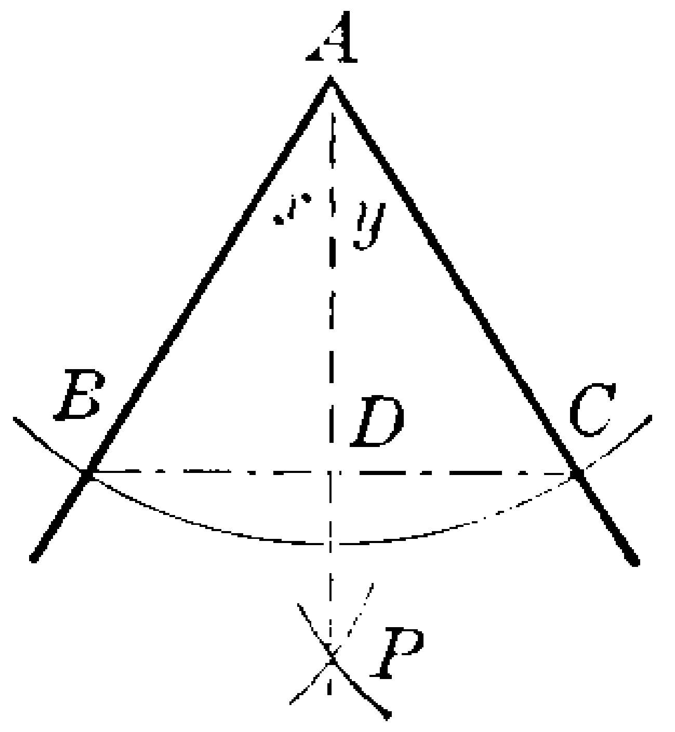
In the following problem, the use of analysis and synthesis is illustrated.

158. PROBLEM. Bisect a given angle.

Analysis. Assume that AD bisects the given angle A. Since we know a number of facts about an isosceles triangle, we draw the isosceles triangle ABC, with the given angle A as the vertex angle. From our Corollary § 138, we know that the bisector of the vertex angle and the altitude to the base are the same. The problem of constructing AD, therefore, is the same as the problem of constructing an altitude of a triangle—a problem that we have already solved. We shall therefore construct an isosceles triangle with the sides of the angle A as the legs, and then construct the altitude to the base.



Synthetic Solution



Given: ∠A
Req.: To bisect ∠A

Construction

1. With A as a center, draw an arc cutting the sides in B and C.
2. Draw BC.
3. Cons. altitude AD.

Conclusion: ∠x = ∠y.

PROOF

Statements	*Reasons*
1. AB = AC	1. Cons. 1.
2. AD the alt. to the base of the isos. △ABC	2. Cons. 3.
3. ∠x = ∠y	3. In an isos. △, the alt. to the base bisects the vertex ∠ (§ 138).

The line BC is drawn only when we want to prove the construction. Otherwise it should be omitted, as it is not needed to locate the point P.

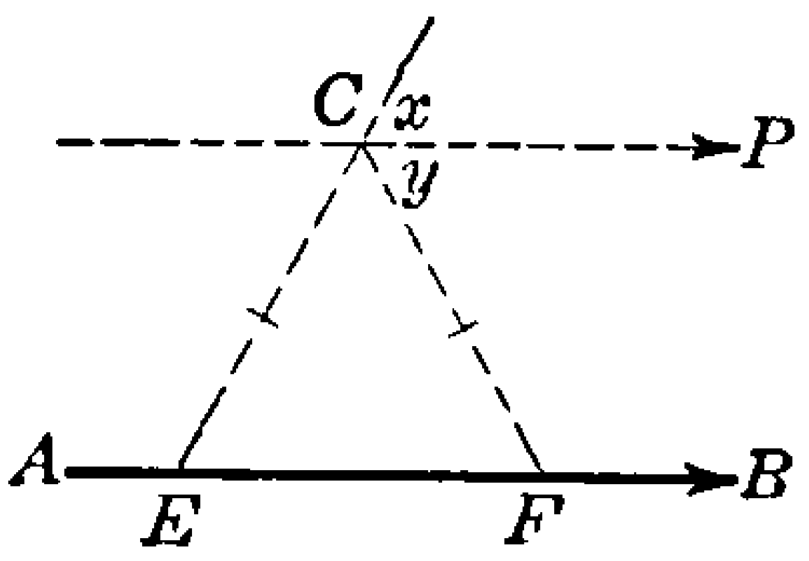
EXERCISES

1. Bisect the three angles of a given triangle and make the bisectors long enough to meet.
2. Bisect the four angles of a given quadrilateral and make the bisectors long enough to meet.
3. Divide a given angle into four equal parts.
4. Bisect each of two given adjacent supplementary angles.
5. Construct an angle of 45°.
6. Bisect two exterior angles of a given triangle and also the nonadjacent interior angle. Make the bisectors long enough to meet.

SUPERIOR WORK

S1. Through a given point C construct a parallel to a given line AB.

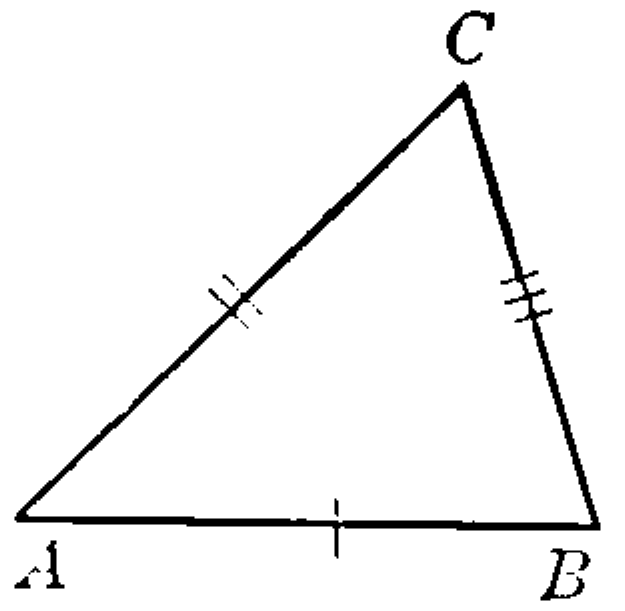
Analysis. Assume that CP ∥ AB. Draw isosceles △CEF. How are the angles x and y related to each other? Using the method suggested by this analysis, give the synthetic solution of the problem and prove it.

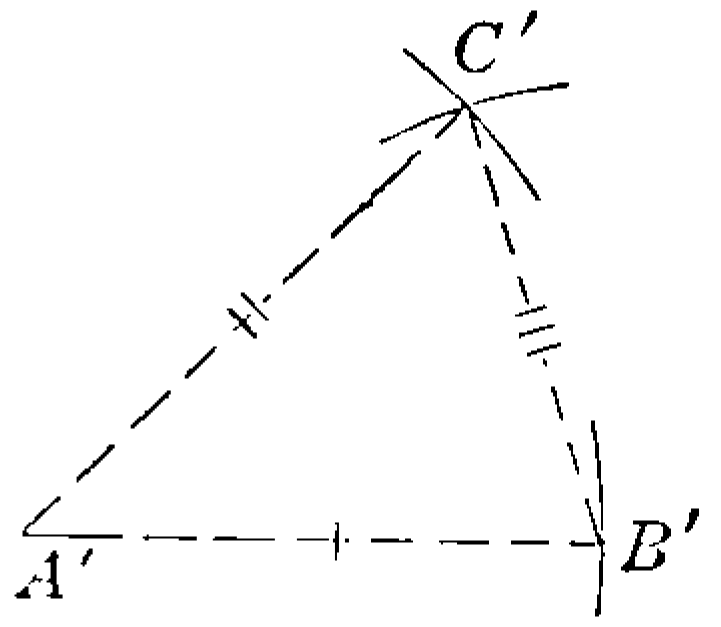


Copying a Triangle

To construct a triangle that is congruent to a given triangle, we copy the three sides as shown in the figure for § 159.

159. PROBLEM. Copy a given triangle.





Given: $\triangle ABC$
Req.: To cons. a $\triangle A'B'C' \cong \triangle ABC$

Construction

1. Cons. $A'B' = AB$.
2. With A' as center and AC as a radius, draw an arc.
3. With B' as center and BC as a radius, draw another arc cutting the first arc in C'.
4. Draw $A'C'$ and $B'C'$.

Conclusion: $\triangle A'B'C' \cong \triangle ABC$.

PROOF

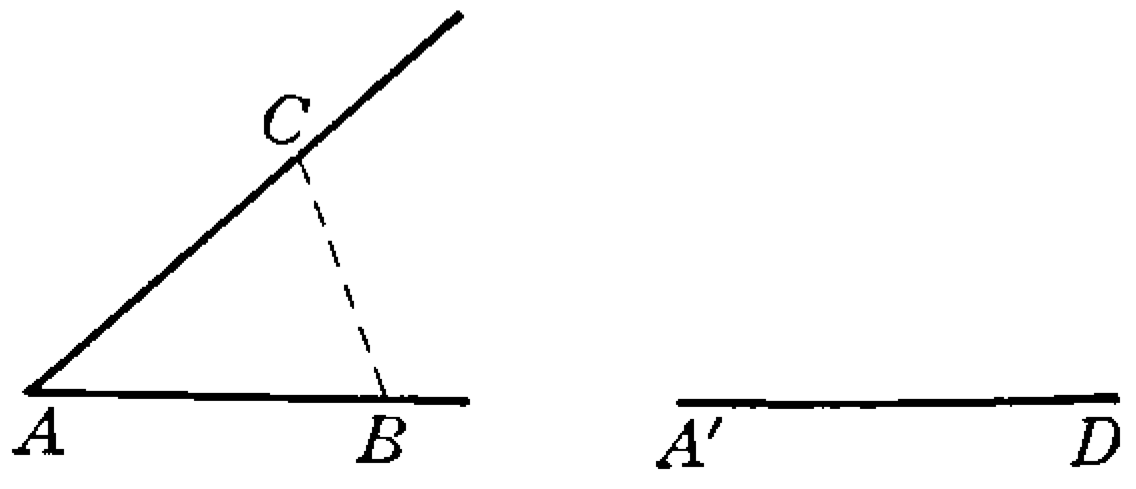
Statements	*Reasons*
1. $A'B' = AB$	1. Cons. 1.
2. $A'C' = AC$	2. Cons. 2.
3. $B'C' = BC$	3. Cons. 3.
4. $\triangle A'B'C' \cong \triangle ABC$	4. s.s.s.

Copying an Angle

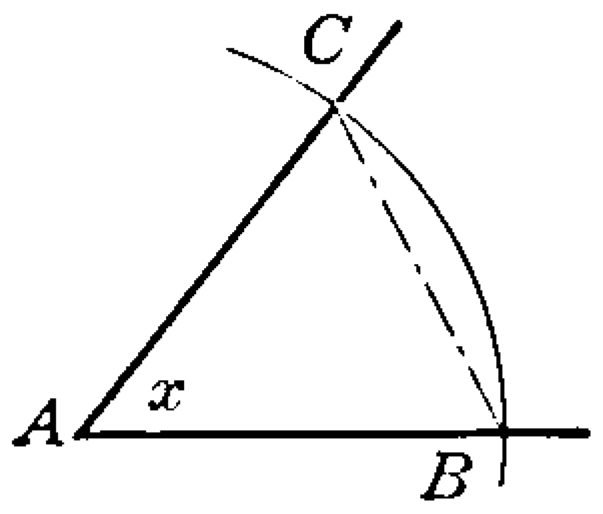
160. PROBLEM. Given one of its sides, construct an angle equal to a given angle.

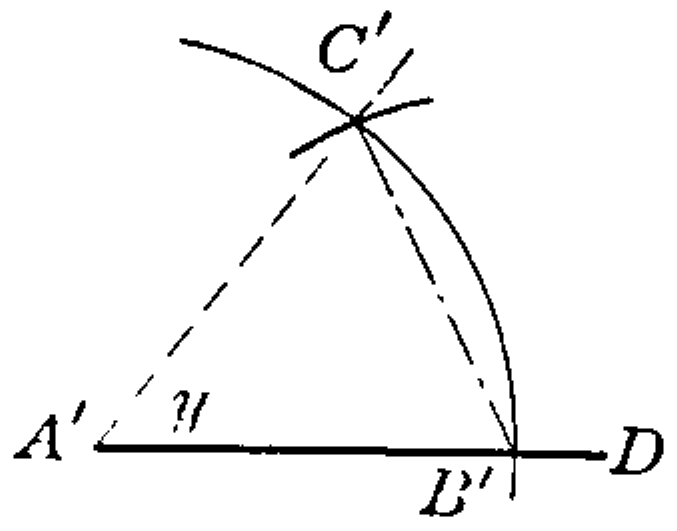
Analysis. Given $\angle A$ and a ray $A'D$, we wish to construct an angle A' equal to $\angle A$ with ray $A'D$ as one side.

We have seen that in order to prove two angles equal, we often had to prove two triangles containing these angles congruent. This is the case here. In order to copy $\angle A$, we copy the whole triangle ABC. In practice it is easier to make $AB = AC$, so that we can use the same setting of the compass.



Synthetic Solution





Given: $\angle A$ and ray $A'D$
Req.: To cons. $\angle DA'C' = \angle A$

Construction

1. Cons. isos. $\triangle ABC$.
2. Cons. $\triangle A'B'C' \cong \triangle ABC$.

Conclusion: $\angle x = \angle y$.

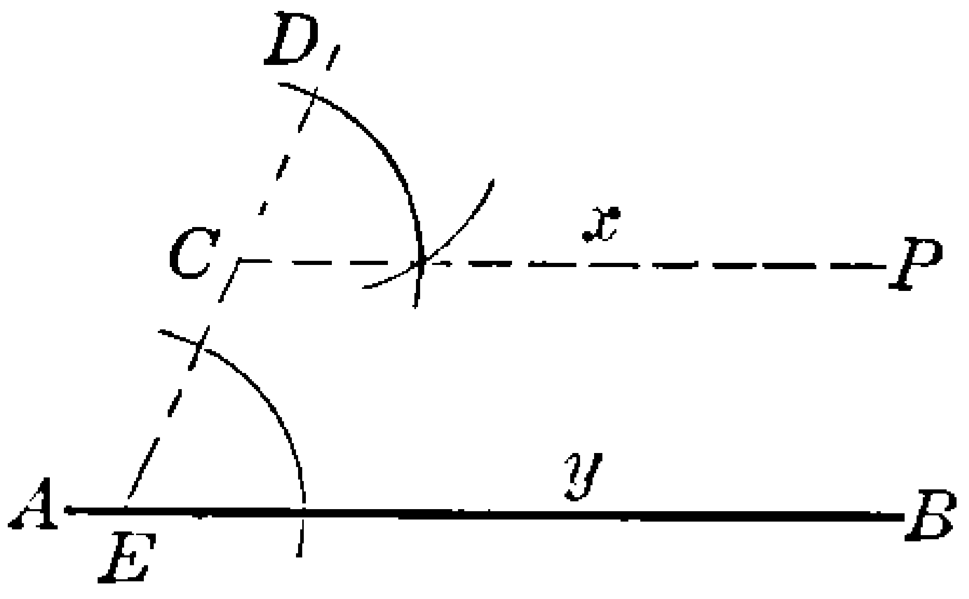
PROOF

Statements	*Reasons*
1. $\triangle ABC \cong \triangle A'B'C'$	1. Cons. 2.
2. $\angle x = \angle y$	2. Def. $\cong$ $\triangle$s.

The lines BC and $B'C'$ need be drawn only when required to prove the construction.

Constructing a Parallel

161. PROBLEM. Through a given point construct a parallel to a given line.



Given: AB and pt. C
Req.: To cons. $CP \parallel AB$

Construction

1. Through C, draw any trans. DE.
2. Cons. $\angle DCP = \angle DEB$.

Conclusion: $CP \parallel AB$.

PROOF

Statements	*Reasons*
1. $\angle DCP = \angle DEB$	1. Cons. 2.
2. $CP \parallel AB$	2. Corr. $\angle$s are =, $\parallel$s.

EXERCISES

1. Given the sides, construct a triangle.

2. Construct an equilateral triangle, given one side.

3. Construct an angle of 60°. Use as few construction lines as possible.

4. Construct an angle of 30°.

5. Construct a triangle, given two of its sides and the included angle.

6. Construct a triangle, given two of its angles and the included side.

7. Construct an angle equal to the sum of two given angles.

8. Construct a triangle, given two of its angles and a nonincluded side.

9. Construct a line that bisects one side of a given triangle and is parallel to another side.

10. Construct a parallelogram.

11. Through a given point on one side of a given triangle, construct parallels to the other two sides.

12. Construct two parallel lines, and bisect a pair of interior angles on the same side of a transversal. Make the bisectors long enough to meet.

SUPERIOR WORK

S1. Construct an equilateral quadrilateral, given one side and one diagonal.

S2. Construct a quadrilateral in which two opposite angles are supplementary.

S3. Construct a triangle, given two of its sides and a nonincluded obtuse angle.

S4. Construct a 30°–60° right triangle, given the hypotenuse.

A Point Equidistant from the Sides of an Angle

162. Definition. The distance from a point to a line is the length of the perpendicular from the point to the line.

It can be proved that the perpendicular is the shortest distance from a point to a line (§ 387).

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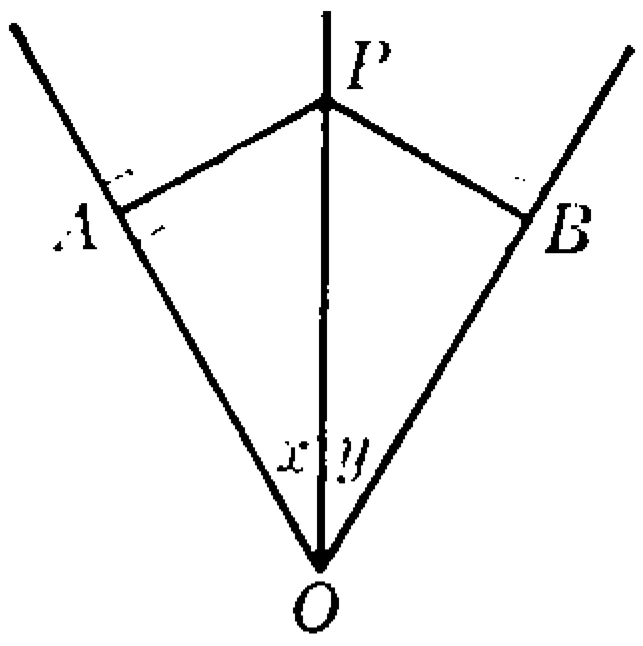
163. THEOREM. If a point is equidistant from the sides of an angle, then the line through that point and the vertex of the angle bisects the angle.



Given: P equidistant from OA and OB (that is, $\perp PA = \perp PB$)

Prove: $\angle x = \angle y$

Write out the proof.

---

164. Corollary. If a point is equidistant from the sides of an angle, it is on the bisector of the angle.

This corollary suggests that the bisector of an angle may be the locus of a point equidistant from the sides. To prove this is true, we must first prove the converse of the corollary.

---

165. CONVERSE THEOREM. If a point is on the bisector of an angle, it is equidistant from the sides of the angle.

Draw a figure and write out the proof.

To summarize Corollary 164 and its Converse 165, we have the following locus theorem.

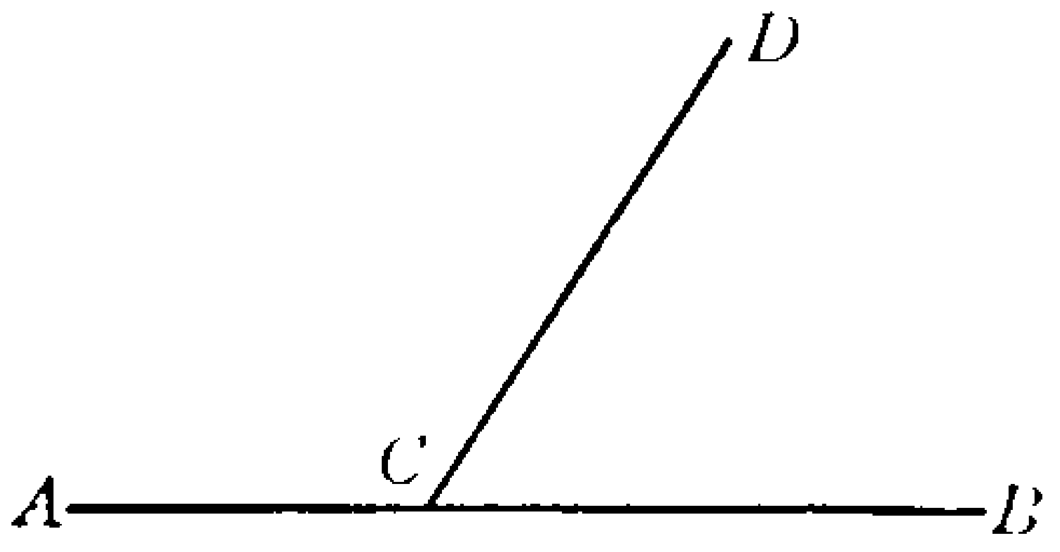
---

166. LOCUS THEOREM. The locus of a point equidistant from the sides of an angle is the bisector of the angle.

EXERCISES

1. Prove that the point where the bisectors of two angles of a triangle meet is equidistant from the sides of the third angle.
2. Prove that the bisectors of the three angles of a triangle meet in a point.
3. Construct the location of a point that is equidistant from three sides of a given quadrilateral.
4. Prove that if the bisectors of three angles of a quadrilateral meet in a point, then that point is equidistant from the four sides of the quadrilateral.

5. Construct the locus of a point equidistant from the ray CD and from the straight line AB.

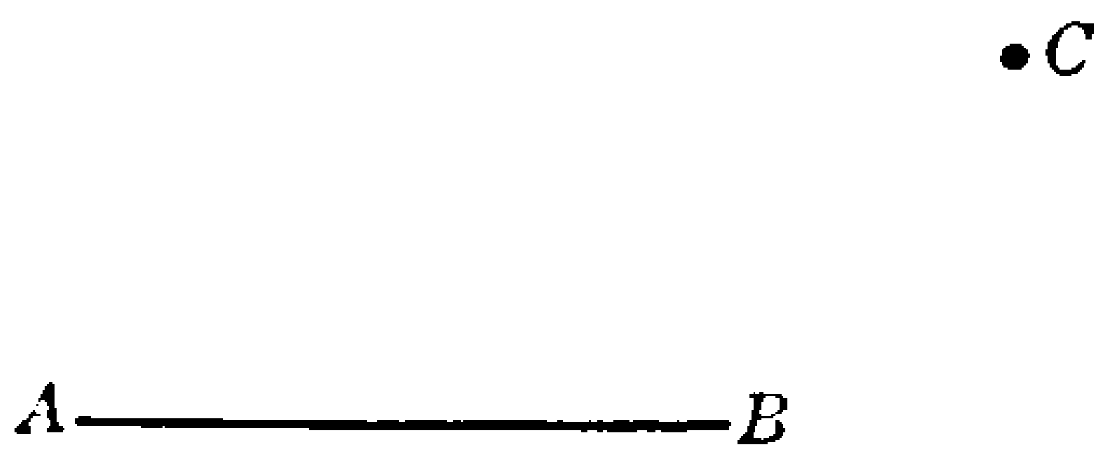


OTHER CONSTRUCTION PROBLEMS

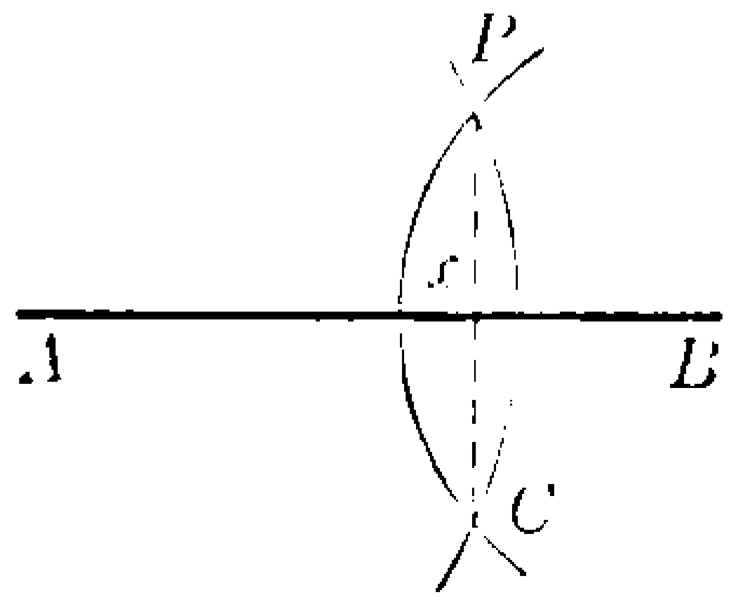
1. Construct an angle of 135°.
2. Construct an angle of $22\frac{1}{2}$°.
3. Construct half the complement of a given angle.
4. Construct a triangle, given one side, making the angle opposite this side 60°, and another angle 30°.
5. Construct a hexagon (a 6-sided polygon) making five of the angles 120° each.
6. From a given point within a given triangle, construct perpendiculars to the three sides of the triangle.
7. From a given point not on a given line, construct a line making a given angle with the given line.
8. Divide a given quadrilateral into four right triangles.

SUPERIOR WORK

S1. Draw a line through C in a direction perpendicular to AB without prolonging AB.

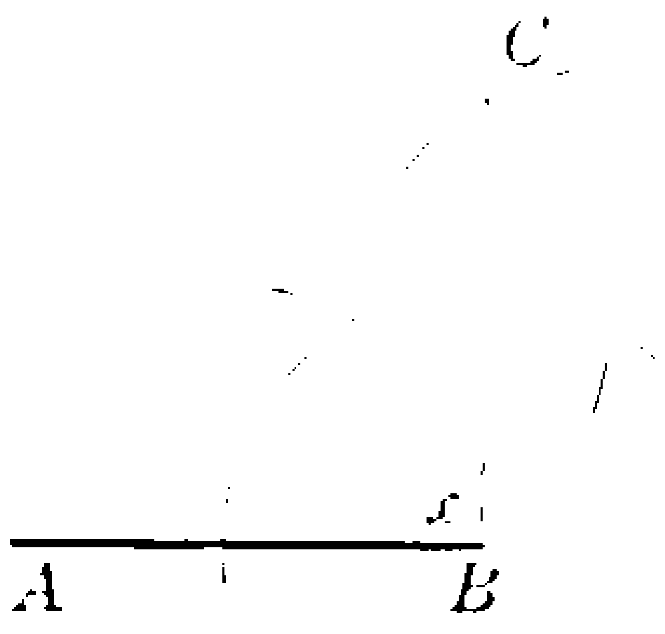


S2. Describe and prove the construction shown.



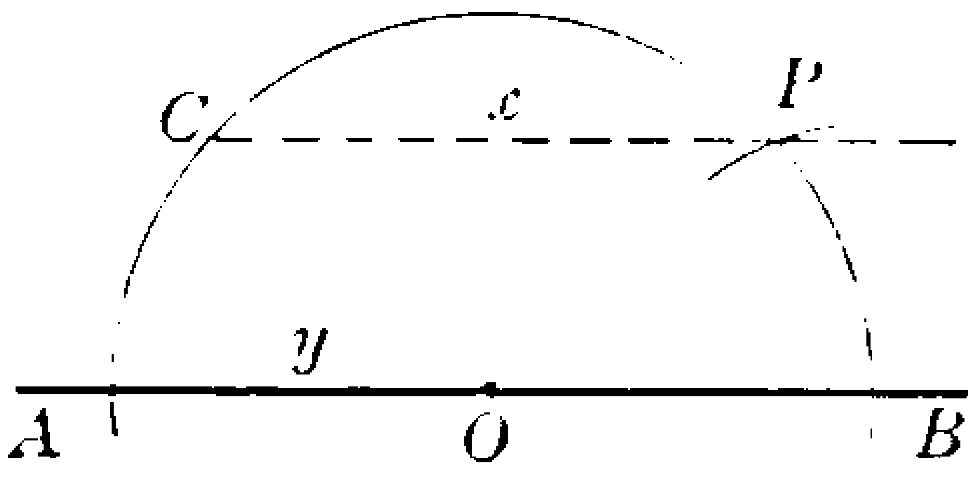
Given:
Pt. P not on AB
Req.:
To cons.
$PC \perp AB$

S3. Describe and prove the construction shown. (All the arcs were drawn with the same radius.)



Given:
Pt. B on AB
Req.:
To cons. $BC \perp AB$ without prolonging AB

S4. Describe and prove the following construction:



Given:
AB and pt. C
Req.:
To cons. $CP \parallel AB$

S5. Construct a triangle, having given two sides and the angle opposite the shorter of these sides. Is the triangle determined?

CLASSIFICATION

Sorting Things

167. We can learn facts more easily and remember them longer if we arrange them in an orderly way. When we do this we *classify* or make a *classification.* The mechanic, the storekeeper, the office worker, the housewife, or anyone who wants good organization classifies his materials. To classify means to sort into groups a number of objects that have something in common.

Objects usually have more than one thing in common. Therefore, they can be classified in more than one way. Each way will have a different purpose. For example, merchandise may be classified according to its use, such as kitchenware or sporting goods; or according to the material of which it is made, such as aluminum or silverware; or according to price, such as five-cent articles and ten-cent articles.

In geometry, triangles are classified according to their sides as scalene, isosceles, or equilateral triangles; or according to their angles as acute, obtuse, or right triangles. Theorems may be classified according to the figures with which they deal, such as triangles, quadrilaterals, or parallelograms; or according to the methods used to prove them, such as congruent triangles, parallel lines, or indirect proof.

In the solution of problems it is sometimes helpful to classify theorems according to their conclusions. This type of classification will be discussed later (§ 200).

In the following paragraphs we shall classify polygons according to some special properties. By classifying what we already know about these figures, we shall discover what new theorems we need and can prove.

The Angles of a Polygon

Keep in mind that we know something about the angles of a triangle, and that a triangle is a special kind of polygon.

We begin with the exterior angles of a polygon. We shall show that it is possible to fit these exterior angles into the space about a single point in the plane. In this way we can see that their sum is two straight angles—no matter how many sides the polygon has.

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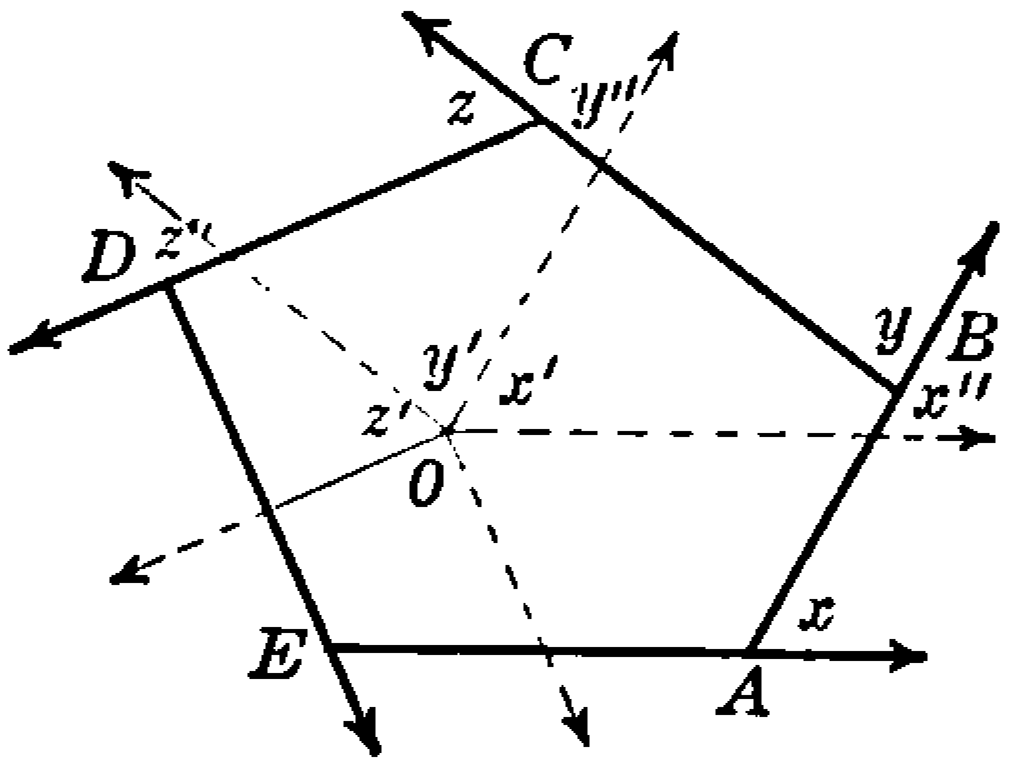
168. THEOREM. The sum of the exterior angles of a polygon is two straight angles.



Given: Polygon $ABCD \cdots$ with ext. $\measuredangle$ $x, y, z, \cdots$

Prove: $\angle x + \angle y + \angle z + \cdots = 2$ st. $\measuredangle$

Cons.: Through any point O, draw rays $\parallel$ to the sides of the polygon.

PROOF

Statements	*Reasons*
1. $\angle x' = \angle x''$, $\angle x'' = \angle x$	1. $\parallel$s, corr. $\measuredangle$ are $=$.
2. $\angle x' = \angle x$ In the same way, $\angle y' = \angle y$, $\angle z' = \angle z, \cdots$	2. Subs. Ax.
3. $\angle x' + \angle y' + \angle z' + \cdots = 2$ st. $\measuredangle$	3. Sum of all the $\measuredangle$ about a pt. $= 2$ st. $\measuredangle$ (§ 136).
4. $\angle x + \angle y + \angle z + \cdots = 2$ st. $\measuredangle$	4. Subs. Ax.

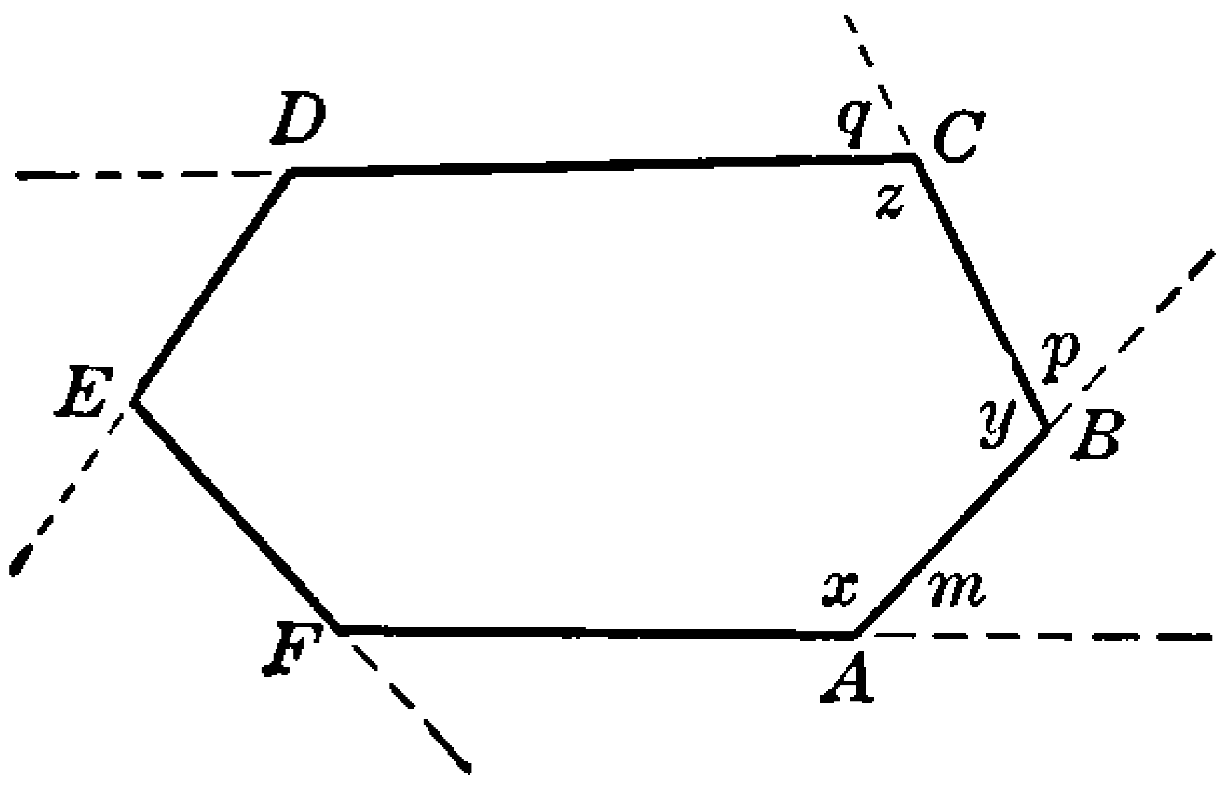
EXERCISES

1. (*a*) What is the relationship between an exterior angle of a polygon and the adjacent interior angle?
   (*b*) If the interior angles of a polygon are equal, what can you say about the exterior angles? Why?
2. (*a*) How many degrees are in each exterior angle of an equiangular quadrilateral? equiangular pentagon (5 sides)? equiangular hexagon (6 sides)? equiangular octagon (8 sides)? equiangular triangle?
   (*b*) How many degrees are in each interior angle of the figures named in (*a*)?
3. (*a*) How many sides does a polygon have if each exterior angle is 20°? 40°? 36°?
   (*b*) How many sides does a polygon have if each interior angle is 150°? 170°? 156°?
4. (*a*) Is it possible to draw a polygon in which each exterior angle is 16°? 8°? 80°?
   (*b*) Is it possible to draw a polygon in which each interior angle is 162°? 130°? 179°?
5. (*a*) What is the sum of all the interior *and* the exterior angles of an octagon? decagon (10 sides)? dodecagon (12 sides)? *n*-gon (a polygon of *n* sides)?
   (*b*) What is the sum of the interior angles alone in each figure named in (*a*)?

The Interior Angles of a Polygon

From the exercises above you have now arrived at the proposition that the sum of the interior angles of a polygon is as many straight angles as the polygon has sides less two. The form that follows will show you how to write out a proof about *n* things, where *n* is *any* number.

169. THEOREM. The sum of the angles of a polygon of *n* sides equals $(n - 2)$ straight angles.



Given: Polygon $ABCD \cdots$ having n sides

Prove: $\angle x + \angle y + \angle z + \cdots = (n - 2)$ st. $\measuredangle$

Cons.: Prolong each side of the polygon.

PROOF

Statements	*Reasons*
1. $\angle x + \angle m = 1$ st. $\angle$ $\angle y + \angle p = 1$ st. $\angle$ $\angle z + \angle q = 1$ st. $\angle$ $\cdots$	1. Def. st. $\angle$.
2. $\angle x + \angle y + \angle z + \cdots + \angle m + \angle p + \angle q + \cdots = n$ st. $\measuredangle$	2. Add. Ax.
3. $\angle m + \angle p + \angle q + \cdots = 2$ st. $\measuredangle$	3. Sum of the ext. $\measuredangle$ of a polygon = 2 st. $\measuredangle$ (§ 168).
4. $\angle x + \angle y + \angle z + \cdots = (n - 2)$ st. $\measuredangle$	4. Subt. Ax.

THE PARALLELOGRAM

Necessary Conditions for a Parallelogram

170. From experimental work, as well as from exercises that we proved, we realize that if a figure is a parallelogram, it is *necessary* that these four conditions be true:

(1) The opposite sides are parallel.
(2) The opposite sides are equal.
(3) The opposite angles are equal.
(4) The diagonals bisect each other.

From the definition of a parallelogram, we know that the first condition is true. We shall now prove that the other three conditions are also true in every parallelogram.

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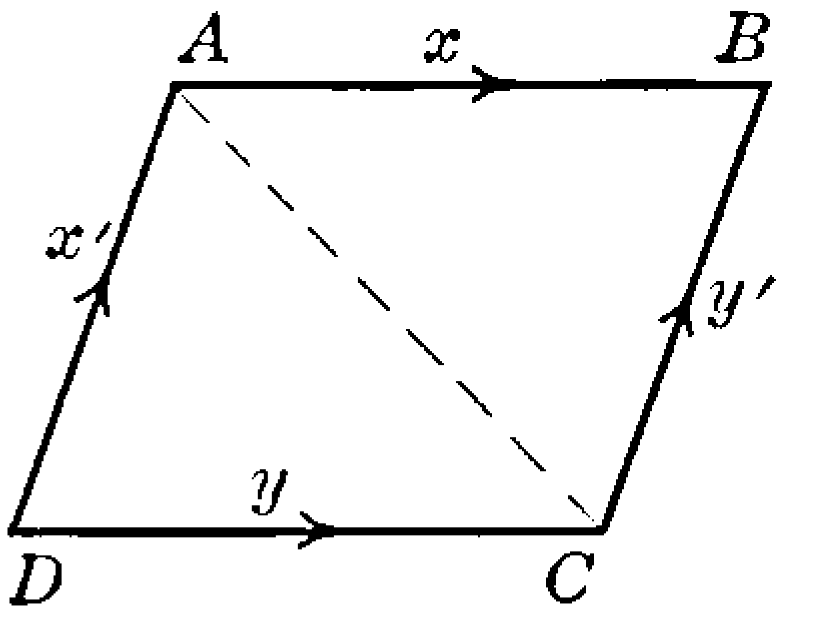
171. THEOREM. The opposite sides of a parallelogram are equal.



Given: $\square ABCD$
Prove: $AB = DC$ and $AD = BC$
Cons.: Draw AC.

Write out the proof.

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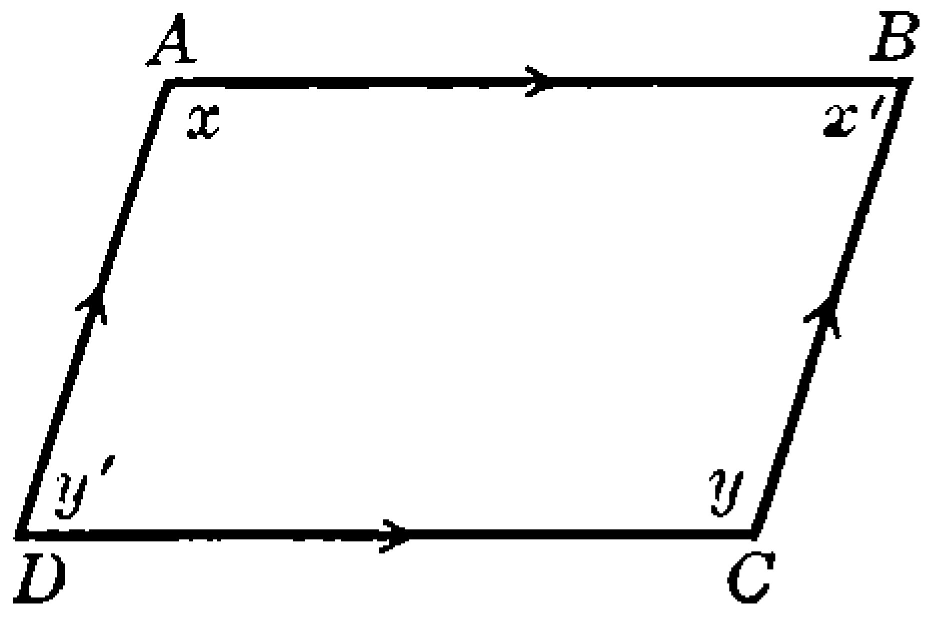
172. THEOREM. The opposite angles of a parallelogram are equal.



Given: $\square ABCD$
Prove: $\angle A = \angle C$ and $\angle B = \angle D$

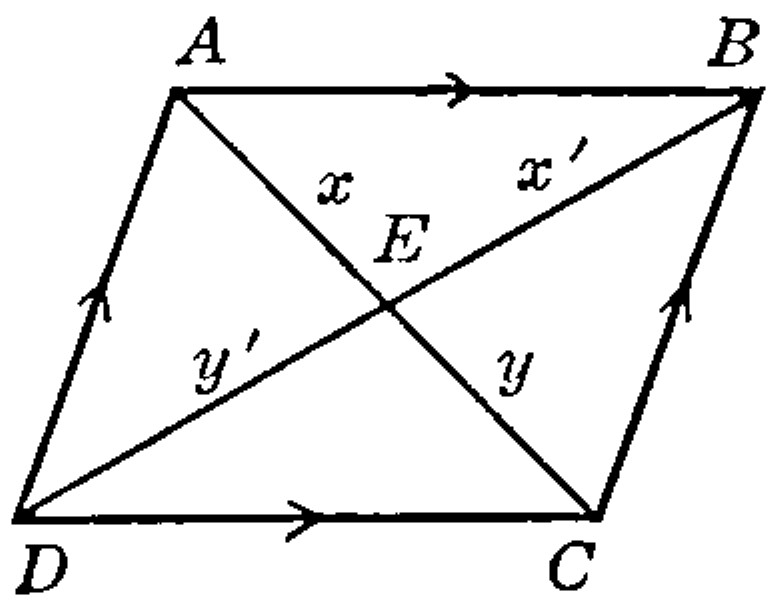
PROOF

Statements	*Reasons*
1. $ABCD$ a $\square$	1. Given.
2. $AB \parallel DC$	2. Def. $\square$.
3. $\angle A + \angle D = 180°$	3. $\|$s, int. $\angle$s sup.

Complete the proof.

---

173. THEOREM. The diagonals of a parallelogram bisect each other.



Given: $\square ABCD$
Prove: $AE = EC$, $BE = ED$

Write out the proof.

Sufficient Conditions for a Parallelogram

174. The words *necessary* and *sufficient* are used in mathematics in the same way as in everyday language. Learn how to use them correctly, and you will find them useful in classifying your facts and in thinking about them clearly. For example, in the simple proposition that vertical angles are equal, the equality of the angles is a *necessary* condition for vertical angles. On the other hand, the condition that the angles are vertical is *sufficient* to make them equal.

A necessary and sufficient condition that two triangles be congruent is that they agree in all their parts. This, in fact, is our definition of congruent triangles. Every definition expresses a necessary and sufficient condition. On the other hand, the condition that two triangles agree in three parts taken at random is necessary if the triangles are congruent, but it is not sufficient to make them congruent.

In a parallelogram certain conditions are necessary. What conditions are sufficient to prove that a quadrilateral is a parallelogram? In other words, what is the least amount of information that we must have in order to prove that a quadrilateral is a parallelogram? From our definition of a parallelogram, we know that two pairs of parallel sides are sufficient to make a quadrilateral into a parallelogram. Are *any* two conditions of the four types listed on page 118 as necessary also sufficient?

In order to set up a plan of study, we list all the possible combinations of these conditions, taking two at a time. Not all of these twelve combinations are sufficient.

1. Two pairs of opposite sides parallel.
2. Two pairs of opposite sides equal.
3. Two pairs of opposite angles equal.
4. Two diagonals bisecting each other.
5. One pair of opposite sides parallel; same sides equal.
6. One pair of opposite sides parallel; other pair of sides equal.
7. One pair of opposite sides parallel; one pair of opposite angles equal.
8. One pair of opposite sides parallel, one diagonal bisected by the other diagonal.
9. One pair of opposite sides equal; one pair of opposite angles equal.
10. One pair of opposite sides equal; one diagonal bisected by the other diagonal.
11. One pair of opposite angles equal; the diagonal through the vertices of these angles bisected by the other diagonal.
12. One pair of opposite angles equal; the diagonal through the vertices of these angles bisects the other diagonal.

Combination 1 is sufficient because of the definition of a parallelogram. We shall now prove that the next four combinations are also sufficient.

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175. THEOREM. If the opposite sides of a quadrilateral are equal, the figure is a parallelogram (Combination 2).

Draw a figure and write out the proof.

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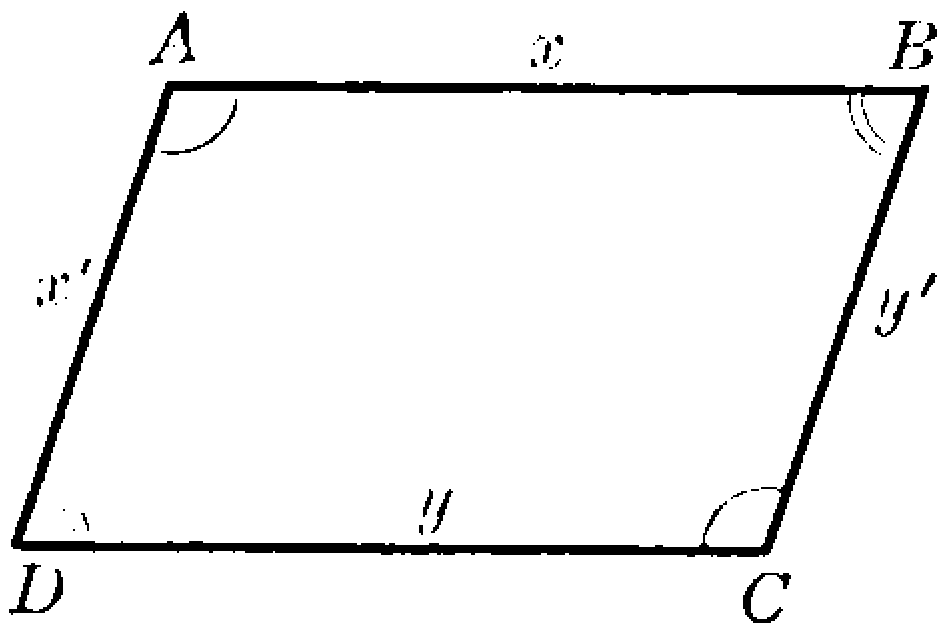
176. THEOREM. If the opposite angles of a quadrilateral are equal, the figure is a parallelogram (Combination 3).



Given: $\angle A = \angle C$
$\angle B = \angle D$
Prove: $ABCD$ a ▱

PROOF

Statements	*Reasons*
1. $\angle A + \angle B + \angle C + \angle D = 2$ st. $\angle$s	1. Sum of the $\angle$s of a polygon $= (n - 2)$ st. $\angle$s (§ 169).
2. $\angle A = \angle C$ $\angle B = \angle D$	2. Given.
3. $\angle A + \overset{\angle D}{(\angle B)} + \overset{\angle A}{(\angle C)} + \angle D = 2$ st. $\angle$s or, $2 \cdot \angle A + 2 \cdot \angle D = 2$ st. $\angle$s	3. Subs. Ax.
4. $\angle A + \angle D = 1$ st. $\angle$	4. Div. Ax.
5. $x \parallel y$	5. Int. $\angle$s sup., $\parallel$s.
6. $\angle A + \overset{\angle B}{(\angle D)} = 1$ st. $\angle$	6. Subs. Ax.
7. $x' \parallel y'$	7. Same as 5.
8. $ABCD$ a ▱	8. Def. ▱.

177. THEOREM. If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram (Combination 4).

Draw a figure and write out the proof.

178. THEOREM. If two sides of a quadrilateral are equal and parallel, the figure is a parallelogram (Combination 5).

Draw a figure and write out the proof.

Do you see that stating the conditions helps us to discover propositions about the parallelogram?

EXERCISES

1. Prove that all the sides of a rhombus are equal.
2. Prove that an equilateral quadrilateral is a rhombus.
3. Prove that perpendiculars to one of two parallel lines included by the parallels are equal.
4. Prove that if three angles of a quadrilateral are right angles, the figure is a rectangle.
5. Prove that if two opposite angles of a parallelogram are supplementary, the figure is a rectangle.

6. Complete the following statements by supplying the correct number:

(*a*) A necessary condition for a figure to be a rectangle is that it contain —?— right angle(s).

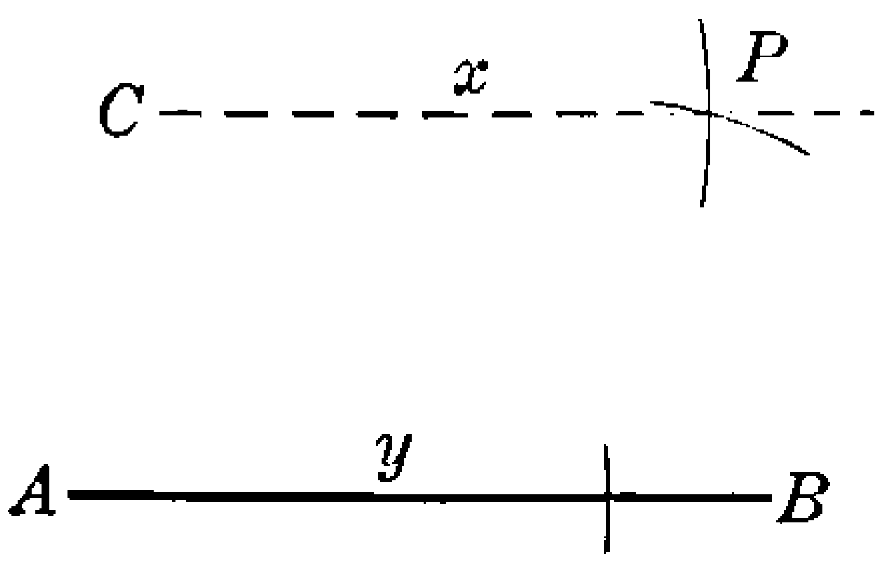
(*b*) A sufficient condition for a quadrilateral to be a rectangle is that it contain —?— right angle(s).

(*c*) A sufficient condition for a parallelogram to be a rectangle is that it contain —?— right angle(s).

7. Complete the following statement by using the word *necessary*, or *sufficient*, or both:

A —?— condition for a quadrilateral to be a rhombus is that the four sides be equal. Justify your statement.

8. Describe and prove the method indicated for constructing a line parallel to a given line through a given point.



Given:
AB and pt. C
Req.:
To cons.
$CP \parallel AB$

This construction is usually more accurate than the construction in § 161. Why?

9. Draw a figure to illustrate that Combination 6 in § 174 is not sufficient to prove that a quadrilateral is a parallelogram. What kind of figure may result from these conditions?

10. Prove that Combination 7 in § 174 is sufficient for a parallelogram.

11. Prove that Combination 8 in § 174 is sufficient for a parallelogram.

SUPERIOR WORK

S1. Draw a figure to illustrate that Combination 11 in § 174 is not sufficient to make a parallelogram.

S2. (*a*) What is a necessary condition that the diagonals of a rectangle (but not of a parallelogram) must satisfy? Prove it.

(*b*) Prove that this condition is sufficient to make a parallelogram into a rectangle.

HONOR WORK

H1. Prove that Combination 9 in § 174 is sufficient for a parallelogram if the equal angles are not acute.

H2. Prove that Combination 10 in § 174 is not sufficient for a parallelogram.

Note: In this and preceding exercises you have solved problems concerning all the combinations in § 174 except Combination 12. A problem concerning Combination 12 will be found on page 166, H2.

Corollaries about Parallelograms

The following are useful corollaries about parallelograms. You have already proved some of them as exercises. Prove the remaining ones informally. (See page 99.)

179. Corollary. A rhombus is equilateral.

180. Corollary. An equilateral quadrilateral is a rhombus.

181. Corollary. The diagonals of a rhombus are perpendicular to each other.

182. Corollary. If the diagonals of a parallelogram are perpendicular to each other, the figure is a rhombus.

183. Corollary. If three angles of a quadrilateral are right angles, the figure is a rectangle.

'184. Corollary. Line segments included between parallels and perpendicular to one of the parallels are equal.

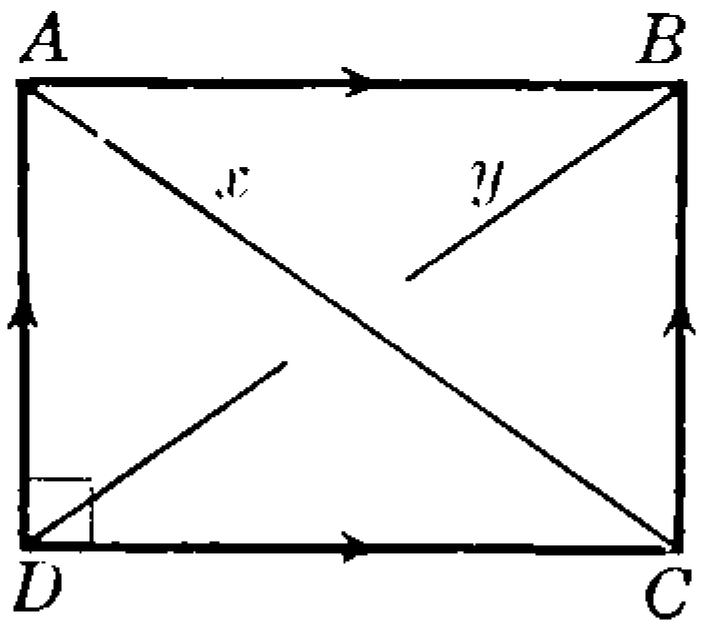
185. Corollary. All the altitudes of a trapezoid are equal.

186. Corollary. In a parallelogram all the altitudes to the same base are equal.

187. Corollary. Parallel lines are everywhere equally distant.

188. Corollary. A diagonal of a parallelogram divides the figure into two congruent triangles.

189. THEOREM. The diagonals of a rectangle are equal.



Given: □$ABCD$
Prove: $AC = BD$

PROOF

Statements	*Reasons*
1. $ABCD$ a □	1. Given.
2. $ABCD$ a ▱	2. Def. □.
3. C and D rt. ∡	3. All ∡ of a □ are rt. ∡ (§ 140).
4. $\angle C = \angle D$	4. All rt. ∡ are =.
5. $AD = BC$	5. Opp. sides of a ▱ are =.
6. $DC = DC$	6. Identity.
7. $\triangle ADC \cong \triangle BCD$	7. s.a.s.
8. $AC = BD$	8. Def. ≅ △.

To prove this theorem and the next one, select a pair of overlapping triangles that have one side of the rectangle in common and prove them congruent. How many such pairs are there?

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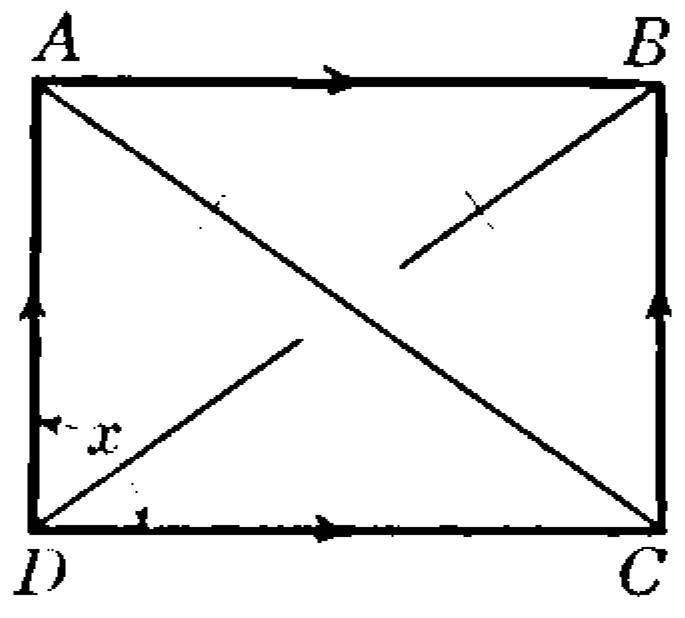
190. THEOREM. If the diagonals of a parallelogram are equal, the figure is a rectangle.

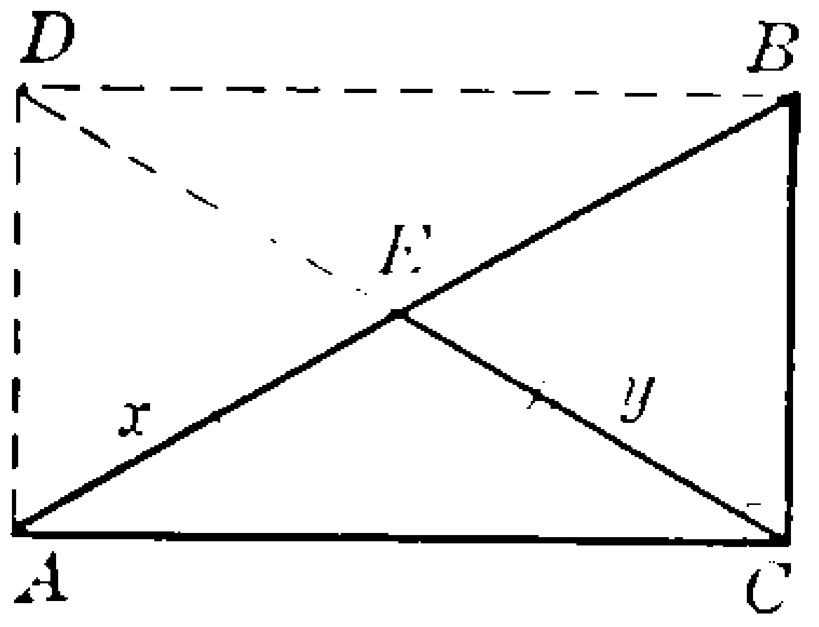


Given: ▱$ABCD$
$AC = BD$
Prove: $ABCD$ a □

PROOF

Statements	*Reasons*
1. $ABCD$ a ▱	1. Given.
2. $AD = BC$	2. Opp. sides of a ▱ are =.
3. $AC = BD$	3. Given.
4. $DC = DC$	4. Identity.
5. $\triangle ADC \cong \triangle BCD$	5. s.s.s.
6. $\angle D = \angle C$	6. Def. ≅ △.
7. $\angle D + \angle C = 180°$	7. ∥s, int. ∡ are sup.
8. $\angle D = 90°$	8. Div. Ax.
9. $ABCD$ a □	9. Def. □.

191. THEOREM. The median to the hypotenuse of a right triangle is one-half the hypotenuse.



Given: Rt. △ABC with CE the median to hypotenuse AB

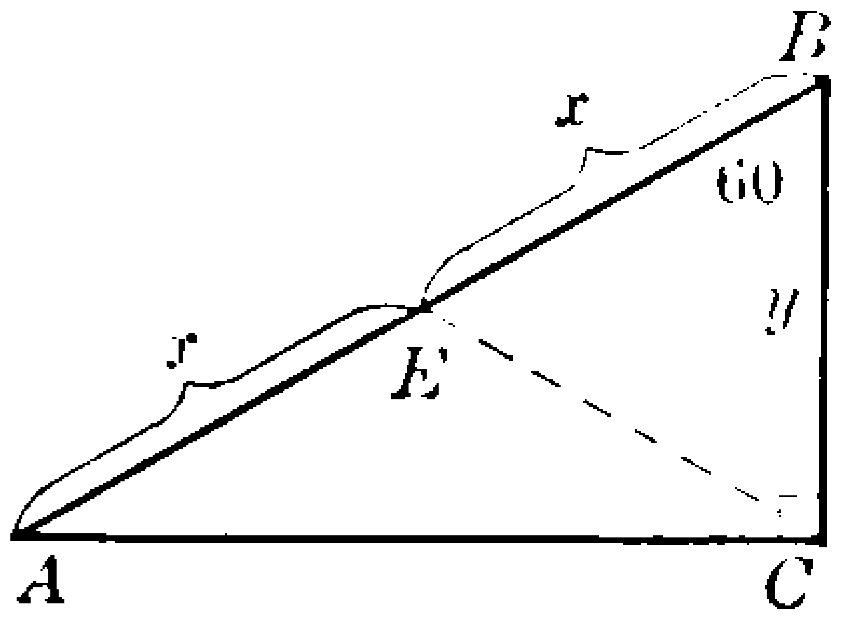
Prove: AE = EC

Cons.: Prolong CE through E, making DE = EC, and draw DA and DB.

Prove: ADBC a ▭ and AB = CD

Write out the proof.

192. THEOREM. If one angle of a right triangle is 60°, the hypotenuse is twice as long as the leg adjacent to the 60° angle.



Given: $\angle C = 90°$

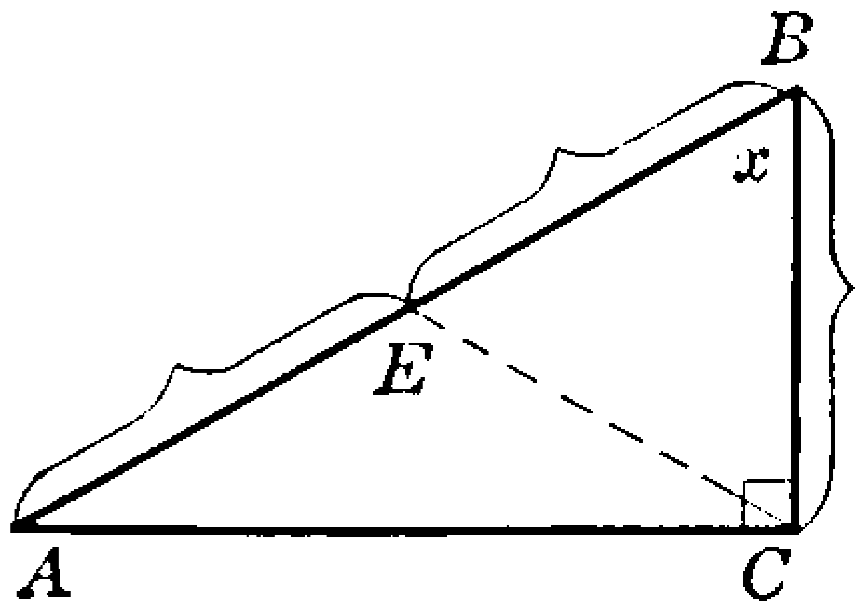
$\angle B = 60°$

Prove: $AB = 2 \cdot BC$

Cons.: Draw median CE.

Write out the proof. (Use Corollary § 133.)

193. CONVERSE THEOREM. If the hypotenuse of a right triangle is twice as long as one leg, the included angle is 60°.



Given: $\angle C = 90°$

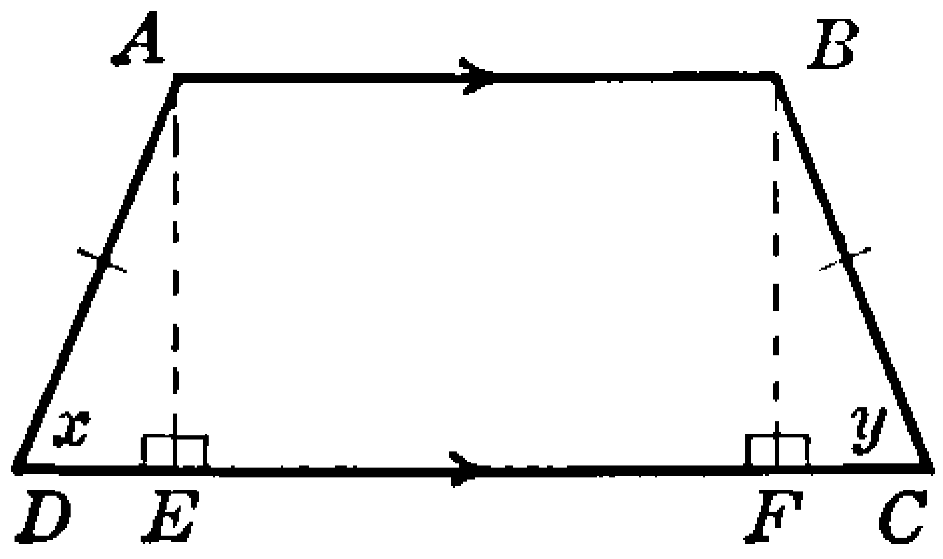
$AB = 2 \cdot BC$

Prove: $\angle B = 60°$

Cons.: Draw median CE.

Write out the proof.

194. THEOREM. The base angles of an isosceles trapezoid are equal.



Given: ABCD an isos. trapezoid

Prove: $\angle x = \angle y$

Cons.: Draw alts. AE and BF.

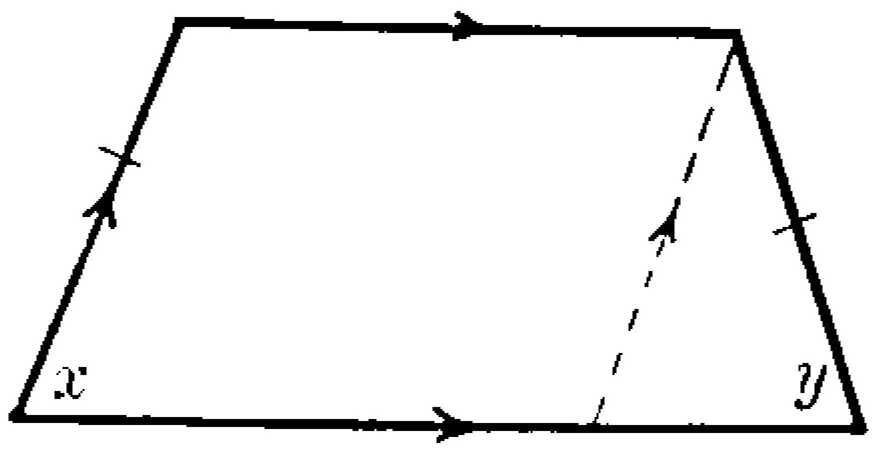
Write out the proof. (Use Corollary § 185.)

195. CONVERSE THEOREM. **If the base angles of a trapezoid are equal, the trapezoid is isosceles.**

Draw a figure and write out the proof.

EXERCISES

1. Complete the following statements by using one or both of the words in parentheses and adding whatever statement is needed:
   (*a*) A (necessary, sufficient) condition for a parallelogram to be a rectangle is that the diagonals . . . .
   (*b*) A (necessary, sufficient) condition for a trapezoid to be isosceles is that the base angles be —?—.
   (*c*) A (necessary, sufficient) condition for one angle of a right triangle to be 60° is that the hypotenuse . . . .
   (*d*) A (necessary, sufficient) condition for two angles of a triangle to be 30° and 60° is that one side be twice another.
   (*e*) The condition that the diagonals be equal is (necessary, sufficient) for a rectangle.
   (*f*) A (necessary, sufficient) condition for a trapezoid to be isosceles is that two opposite angles be —?—.
   (*g*) A (necessary, sufficient) condition for a quadrilateral to be an isosceles trapezoid is that the opposite angles be —?—.
   (*h*) A (necessary, sufficient) condition for two line segments included between two parallels to be equal is that they be parallel.
2. Add whatever is necessary to complete this statement: For a quadrilateral to be a rectangle it is sufficient that the diagonals . . . .
3. Change the following statement into two theorems and write the proof of each: A necessary and sufficient condition for a trapezoid to be isosceles is that the diagonals be equal.
4. Complete the following statements:
   (*a*) If two sides of a quadrilateral are parallel, and a pair of opposite angles are supplementary, the figure is either a(n) —?— or a(n) —?—.
   (*b*) If two sides of a quadrilateral are parallel and the diagonals are equal, the figure is either a(n) —?— or a(n) —?—.
5. Prove the theorem that the base angles of an isosceles trapezoid are equal and its converse (§ 195) by making the construction indicated in the diagram.



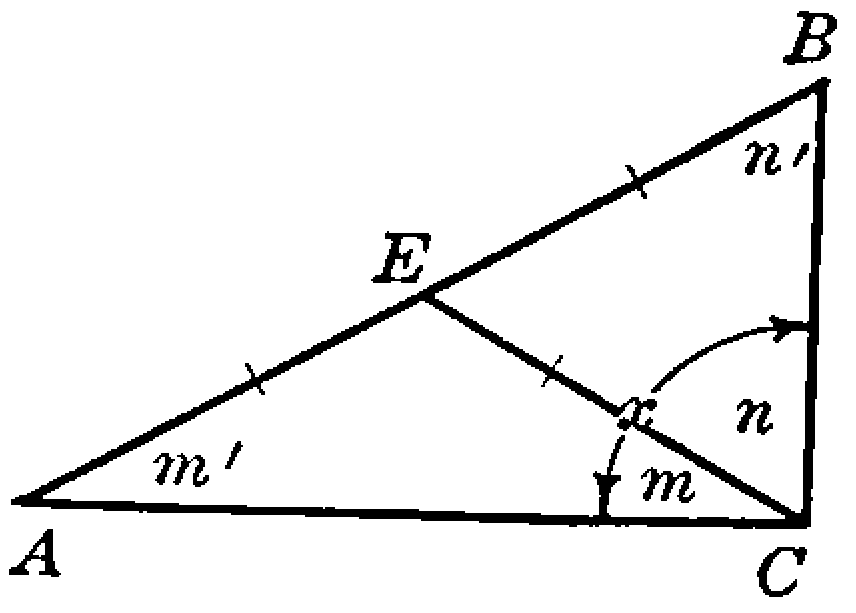
SUPERIOR WORK

S1. (*a*) In general, how many conditions determine that a quadrilateral is a parallelogram? a rectangle? an isosceles trapezoid? a rhombus? a square?

(*b*) State some of the conditions that determine the last four figures.

S2. A necessary condition for a median of a triangle to be half as long as the side to which it is drawn, is that the side be the hypotenuse of a right triangle.

We have already proved that this condition is sufficient (§ 191). Now prove that it is also necessary.



Given:
$AE = EB = EC$

Prove:
$\angle C = 90°$

Help: $\angle m + \angle m' + \angle n + \angle n' = 180°$

S3. Complete each of the following statements by using the words *necessary* and or *sufficient:*

(*a*) A —?— condition for a citizen to be president of the United States is that he be 35 years old.

(*b*) Universal suffrage is a —?— condition for a democracy.

(*c*) A —?— condition for a nation to avoid war is for that nation to hate war.

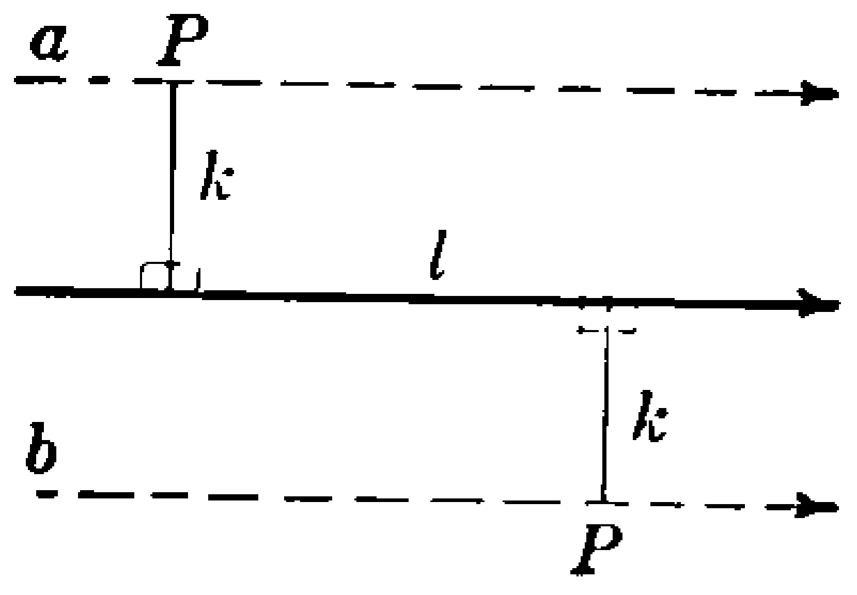
(*d*) For two angles to be equal it is —?— that they be right angles.

(*e*) For a quadrilateral to be a trapezoid it is —?— that two sides be parallel.

(*f*) For a trapezoid to be isosceles, it is —?— that a pair of opposite angles be supplementary.

A Point at a Given Distance from a Given Line

196. In § 146 it is stated that the locus of points at a given distance from a given point is a circle. From Corollary 187 we conclude that the locus of points at a given distance from a given line *l* consists of two lines *a* and *b* parallel to *l* at the given distance from it on either side.

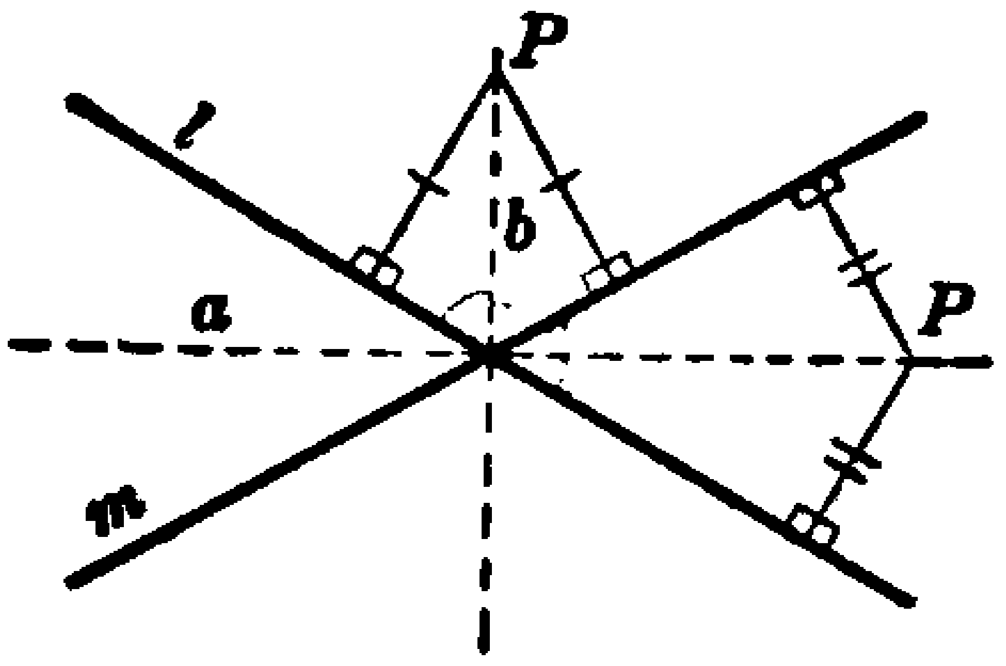


197. LOCUS THEOREM. The locus of a point a given distance from a given line is made up of two lines parallel to the given line at the given distance from it.

A Point Equidistant from Two Intersecting Lines

We have proved in § 166 that the locus of a point equidistant from two *rays* forming the sides of an angle is the bisector of the angle. Since two intersecting *lines* are made up of four such rays, we may arrive at the following more general theorem.

198. LOCUS THEOREM. The locus of a point equidistant from two intersecting lines is made up of the bisectors of the angles formed by these lines.

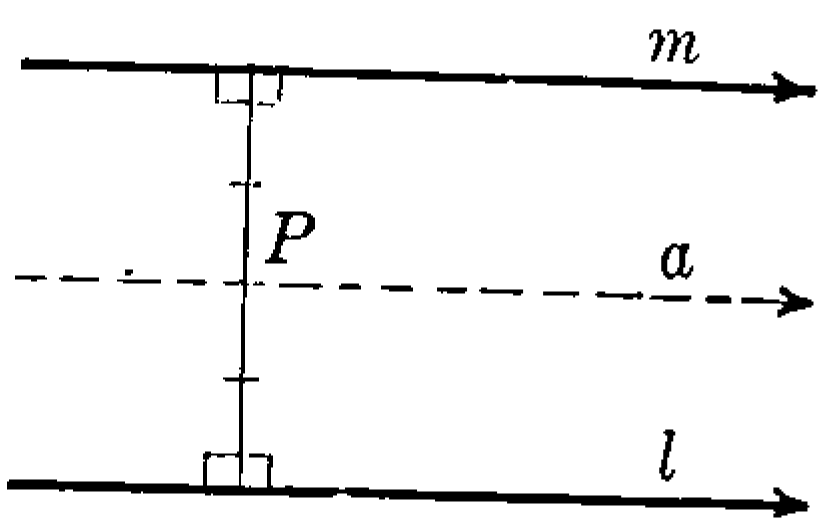


What is the relationship between the lines that make up this locus?

A Point Equidistant from Two Parallel Lines

In the last locus problem, we may think of the point of intersection of the two lines *l* and *m* as moving away indefinitely far along the bisector *a*. Two things happen: The lines *l* and *m* approach a position where they become parallel, and the angle bisector *b* disappears. Thus the following theorem results.

199. LOCUS THEOREM. **The locus of a point equidistant from two parallel lines is another parallel midway between them.**



EXERCISES

1. Construct the locus of a point that is a given distance k from a given line l.
2. Construct the locus of a point that is equidistant from two given intersecting lines.
3. Construct two parallel lines l and m. Then construct the locus of a point equidistant from l and m.

Classifying Theorems by Conclusions

200. When we use the method of analysis to solve a problem, we begin with the conclusion and work backwards. For this reason we are now classifying our fundamental theorems and other useful propositions according to their conclusions. Here we shall put into one group all the theorems that prove two angles equal, into another group all the theorems that prove two lines parallel, and so on. A good mechanic arranges his shelves in this way, each shelf containing the tools to do a certain job.

TABLE I. Proving Line Segments Equal

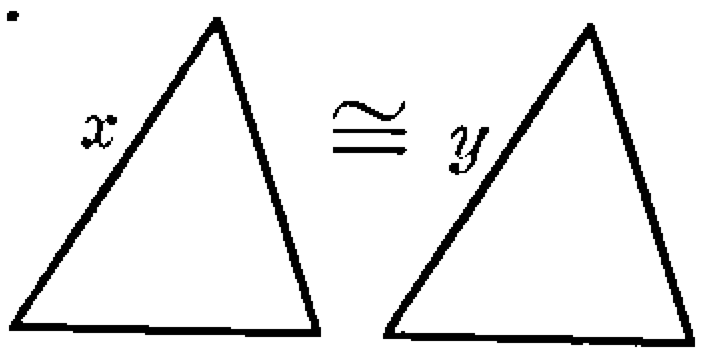
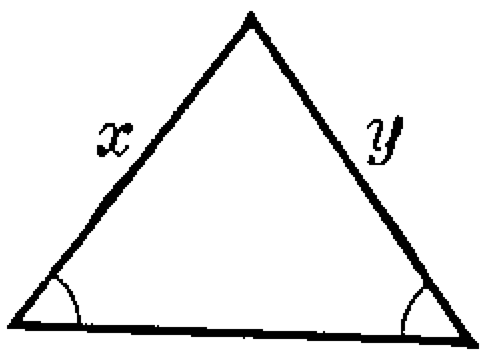
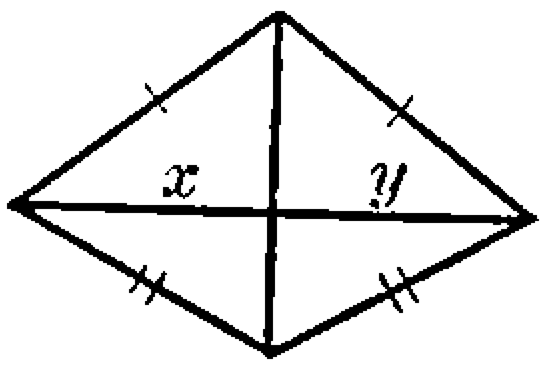
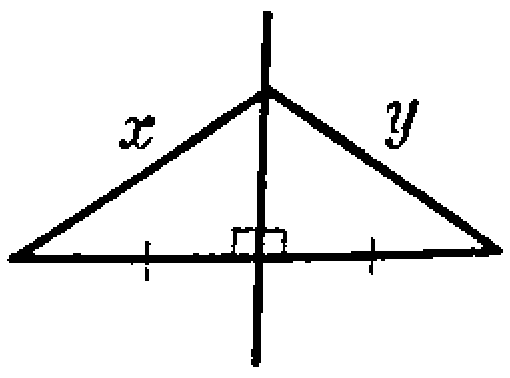
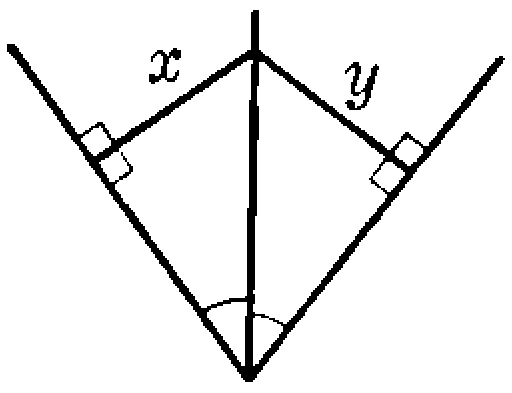
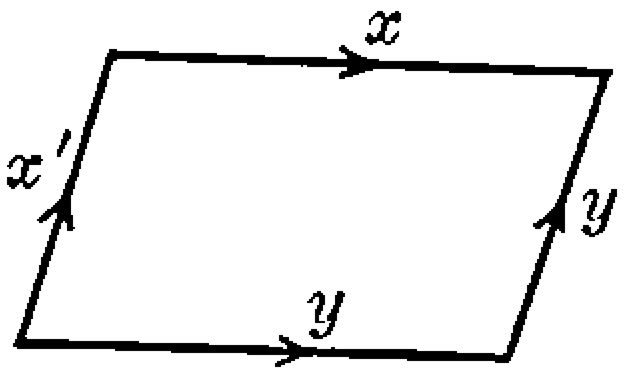
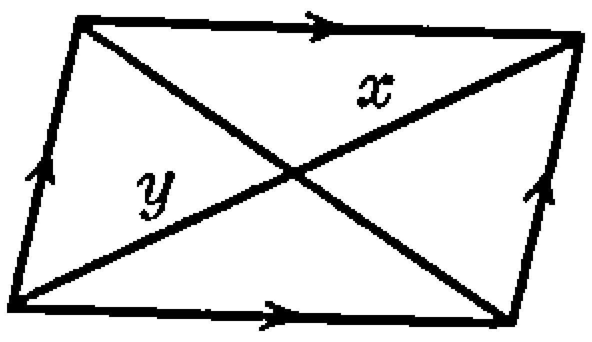
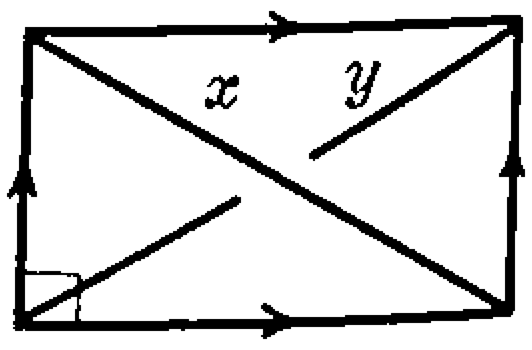
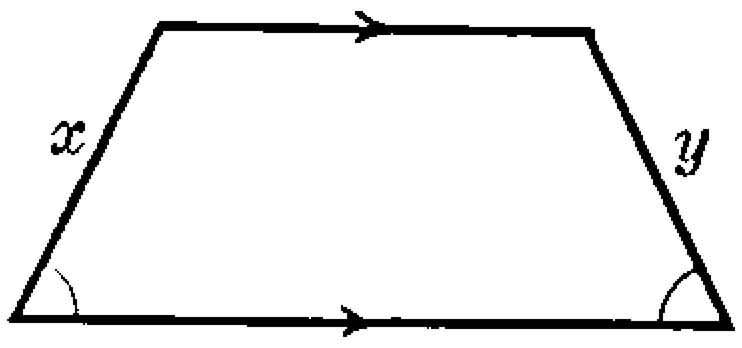
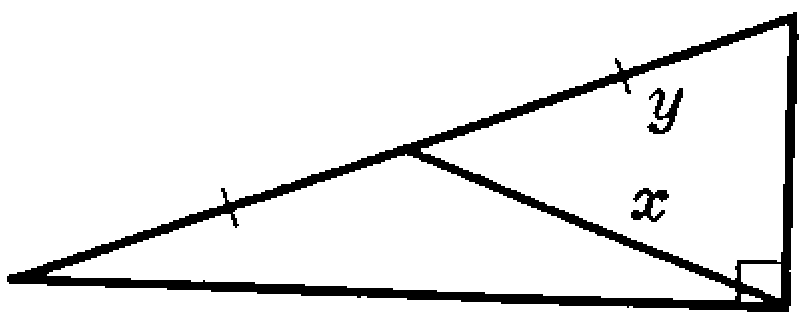
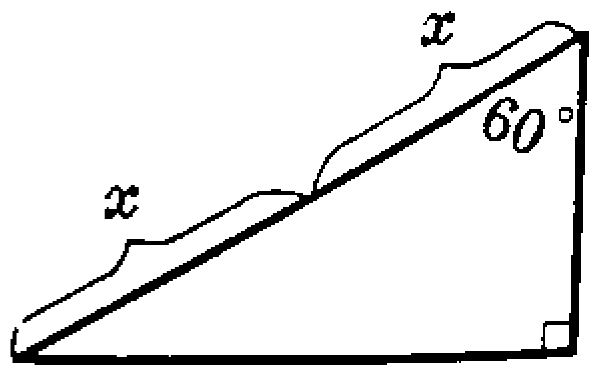
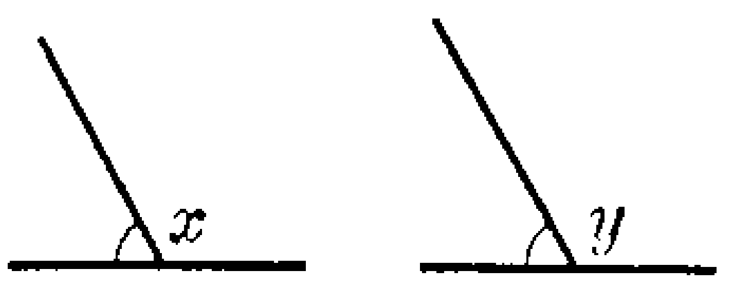
1.  Def. ≅ △s §111	2.  If 2 ∠s of a △ are =, sides opp. are =. §119	3.  Two-Pt. Theorem. §151	4.  A pt. on ⊥ bisector of a line segment is equidistant from ends. §153
5.  A pt. on bisector of ∠ is equidistant from sides of ∠. §165	6.  Opp. sides of ▱ are =. §171	7.  Diags. of ▱ bisect each other. §173	8.  Diags. of □ are =. §185
9.  If base ∠s of trap. are =, trap. is isos. §195	10.  Median to hypotenuse of rt. △ = ½ hypotenuse. §191	11.  Hypotenuse of 30°–60° rt. △ = 2 · shorter leg. §192	

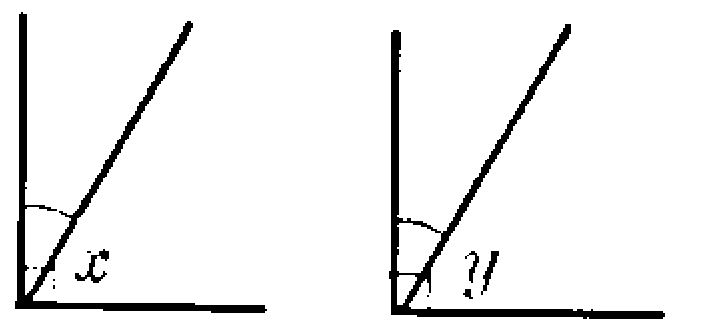
TABLE II. Proving Angles Equal

1.



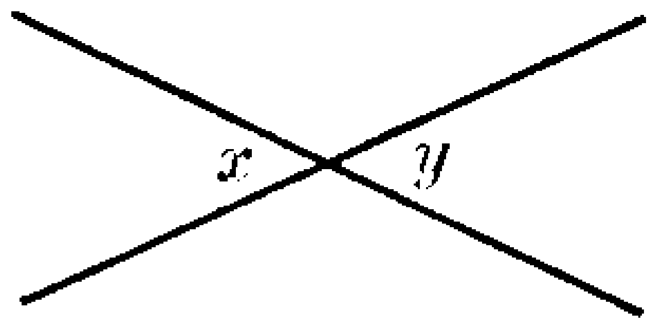
Sups. of = ∠s are =. §43

2.



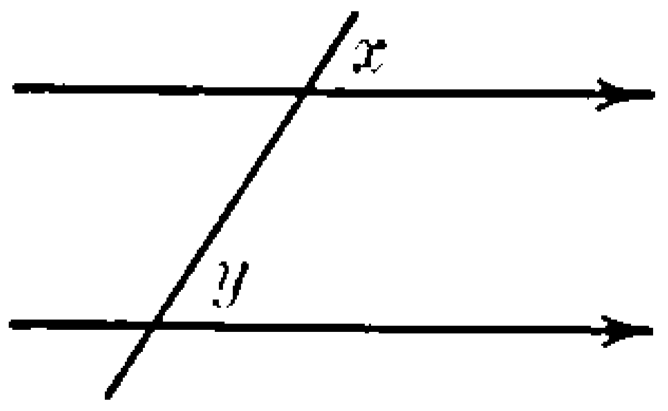
Comps. of = ∠s are =. §49

3.



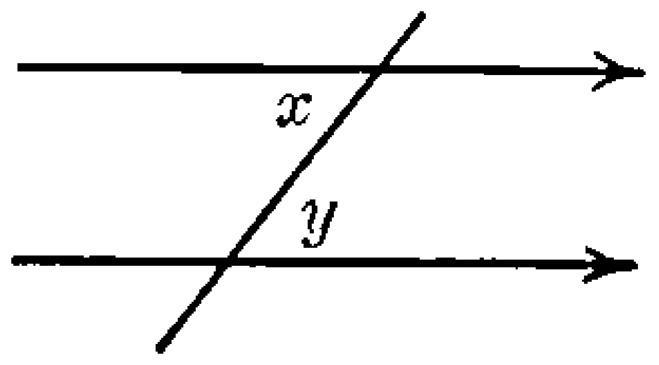
Vert. ∠s are =. §46

4.



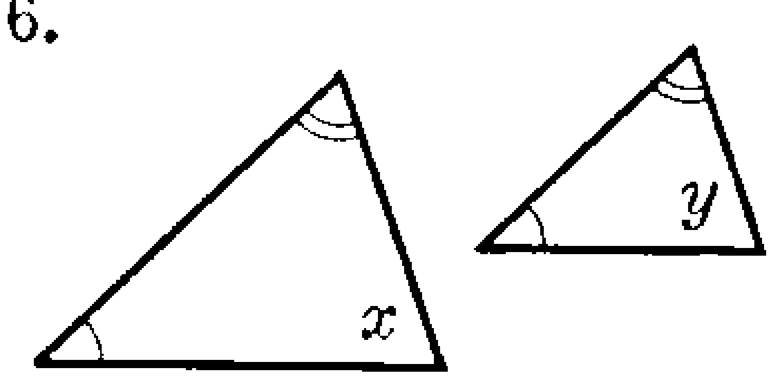
||s, corr. ∠s are =. §54, 2

5.



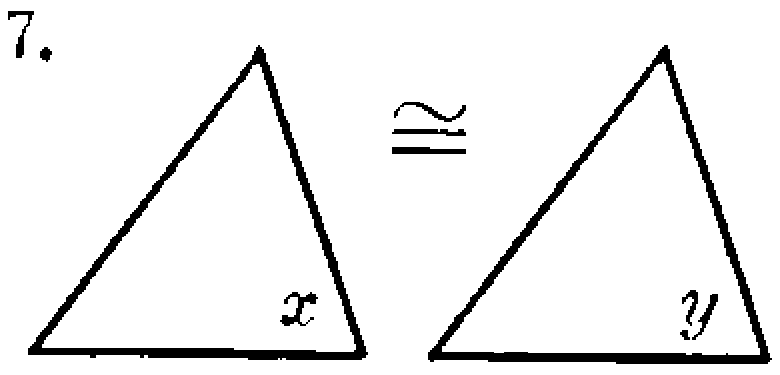
||s, alt. int. ∠s are =. §58

6.



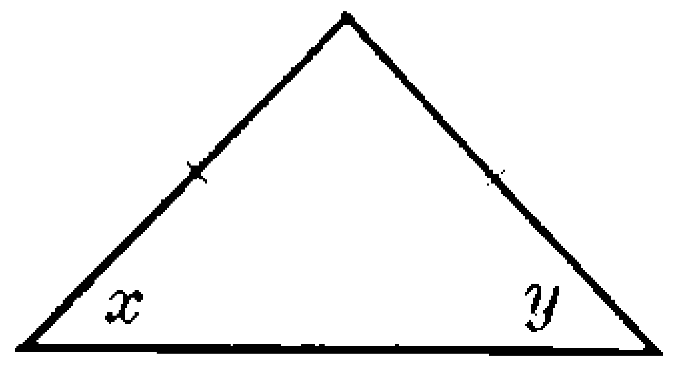
If 2 △s agree in 2 ∠s, they agree in remaining ∠. §65

7.



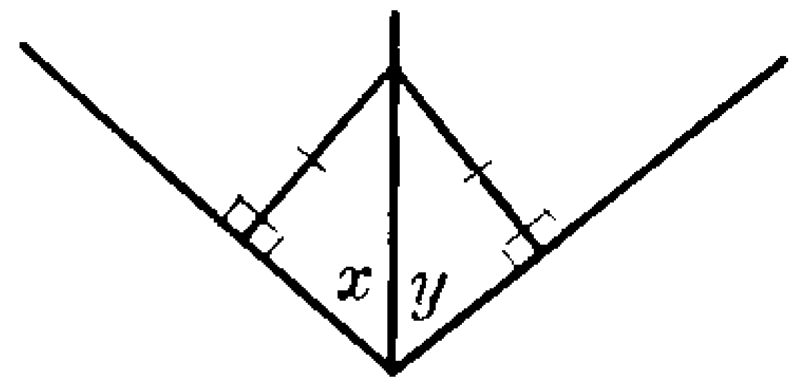
Def. ≅ △s. §111

8.



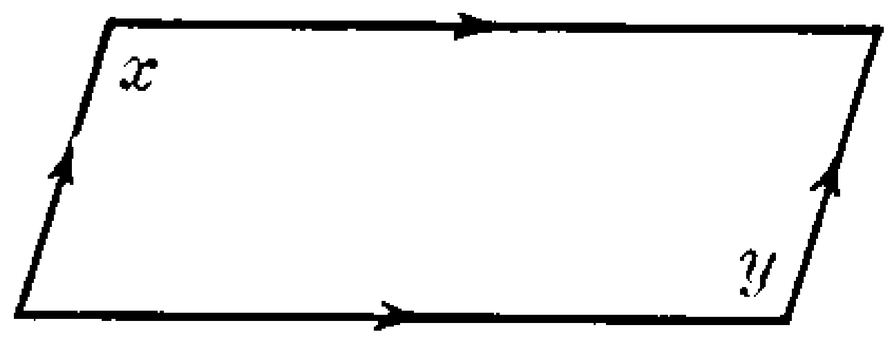
Base ∠s of isos. △ are =. §118

9.



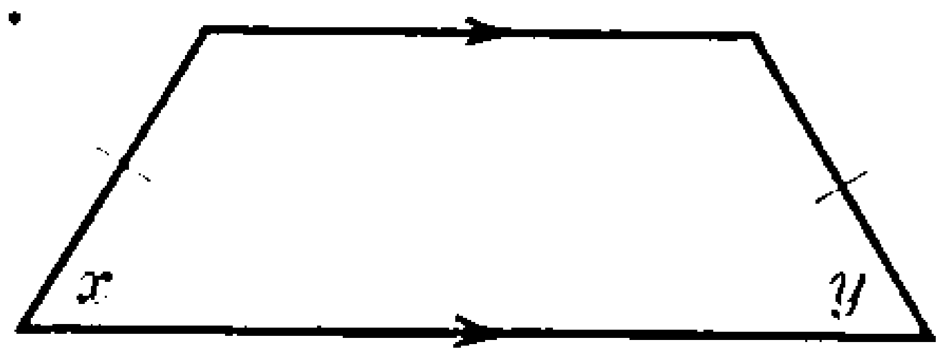
A pt. equidistant from sides of ∠ is on bisector of ∠. §163

10.



Opp. ∠s of ▱ are =. §172

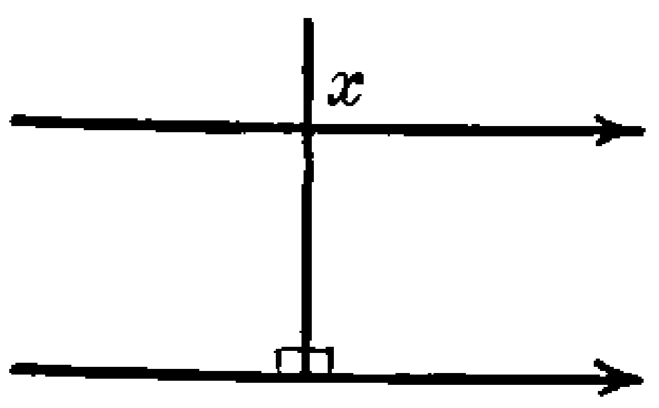
11.



Base ∠s of isos. trap. are =. §194

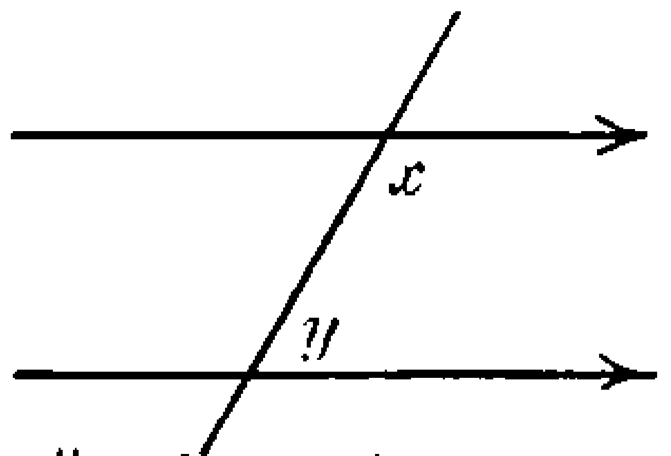
TABLE III. Evaluating Angles

1.



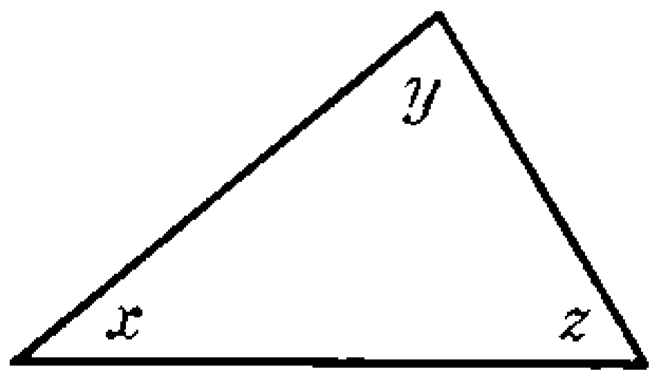
A ⊥ to one of 2 ||s is ⊥ to other. §39

2.



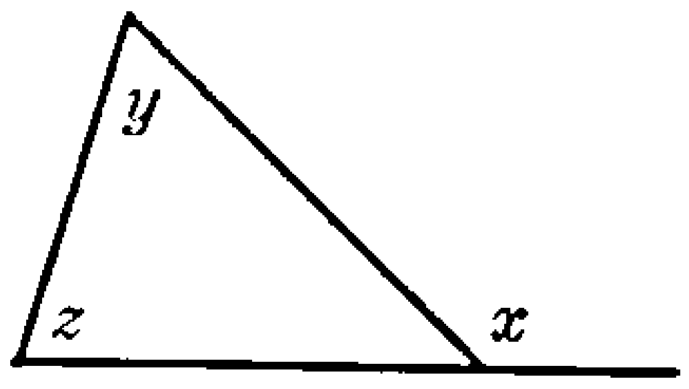
||s, int. ∠s same side of trans. sup. §60

3.



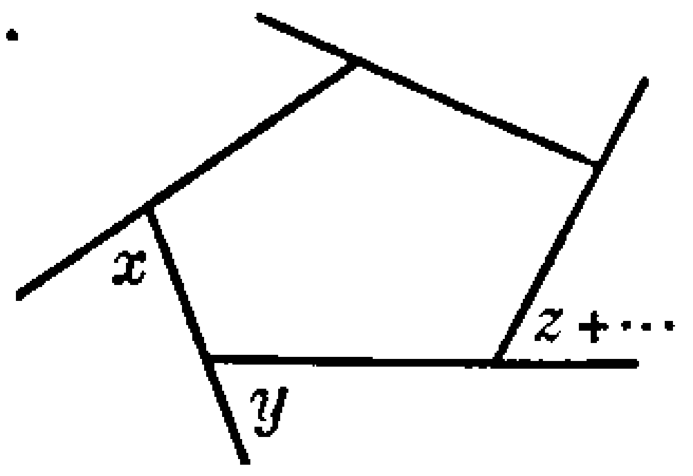
Sum of ∠s of a △ = 180°. §62

4.



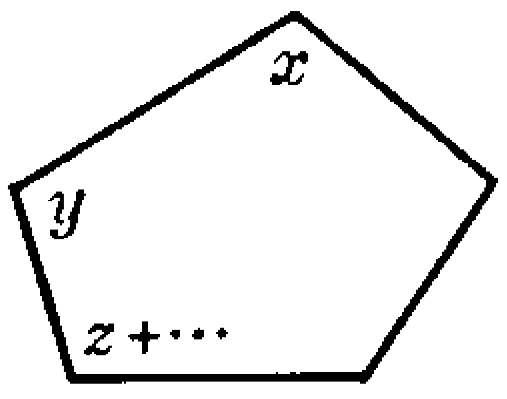
Ext. ∠ of a △ = sum 2 remote int. ∠s. §63

5.



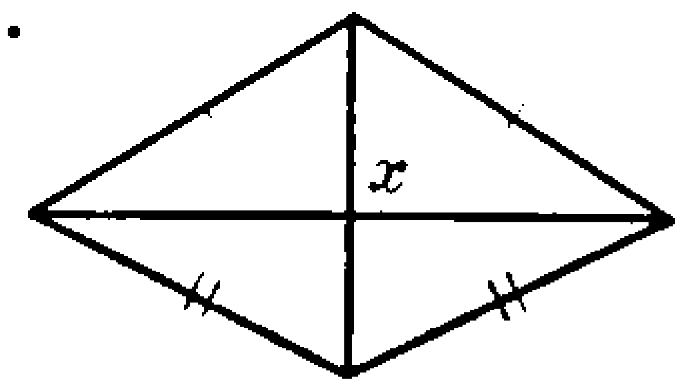
Sum ext. ∠s of polygon = 360°. §168

6.



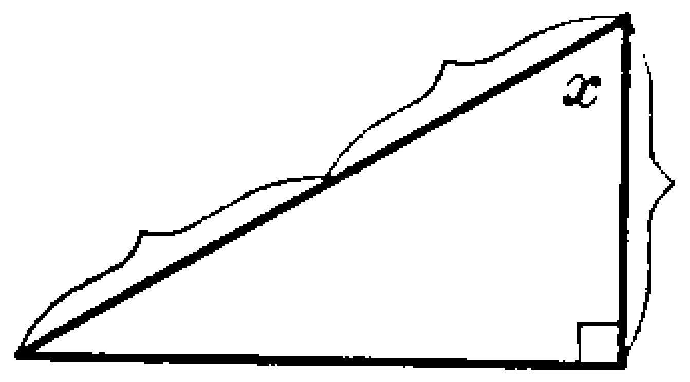
Sum ∠s of n-gon = (n-2) st. ∠s. §169

7.



Two-Pt. Theorem. §151

8.



If hypotenuse = twice one leg, included ∠ = 60°. §193

TABLE IV. Proving Lines Parallel

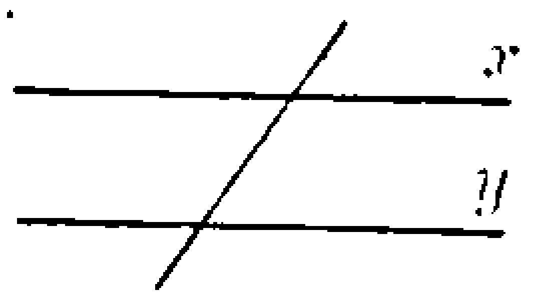
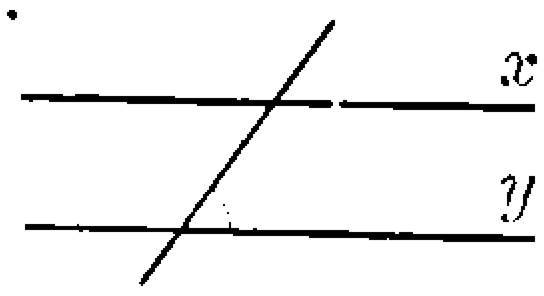
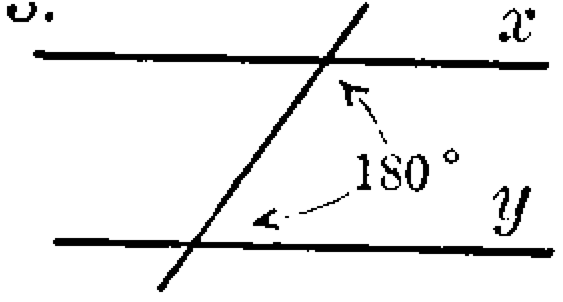
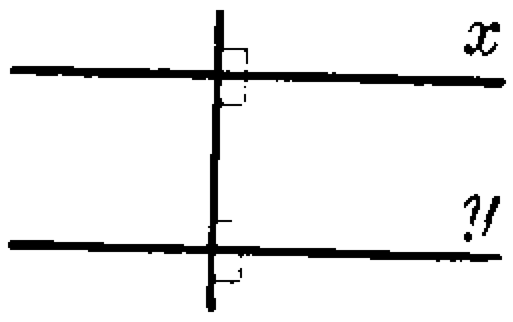
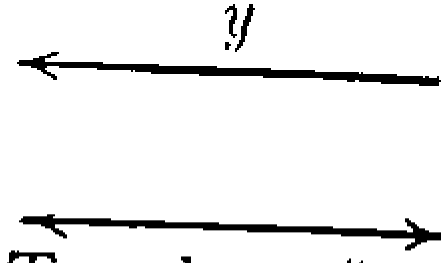
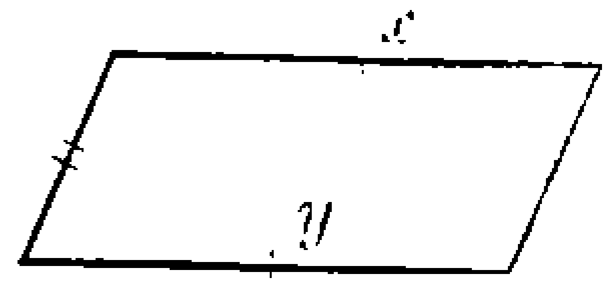
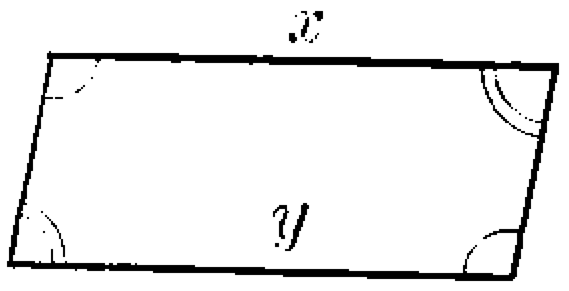
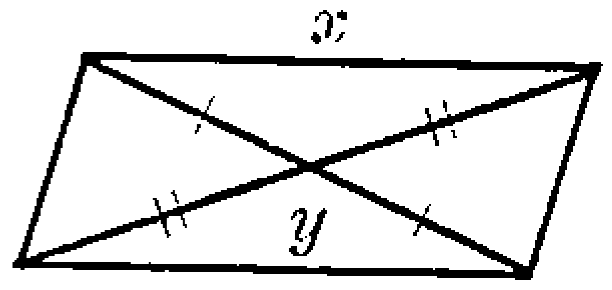
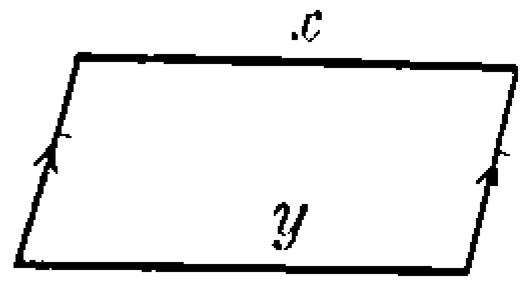
1.  Corr. ∡ are =, ∥s. §54, 1	2.  Alt. int. ∡ are =, ∥s. §57	3.  Int. ∡ same side trans. sup., ∥s. §59	4.  Two lines ⊥ to same line are ∥. §55	5.  Two lines ∥ to same line are ∥. §61
6.  If opp. sides are =, quad. is ▱. §175	7.  If opp. ∡ are =, quad. is ▱. §176	8.  If diags. bisect each other, quad. is ▱. §177	9.  Two sides are = and ∥, quad. is ▱. §178	

TABLE V. Proving Triangles Congruent

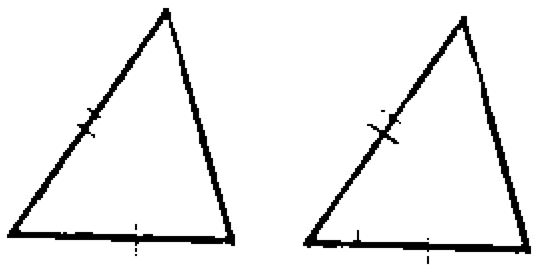
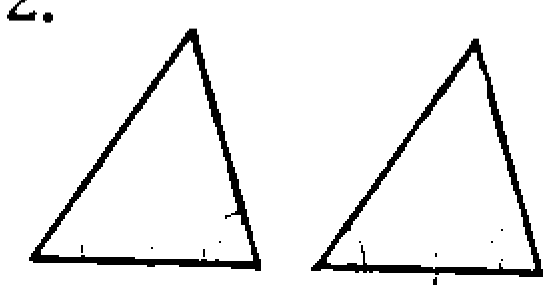
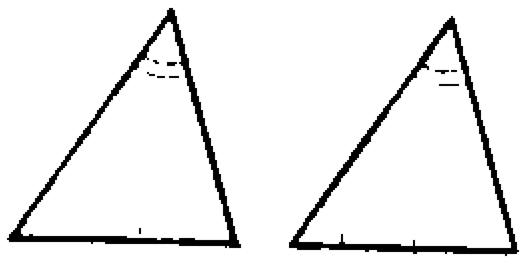
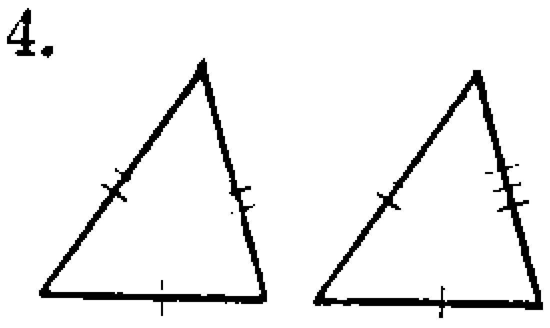
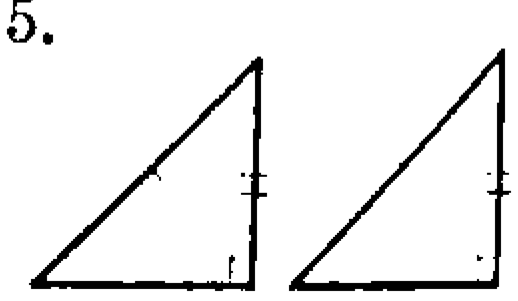
1.  s. a. s. §113	2.  a. s. a. §114	3.  s. a. a. §116	4.  s. s. s. §120	5.  h. l. §122

TABLE VI. Loci

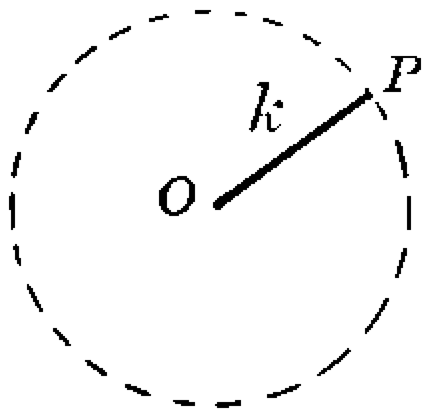
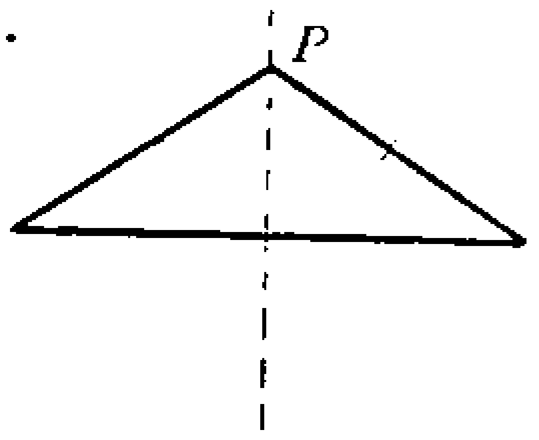
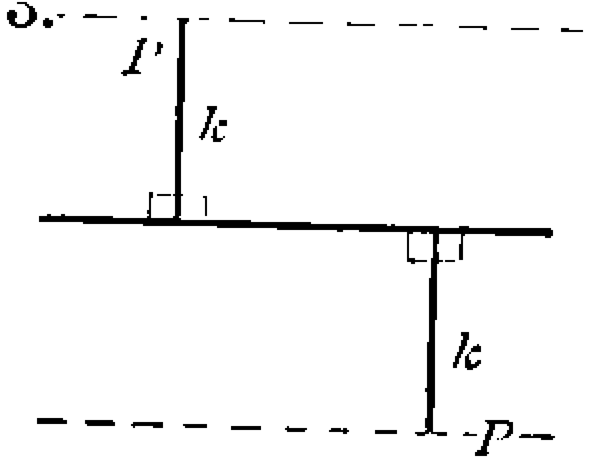
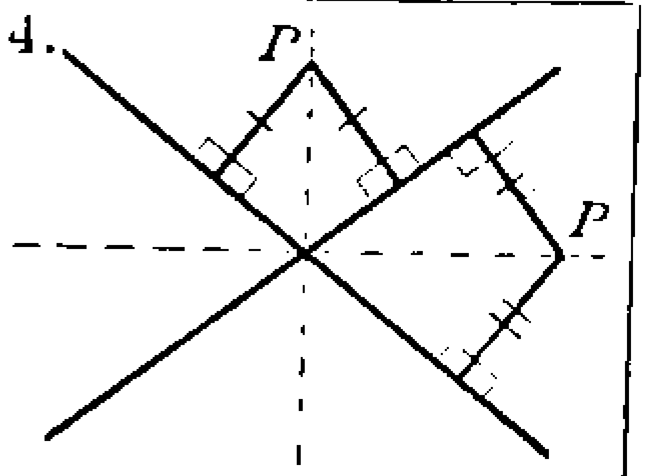
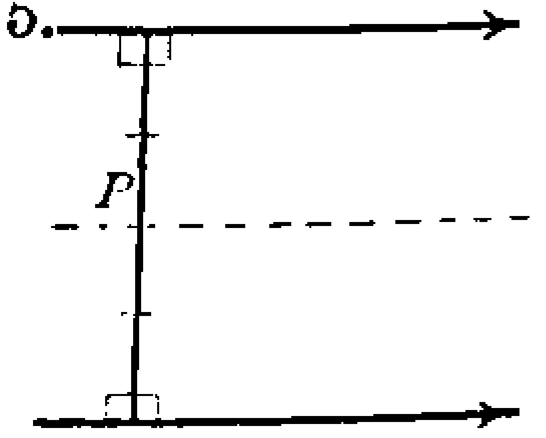
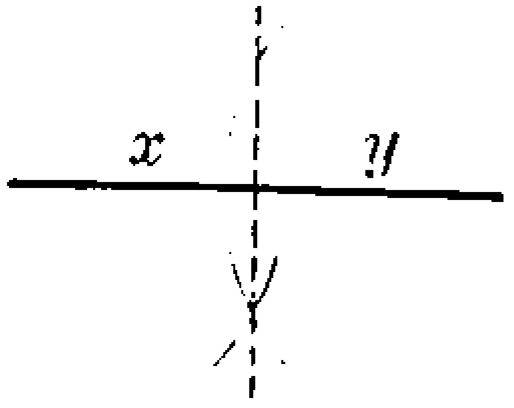
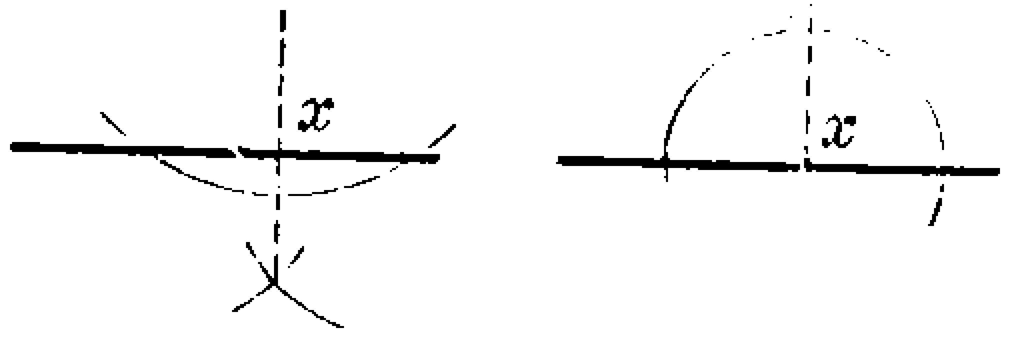
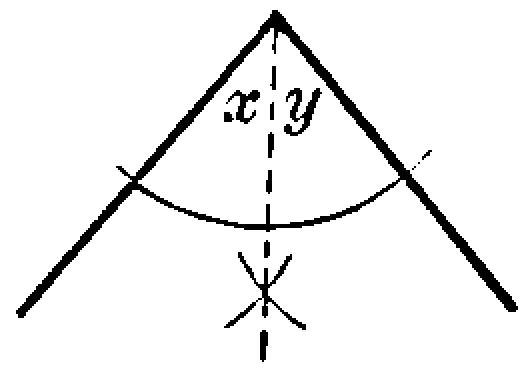
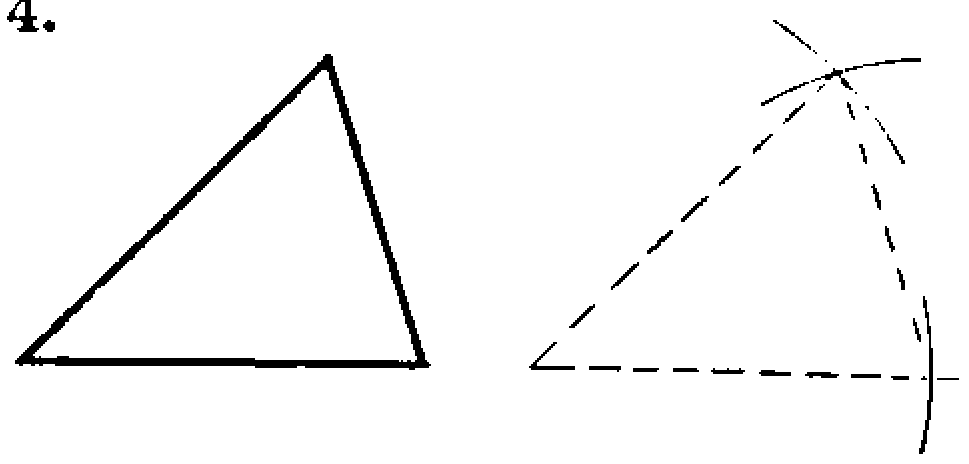
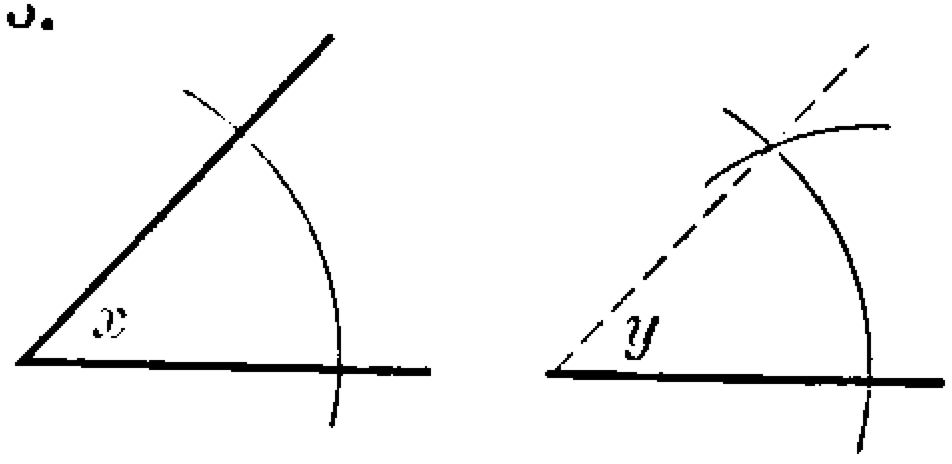
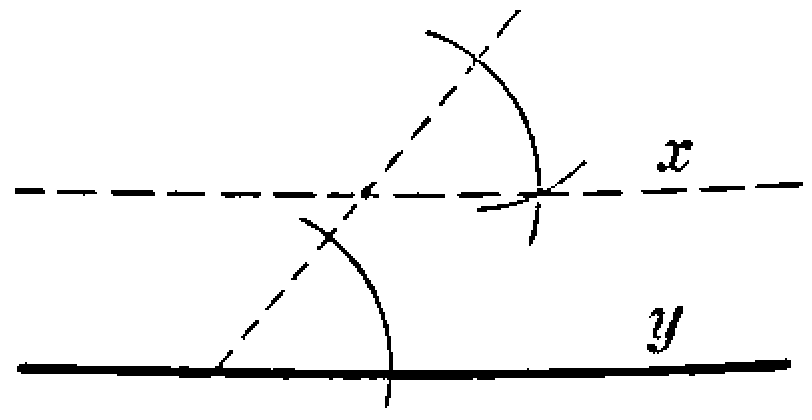
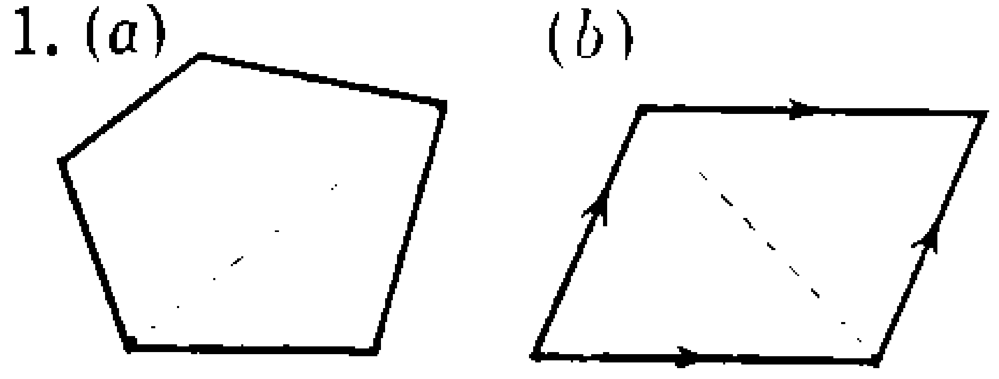
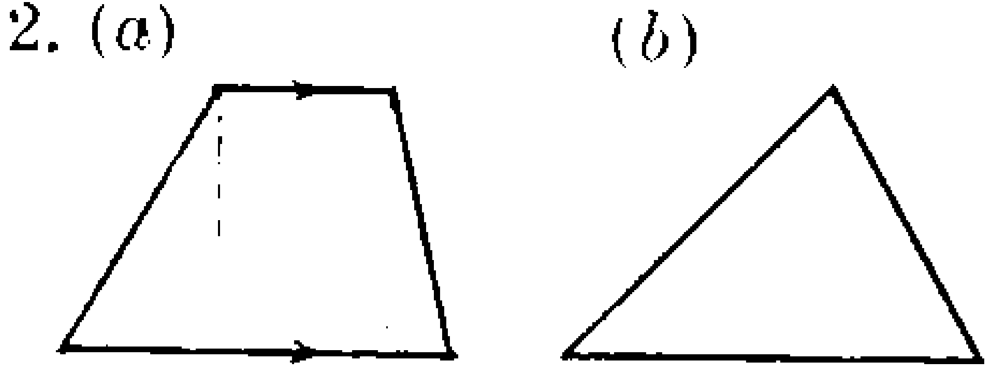
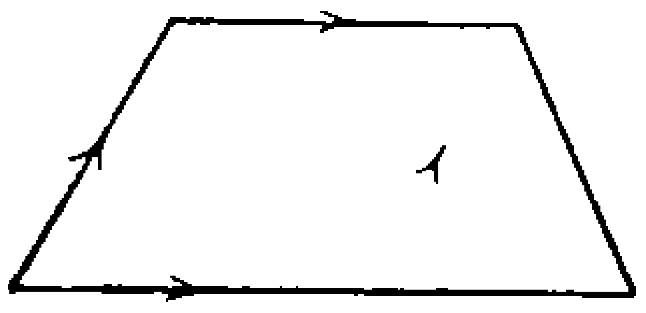
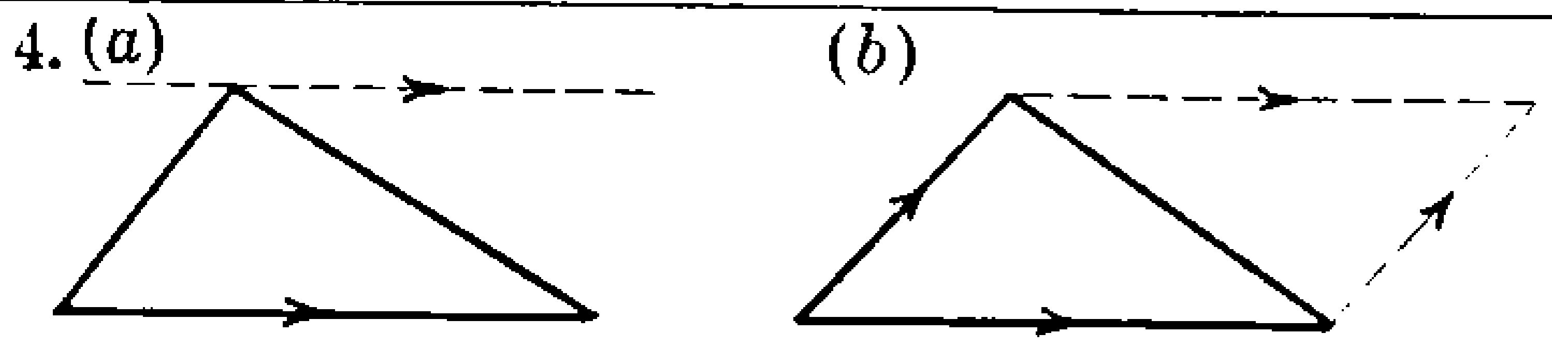
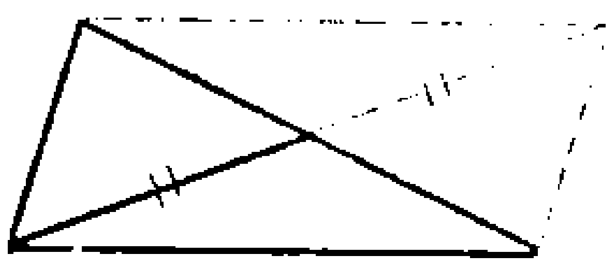
1.  Locus of a pt. given distance from given pt. §146	2.  Locus of a pt. equidistant from ends of line segment. §147	3.  Locus of a pt. given distance from given line. §197	4.  Locus of a pt. equidistant from 2 intersecting lines. §198	5.  Locus of a pt. equidistant from 2 ∥ lines. §199

TABLE VII. Constructions

1.  Bisect a line segment. §155	2.  Cons. ⊥ to given line through given pt. §156	3.  Bisect an angle. §158
4.  Copy a △. §159	5.  Copy an angle. §160	6.  Cons. ∥ to given line through given pt. §161

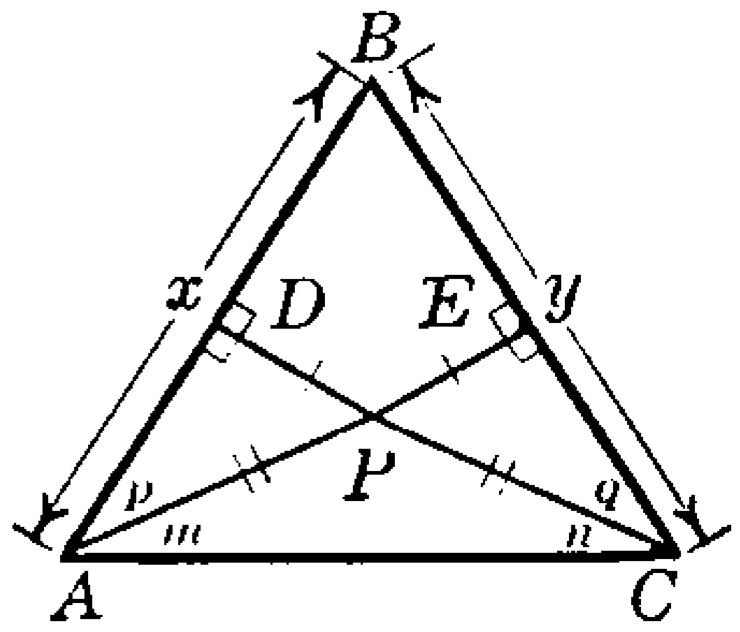
The following table contains certain constructions that are often helpful in proving propositions.

TABLE VIII. Useful Auxiliary Constructions

<table>
<tr><th colspan="3">Breaking Down Figures</th></tr>
<tr><td>1. (a) (b)

Using diagonals.</td><td>2. (a) (b)

Using altitudes.</td><td>3

Using parallels.</td></tr>
<tr><th colspan="3">Building Up Figures</th></tr>
<tr><td colspan="2">4. (a) (b)

Drawing parallels through the vertices of a triangle.</td><td>5.

Forming a parallelogram by prolonging a median its own length.</td></tr>
</table>

EXERCISES ABOUT TRIANGLES

Illustrative Example. If a point within a triangle is equidistant from two sides of the triangle and also from the ends of the third side, then the triangle is isosceles.

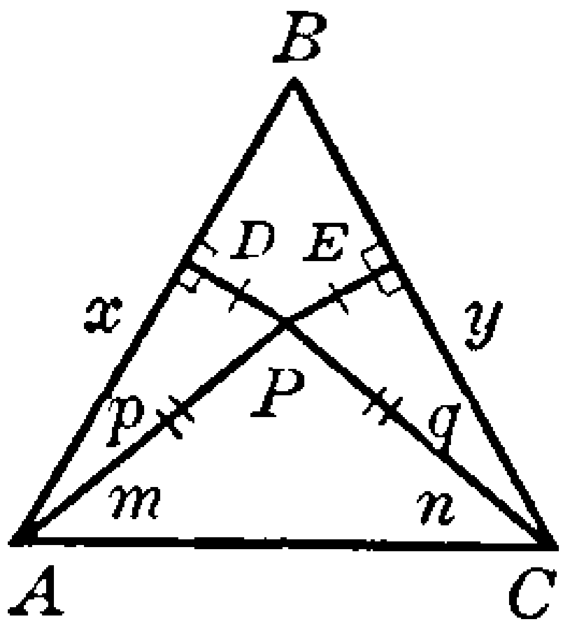


Analysis. Since we wish to prove $AB = BC$, we refer to Table I, which lists the main theorems for proving line segments equal. By comparing our figure with those in the table, we find that No. 2 is the most promising; that is, we can prove $AB = BC$ if we can show that $\angle A = \angle C$. This we can do if we can show that $\angle m = \angle n$, and $\angle p = \angle q$.

Since our problem now is to prove angles equal, we scan Table II. We soon discover that No. 8 will enable us to prove that $\angle m = \angle n$; also that No. 7 will help us to prove that $\angle p = \angle q$, if we can prove $\triangle APD \cong \triangle CPE$.

Finally, we look over Table V for the method of proving our triangles congruent. We find that No. 5 fits our case.

Deformalization. We now reverse the steps of the analysis and give the synthetic proof. The synthetic proof that follows has been *deformalized*. This means that some of the formal steps that are obvious are omitted.

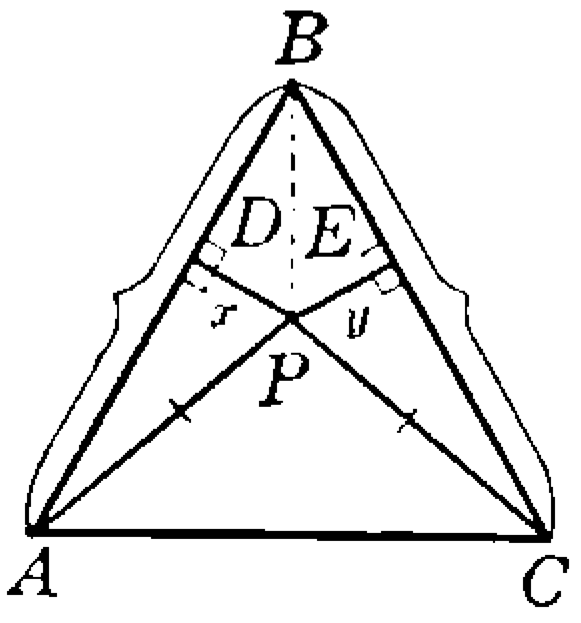


Given:
$PD \perp AB$
$PE \perp BC$
$PD = PE$
$PA = PC$
Prove:
$AB = BC$

SYNTHETIC PROOF

Statements	*Reasons*
1. $\triangle APD \cong \triangle CPE$	1. h.l.
2. $\angle p = \angle q$	2. Def. $\cong$ $\triangle$s.
3. $\angle m = \angle n$	3. Base $\angle$s of an isos. $\triangle$ =.
4. $\angle A = \angle C$	4. Add. Ax.
5. $AB = BC$	5. If 2 $\angle$s of a $\triangle$ are =, the sides opposite are =.

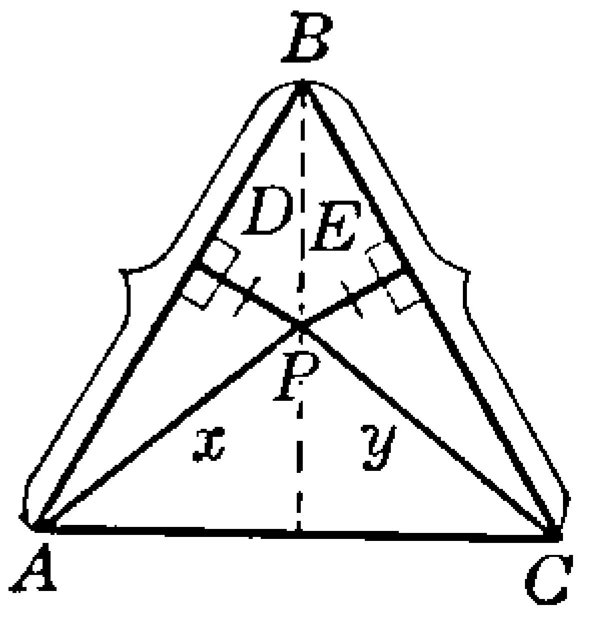
Converse 1. By interchanging one of the given statements with the conclusion, we may obtain a converse. In the figure, we have given that the point P is equidistant from the ends of the base of the isosceles $\triangle ABC$. We wish to prove that P is equidistant from the legs.



Analysis. From Table I, No. 5, we see that we can prove $x = y$ if we can prove that P is on the bisector of $\angle B$. We therefore draw BP. Now, from Table V, triangles ABP and BCP will be congruent by s.s.s.

Write out the synthetic proof.

Converse 2. We shall prove another converse: A point that is equidistant from the legs of an isosceles triangle is also equidistant from the ends of the base.



Analysis. To prove $AP = PC$, we may try to prove $\triangle APC$ isosceles or $\triangle APD \cong \triangle CPE$. There is not sufficient data to do either. Since P is equidistant from AB and BC, it is on the bisector of $\angle B$. So we draw BP and we know that it is the bisector of $\angle B$. In an isosceles triangle, the bisector of the vertex angle is the perpendicular bisector of the base (§ 138). Now, from Table I, No. 4, we have $x = y$.

1. (*a*) Prove: If, through any point in the bisector of an angle, a line is drawn parallel to one side of the angle, then an isosceles triangle is formed.
   (*b*) Form and prove two converses by interchanging each of the two given statements with the conclusion.

2. (*a*) Prove: If, from any point in the base of an isosceles triangle, perpendiculars are drawn to the legs, they form equal angles with the base.
   (*b*) State and prove the converse in which the conclusion is that the triangle is isosceles.
   (*c*) Prove the following converse: If, from a point in the base of an isosceles triangle, lines are drawn making equal angles with the base, and one of these lines is perpendicular to a leg, then the other line is perpendicular to the other leg.
   (*d*) Prove the following extension of (*a*): If, from any point in the prolongation of the base of an isosceles triangle, perpendiculars are drawn to the legs (one or both legs extended), they form equal angles with the base.

3. (*a*) Prove: If, from any point in the base of an isosceles triangle except the midpoint, a perpendicular is drawn to the base meeting the legs (one of them extended), then another isosceles triangle is formed.
   (*b*) Form and prove two converses of the preceding proposition.
   (*c*) Extend the proposition in (*a*) to read: If, from any point in the prolongation of the base of an isosceles triangle, a perpendicular is drawn to the base, meeting the legs extended, then another isosceles triangle is formed.

4. (*a*) Prove: If, from the midpoint of the base of an isosceles triangle, perpendiculars are drawn to the legs, they are equal.
   (*b*) Prove the converse in which the conclusion is that the triangle is isosceles.
   (*c*) Prove the converse having as its conclusion that the base is bisected.

5. (*a*) Prove: The altitudes to the legs of an isosceles triangle are equal.
   (*b*) Prove: If two altitudes of a triangle are equal, the triangle is isosceles.

6. Prove: The medians to the legs of an isosceles triangle are equal.

7. Prove: The bisectors of the base angles of an isosceles triangle are equal.

8. (*a*) Prove: The altitude to a leg of an isosceles triangle makes an angle with the base which is one-half as large as the vertex angle. (*Help:* Bisect the vertex angle.)

(*b*) Prove the two converses of this proposition.

9. (*a*) Prove: If, from the ends of one side of a triangle, perpendiculars are drawn on the median (prolonged) to that side, these perpendiculars are equal.

(*b*) Prove the converse in which the two perpendiculars are given equal.

10. (*a*) Prove: The bisector of the exterior angle at the vertex of an isosceles triangle is parallel to the base.

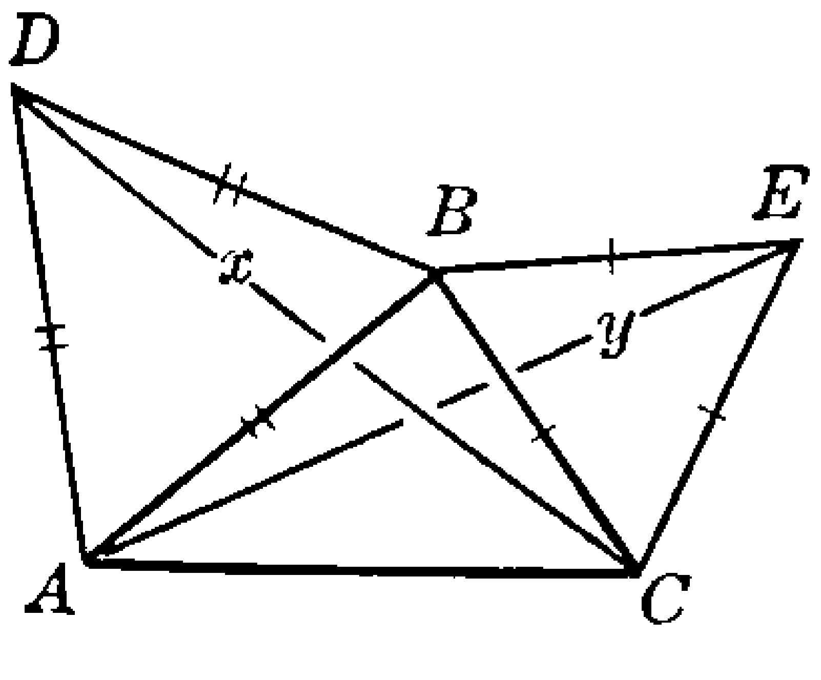
(*b*) Prove the two converses of this proposition.

11. Draw an isosceles triangle with AB and BC as the legs. Choose any point M on AC. Extend BA a distance AN equal to AM. Draw a line from N through M, cutting BC in R. Prove: $\angle BRN = 3 \cdot \angle N$.

12. On the legs AB and BC of an isosceles triangle, lay off $AD = CE$. Draw AE and CD intersecting in F. Prove that BF bisects $\angle B$.

SUPERIOR WORK

S1. If, on two sides of a triangle as bases, equilateral triangles are constructed, then the line segments joining the vertices of the equilateral triangles with the vertices of the original triangle are equal.



Given:
$AD = DB = BA$
$BE = EC = BC$
Prove:
$DC = AE$

S2. (*a*) Prove that if two triangles are congruent, their corresponding altitudes are equal.

(*b*) The general converse of this proposition is that if two triangles agree in two of their parts, and also in a corresponding altitude, they are congruent. Depending on the parts of the triangle that are associated with the altitude, there are a number of cases. Draw some of these cases and write a proof for each.

S3. (*a*) Prove that if two triangles are congruent, their corresponding medians are equal.

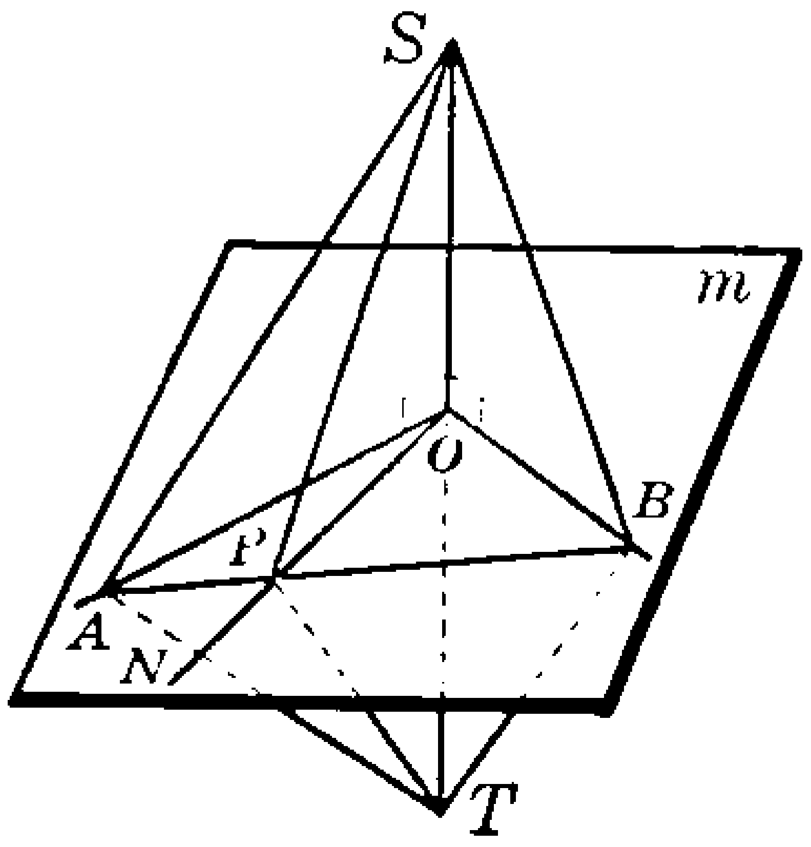
(*b*) The general converse of this proposition is that if two triangles agree in two of their parts, and also in a corresponding median, they are congruent. Draw some of these cases and write a proof for each.

S4. (*a*) Prove that if two triangles are congruent, their corresponding angle bisectors are equal.

(*b*) Investigate the converses of this proposition as you did in S2 and S3.

S5. Classify some personal qualifications according to two or three vocations that require them.

S6. *Theorem.* If a line is perpendicular to two intersecting lines at their point of intersection, it is perpendicular to the plane of the lines.



Given: In plane m, OA and OB intersect at O

$SO \perp OA$
$SO \perp OB$

Prove: $SO \perp$ plane m

Cons.: In plane m, draw *any* line ON. Also in plane m, draw a line cutting OA, ON, and OB in pts. A, P, and B. Prolong SO, making $OT = OS$. Draw SA, SP, SB, TA, TP, and TB.

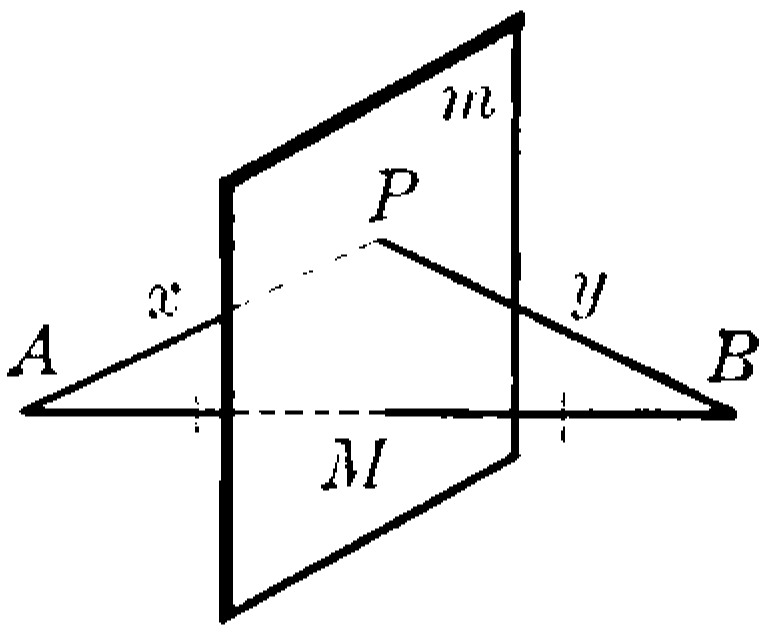
Analysis: To prove $OS \perp$ plane m, we must prove it perpendicular to any line such as ON in the plane m through the foot of OS (page 27, S4). By construction, O is equidistant from S and T. If we can prove that P is also equidistant from S and T, we can use the two-point theorem to prove that ON is the perpendicular bisector of ST.

Write out the synthetic proof.

Using a carpenter's square, how can you test whether a tent pole is upright?

S7. (*a*) Given: Plane $m \perp AB$
Plane m bisects AB

Prove: Any point P in plane m is equidistant from A and B



Write out the proof.

(*b*) What is the locus of points in space that are equidistant from two given points?

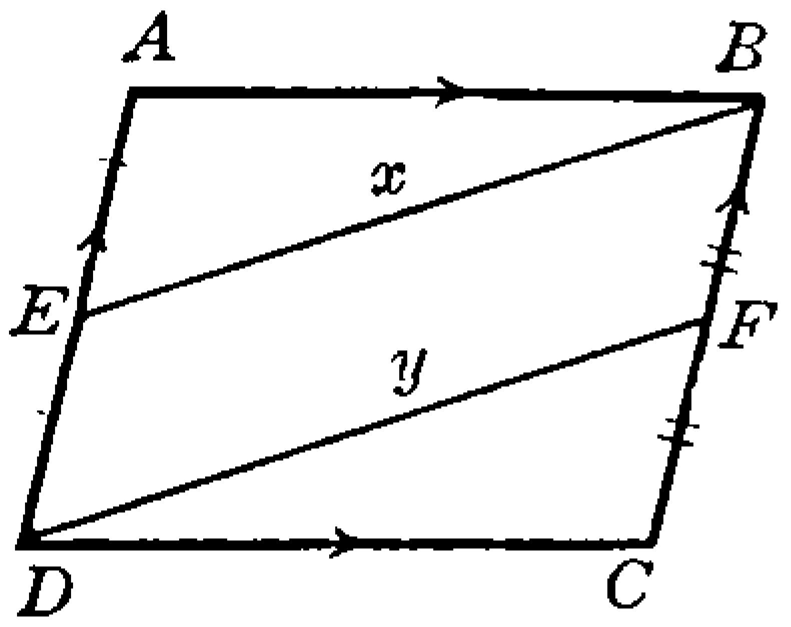
(*c*) What is the locus of points that are equidistant from the faces of a dihedral angle? (See page 27, S3.)

S8. Two line segments from the vertices of a triangle to the opposite sides cannot bisect each other. (Prove this indirectly.)

S9. A transversal cuts two parallel lines in R and S. The bisectors of two interior angles on the same side of the transversal cut the parallels again at B and M. B and M are joined. Prove that the quadrilateral $RMBS$ is a rhombus.

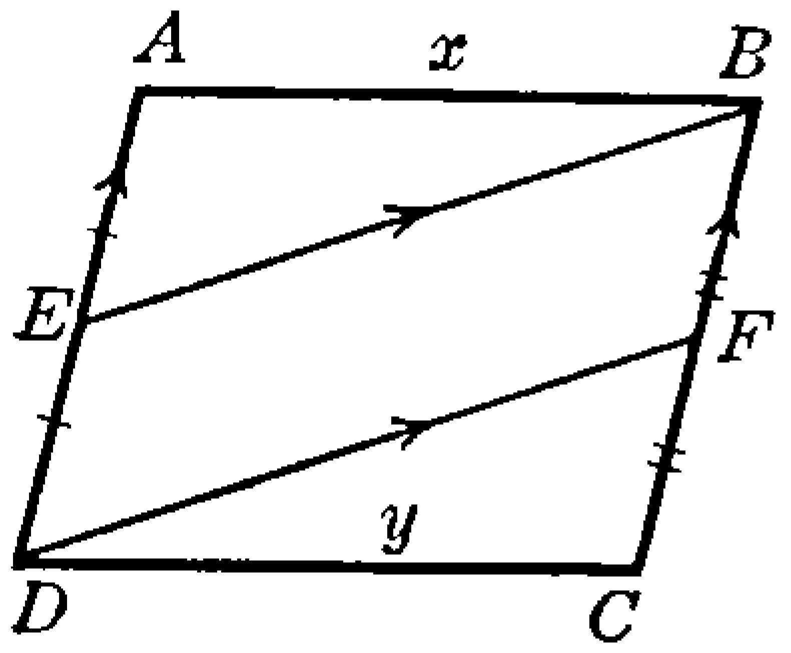
EXERCISES ABOUT QUADRILATERALS

Illustrative Example. If, from two opposite vertices of a parallelogram, line segments are drawn to the midpoints of two opposite sides, then these line segments are parallel.



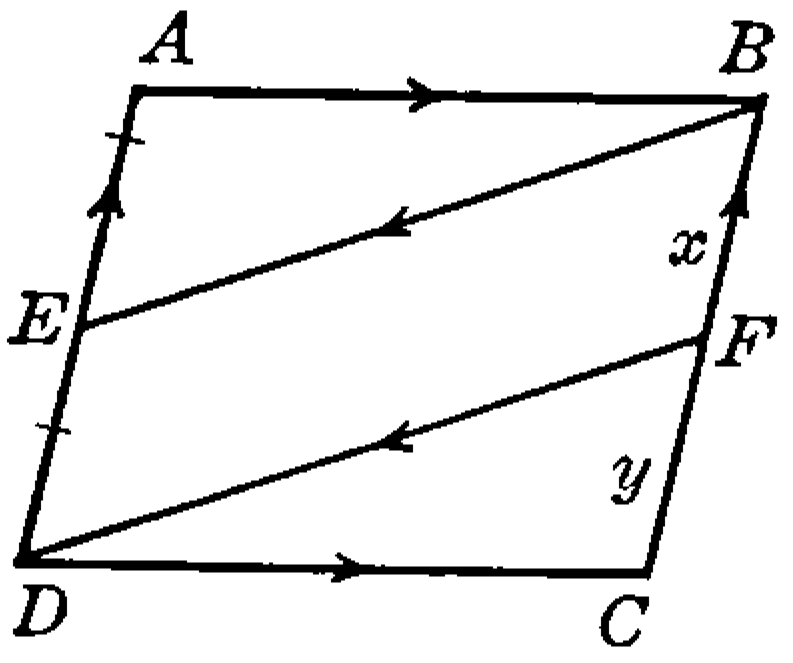
Given:
$AB \parallel DC$ (1)
$AD \parallel BC$ (2)
$AE = ED$ (3)
$BF = FC$ (4)
Prove:
$EB \parallel DF$ (5)

Analysis. Since we are trying to prove lines parallel, we study Table IV (page 128). No. 9 will enable us to prove $EB \parallel DF$, provided we can prove $ED = BF$. Referring to Table I, we see that No. 6 will prove $AD = BC$. We need only divide this equation by 2. Write out the synthetic proof.



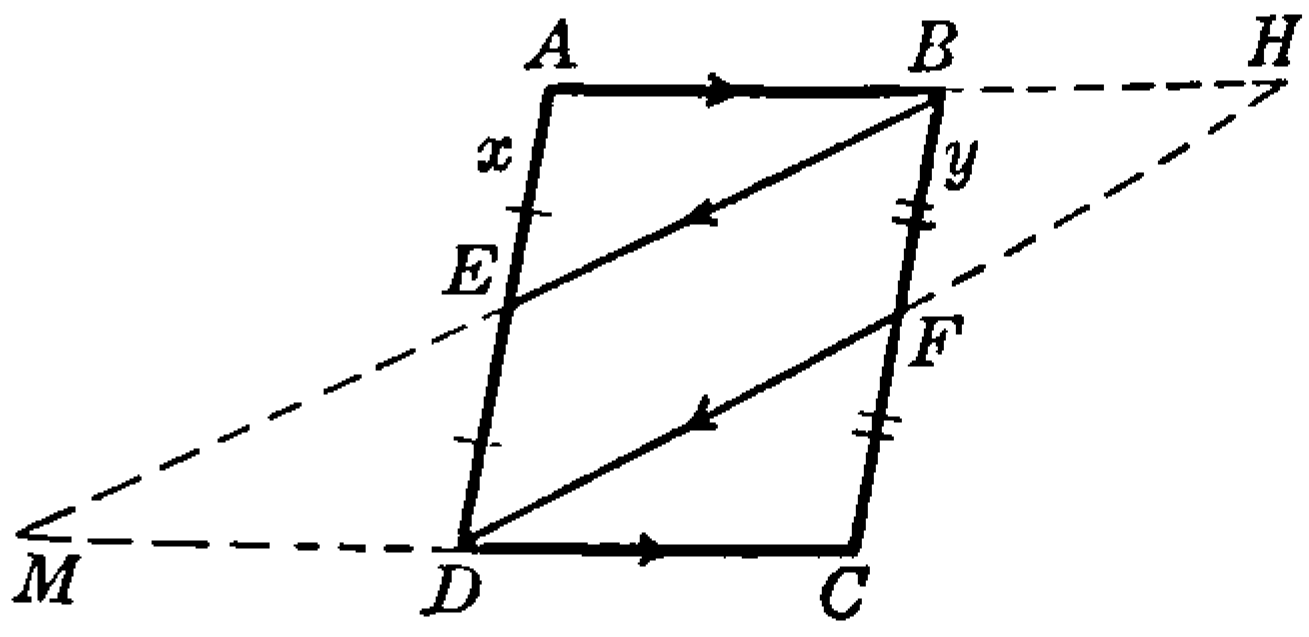
Given:
$EB \parallel DF$ (5)
$AD \parallel BC$ (2)
$AE = ED$ (3)
$BF = FC$ (4)
Prove:
$AB \parallel DC$ (1)

Converse 1: *Analysis.* The proof of this is similar to the original. Begin with ED and BF as the opposite sides of $\square EBFD$, and multiply the equation by 2.



Given:
$AB \parallel DC$ (1)
$AD \parallel BC$ (2)
$AE = ED$ (3)
$EB \parallel DF$ (5)
Prove:
$BF = FC$ (4)

Converse 2: *Analysis.* This time we have two parallelograms. From them, $AD = BC$ and $ED = x$. Subtracting, we obtain $AE = y$.



Given: $AB \parallel DC$ (1)
$EB \parallel DF$ (5)
$AE = ED$ (3)
$BF = FC$ (4)
Prove: $AD \parallel BC$ (2)

Converse 3: *Analysis.* In Table IV, No. 9 looks promising. To use this, however, we must prove $AB = DC$. We have given two sets of parallel lines. These should form a parallelogram. We therefore prolong them as shown in the figure, and we see that the sides BH and MD of $\square BHDM$ are equal. From the congruent triangles we may now prove that $AB = MD$ and $DC = BH$.

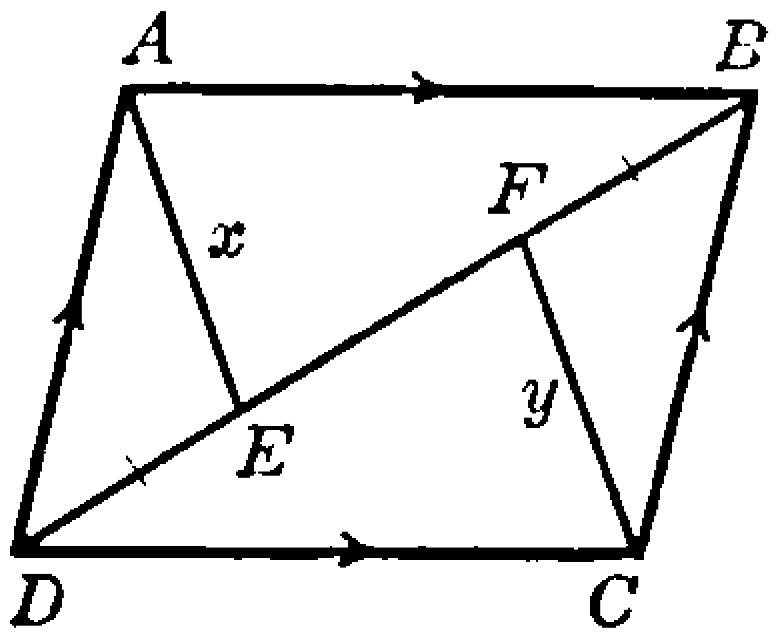
Write out the synthetic proof.

1. (*a*) If, from two opposite vertices of a parallelogram, perpendiculars are drawn to the diagonal connecting the other two vertices, then the perpendiculars are equal.

(*b*) Prove the converse in which the conclusion is that a pair of opposite sides of the quadrilateral are parallel.

2. Prove: A line segment through the point of intersection of the diagonals of a parallelogram, terminating in the sides of the parallelogram, is bisected by the diagonals.

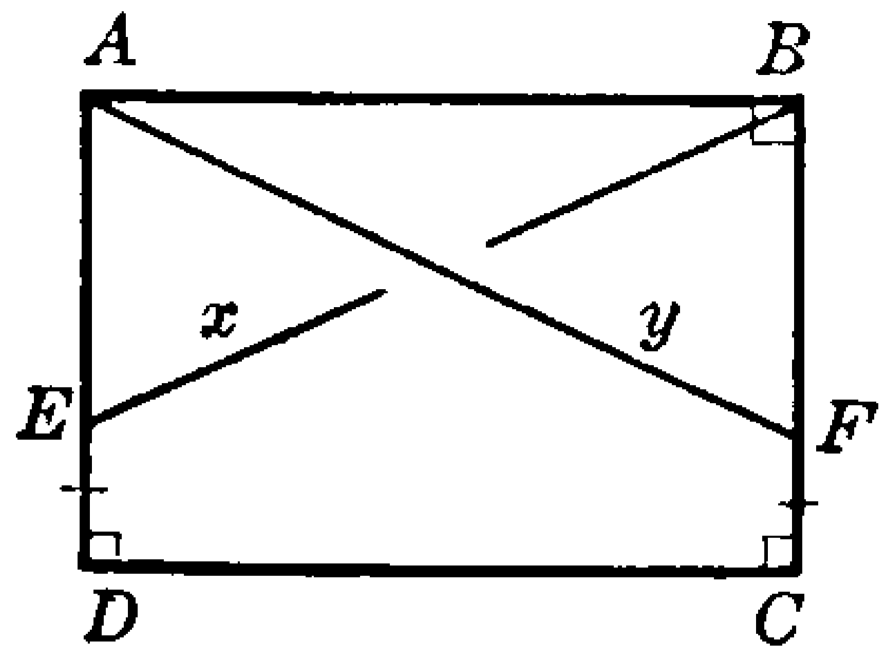
3. (*a*)



Given:
$AB \parallel DC$
$AD \parallel BC$
$DE = FB$
Prove:
$AE \parallel FC$

(*b*) Prove one or more converses.

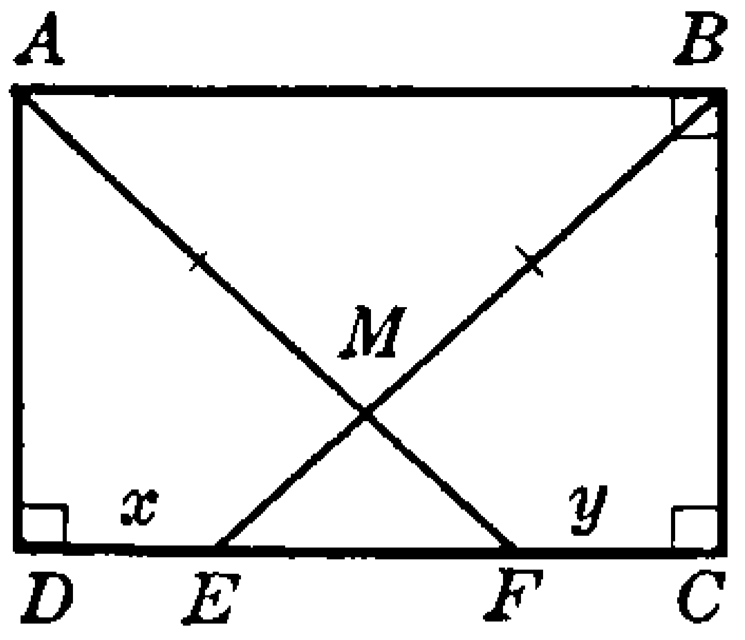
4. (*a*)



Given:
$\angle D = 90°$
$\angle C = 90°$
$\angle B = 90°$
$ED = FC$
Prove:
$AF = EB$

(*b*) Prove one or more converses.

5. (*a*)



Given:
$\angle D = 90°$
$\angle C = 90°$
$\angle B = 90°$
$AM = MB$
Prove:
$DE = FC$

(*b*) Prove one or more converses.

6. (*a*) A point that is equidistant from two consecutive vertices of a rectangle is equidistant from the other two vertices.

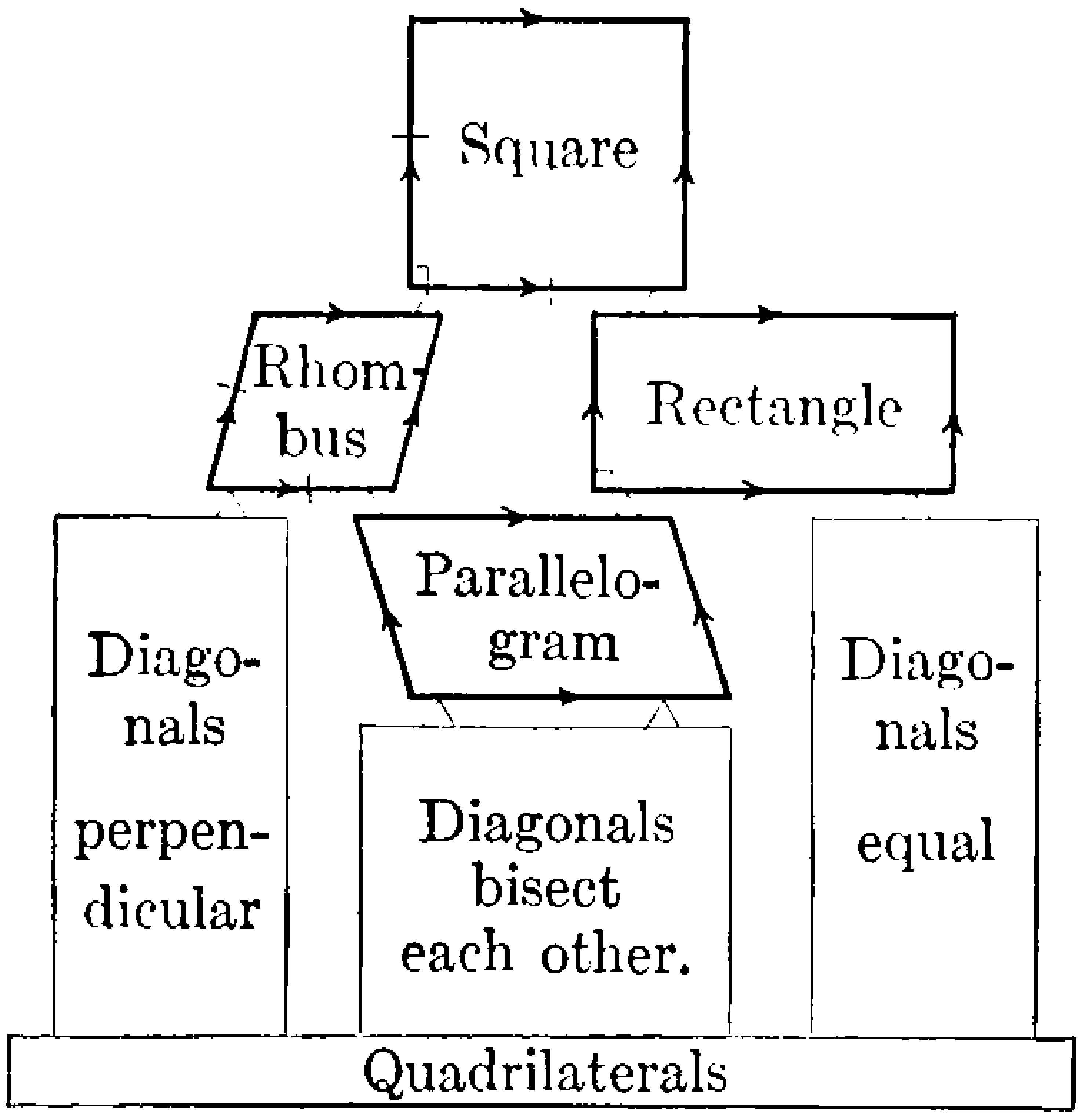
(*b*) Prove one or more converses.

7. Prove: The line through the midpoints of the bases of an isosceles trapezoid is perpendicular to them.

8. Prove: If the bisectors of a pair of base angles of an isosceles trapezoid meet on the opposite base, then that base equals the sum of the legs.

SUPERIOR WORK

S1. The diagram is a mechanical classification of quadrilaterals in terms of three relationships between their diagonals. From it we can see that the condition that the diagonals bisect each other is both necessary and sufficient for a quadrilateral to be a parallelogram, for that column *alone* supports the parallelogram. The condition that the diagonals are equal is necessary for the rectangle, because if that column were removed the rectangle would topple. However, it is not sufficient, since it is not the sole support of the rectangle.



Determine all the necessary and sufficient conditions implied in the diagram, and prove each of them.

S2. (*a*) If two opposite angles of a quadrilateral are right angles, then the bisectors of the other two angles are parallel.

(*b*) Prove one or more converses.

S3. (*a*) The bisectors of the angles of a quadrilateral form another quadrilateral, in which the opposite angles are supplementary.

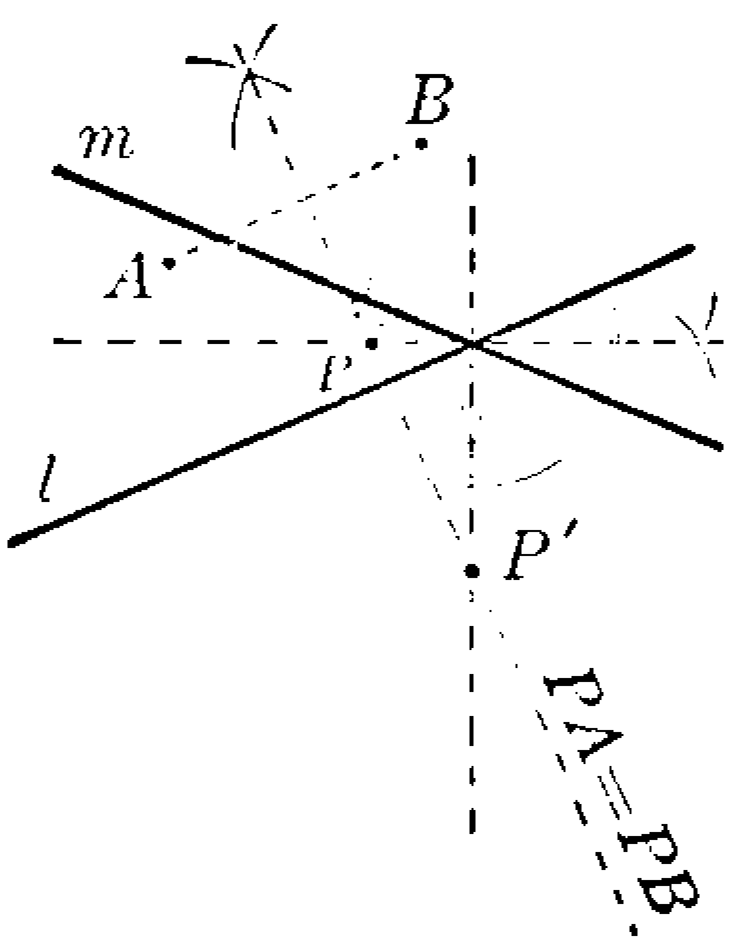
(*b*) What kind of quadrilateral is formed by the bisectors of the angles of a parallelogram? of a rectangle?

CONSTRUCTION EXERCISES

Illustrative Example. Locate a point that is equidistant from two given lines and also from two given points.

To solve this problem we must construct the locus of a point equidistant from two given lines l and m, and the locus of a point equidistant from two given points A and B ($PA = PB$). The intersection of these two loci is the point we are seeking.

Case I. The Given Lines Intersect



Given: Lines l and m and pts. A and B

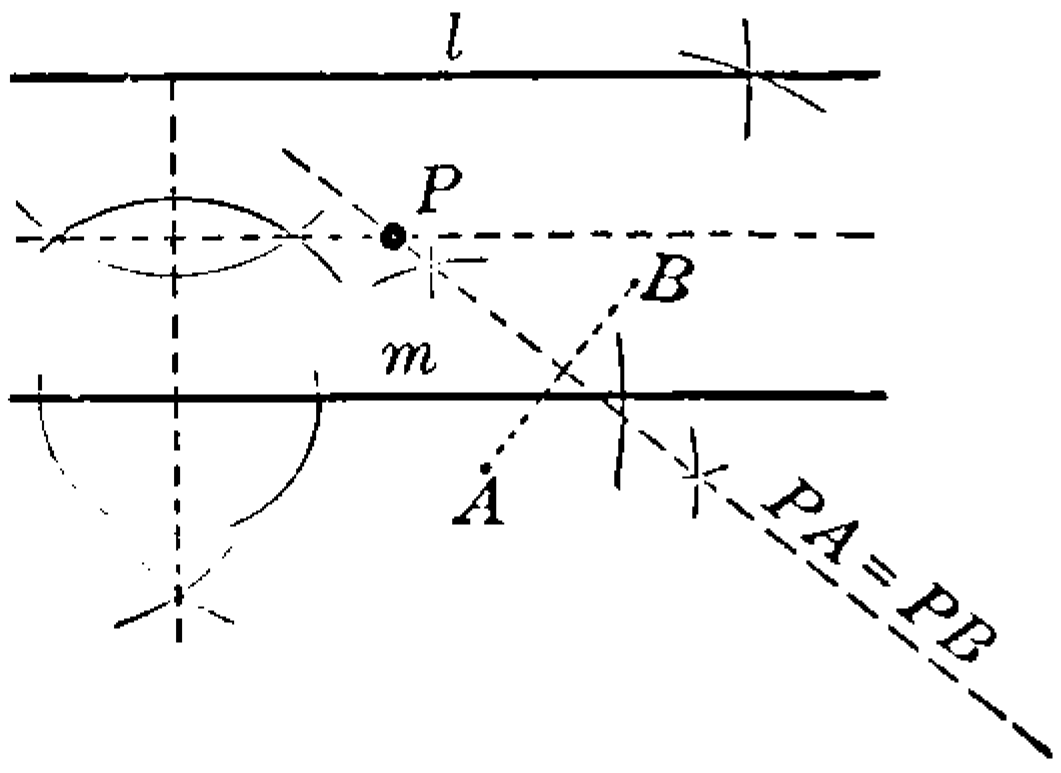
Req.: To locate a point P equidistant from l and m, also from A and B

Conclusion: P or P' the required point.

Discussion: Under what conditions can the problem not be solved at all, and under what conditions will the problem have one or more solutions? If the perpendicular bisector is parallel to one of the angle bisectors, there will be but one point of intersection. If the perpendicular bisector is not parallel to an angle bisector (the general case), there will be two points of intersection. Finally, if the perpendicular bisector coincides with one of the angle bisectors, there will be an infinite number of points that will satisfy the problem. There must always be at least one solution.

The discussion may be summarized as follows: *Possible Solutions:* 1, 2, or infinite.

Case II. The Given Lines Are Parallel



Given: $l \parallel m$ and pts. A and B

Req.: To locate a point P equidistant from l and m, also from A and B

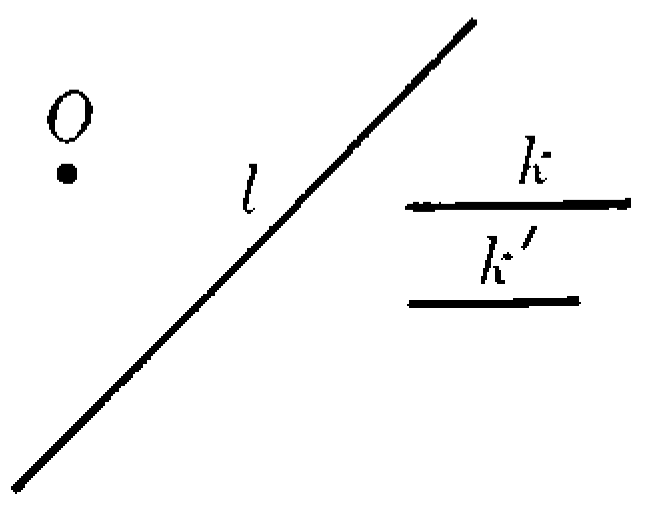
Conclusion: P the required point

Possible Solutions: 0, 1, or infinite.

1. Locate a point that is a given distance from a given point and another given distance from a given line.

Given: Pt. O, line l, and distances k and k'

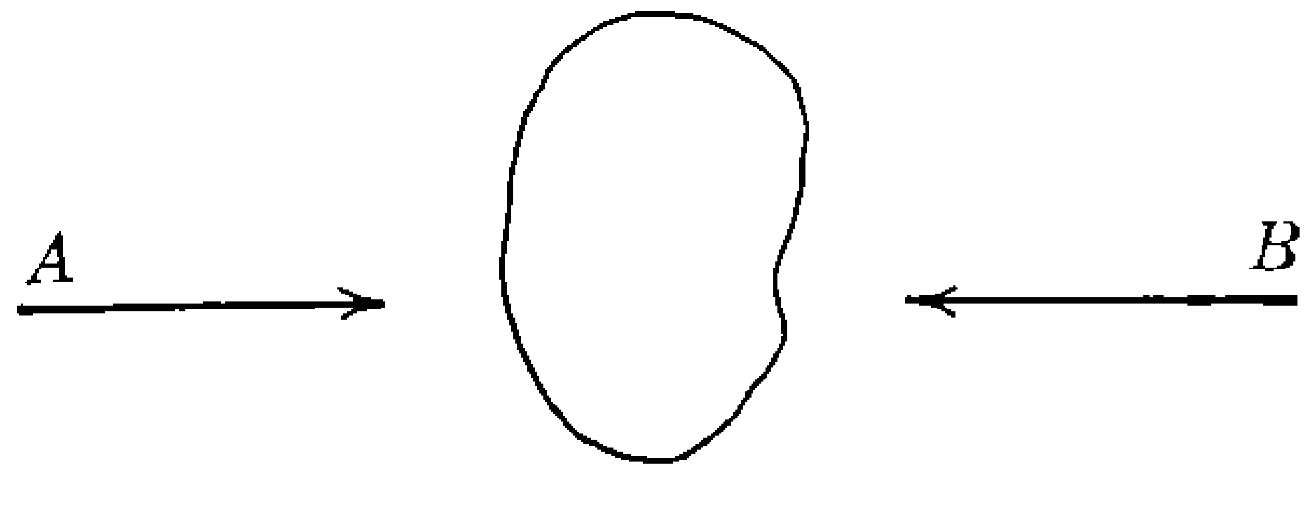
Req.: To locate a pt. that is a distance k from O, and k' from l



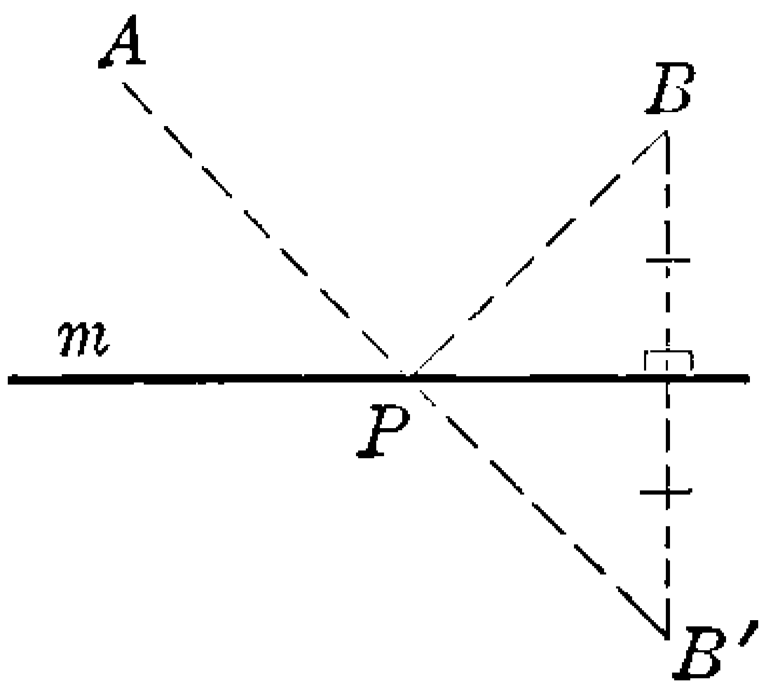
2. Given three points. Locate another point that is a given distance from one of the three given points and equidistant from the other two.
3. Locate a point that is a given distance from a given point and equidistant from two given lines.
4. Locate a point that is a given distance from a given line and equidistant from two given points.
5. Locate a point that is a given distance from one of three given lines and equidistant from the other two.
6. Locate a point that is equidistant from two given points and also equidistant from two other given points.
7. Locate a point that is equidistant from two given lines and also equidistant from two other given lines.
8. Locate a point equidistant from three given points.
9. Locate a point that is equidistant from three given lines.
10. Locate a point that is a given distance from each of two given lines.

SUPERIOR WORK

S1. Given two points A and B, with an obstruction between them so that it is impossible to lay a straightedge from one point to the other, devise a construction for drawing two lines through A and B that will fall in one straight line.



S2.



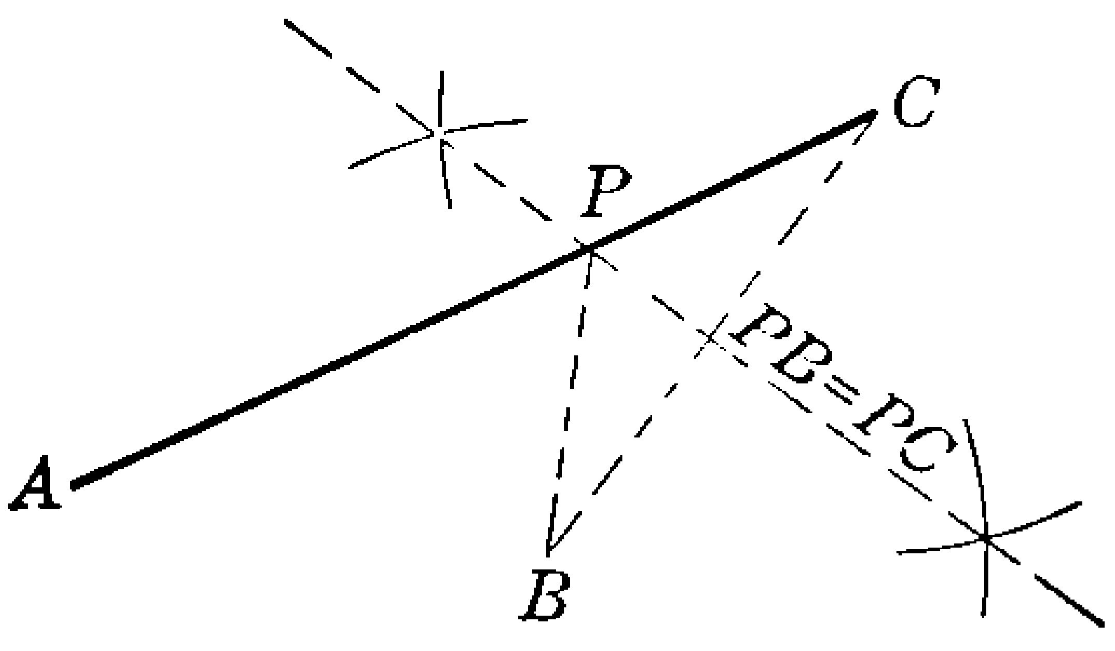
Given:
Line m
Pts. A and B
Req.:
To cons. the shortest distance from A to m to B

Analysis. Find a point B' so that m is the perpendicular bisector of BB'. Since m is the locus of a point equidistant from B and B', the path APB is the same length as the path APB'. Now, we know that the shortest distance from A to B' is a straight line. Therefore, we can locate the point P on m.

(*a*) Make the construction.

(*b*) A ray of light follows the shortest path from A to a mirror m and then to B. Can you find any relationship between the angles that the rays AP and PB make with mirror m?

S3. An aviator has enough fuel to fly a course from A to C. At what point in the course must he turn off so that he will have enough fuel to reach an alternate airport B?



We must find a point P on AC that is equidistant from B and C. This point is the intersection of AC with the locus of $PB = PC$.

A pilot leaves an airport with enough fuel to fly a course of 300 miles in a direction of 140° (page 41, S5). How far from his starting point must he turn off, so that he will have enough fuel to reach an airport 250 miles directly east of the first airport? (Make no allowance for wind.)

Make the construction to scale, using a protractor and a suitable scale for the distances. In what direction must the pilot head to get to the second airport?

S4. An aviator leaves an aircraft carrier that is sailing due north at 23 knots (23 nautical mph). He flies at 150 knots with only enough fuel for 6 hours. He is ordered to scout a course of 320° as long as possible before returning to the carrier. After how many hours must he turn and in what direction is the return leg of the journey? Use the method of the previous problem.

SUMMARY

201. A problem is a task to be performed when the method for doing it is unknown. In a construction problem, we must first locate certain points. The location of points is called the locus of the points. A locus may also be thought of as the path of a point that moves according to a certain restriction.

All geometric constructions depend on two postulates: (1) a straight line can be drawn through two points; and (2) a circle can be drawn if the center and the radius are given. The only instruments we are permitted to use in constructing a straight line and a circle are the straightedge and the compass.

The two-point theorem states that two points, each equidistant from the ends of a line segment, determine the perpendicular bisector of the line segment. The locus of points equidistant from two given points is the perpendicular bisector of the line segment connecting the two given points. The two-point theorem is used to construct the perpendicular bisector of a given line segment, and to construct the perpendicular to a given line through a given point. An angle is bisected by making it the vertex angle of an isosceles triangle and then constructing the altitude to the base. A triangle is copied by copying the three sides. To copy an angle, the triangle is completed and then copied. To construct parallel lines, one of the corresponding angles is made a copy of the other.

Classification helps to recall theorems, to select theorems for proving problems, and to discover new theorems.

In a polygon, each interior angle is the supplement of the adjacent exterior angle. The sum of the interior angles *and* the exterior angles of a polygon of n sides is therefore n straight angles. Of these, the sum of the exterior angles is always two straight angles, while the remaining $(n - 2)$ straight angles is the sum of the interior angles.

In every parallelogram it is *necessary* that the opposite sides be parallel and equal, that the opposite angles be equal, and that the diagonals bisect each other. Usually two of these conditions are all we need to know about a quadrilateral to prove that it is a parallelogram; that is, they are *sufficient* to make a quadrilateral into a parallelogram. We proved four of these cases as theorems:

A quadrilateral is a parallelogram if:
- (*a*) The opposite sides are equal.
- (*b*) Two sides are equal and parallel.
- (*c*) The opposite angles are equal.
- (*d*) The diagonals bisect each other.

In a rectangle it is *necessary* that the diagonals be equal. Conversely, to prove that a parallelogram is a rectangle, it is *sufficient* to know that the diagonals are equal.

In an isosceles trapezoid the base angles are equal; and, conversely, if the base angles of a trapezoid are equal, the trapezoid is isosceles.

Locus propositions include the locus of points at a given distance from a given point or a given line, and the locus of points equidistant from two given points or two given lines.

Analysis is a device that often helps to discover the solution of a problem. To analyze a problem, assume that the problem has been solved and then work backwards. After discovering the solution, arrange it in its logical order—beginning with the hypothesis and ending in the conclusion. Putting the solution together in this way is called *synthesis.*

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

problem § 142
locus § 144, § 147
circle (⊙) § 145
  radius (r) § 145
  center § 145, arc § 145
concentric circles § 145
construction (cons.) § 149
distance from a point to a line § 162
analysis and synthesis § 157
classification § 167
necessary and sufficient conditions § 174

TEST

1. (*a*) Construct two equilateral triangles with a common side.
(*b*) Prove that your figure is a rhombus.

2. (*a*) Prove that the diagonals of a rhombus are perpendicular to each other.
(*b*) Prove that a point on one diagonal of a rhombus is equidistant from the ends of the other diagonal.

3. (*a*) Construct a 30°–60° right triangle and the altitude on the hypotenuse.
(*b*) Prove that one segment of the hypotenuse cut off by the altitude equals one-fourth of the whole hypotenuse.

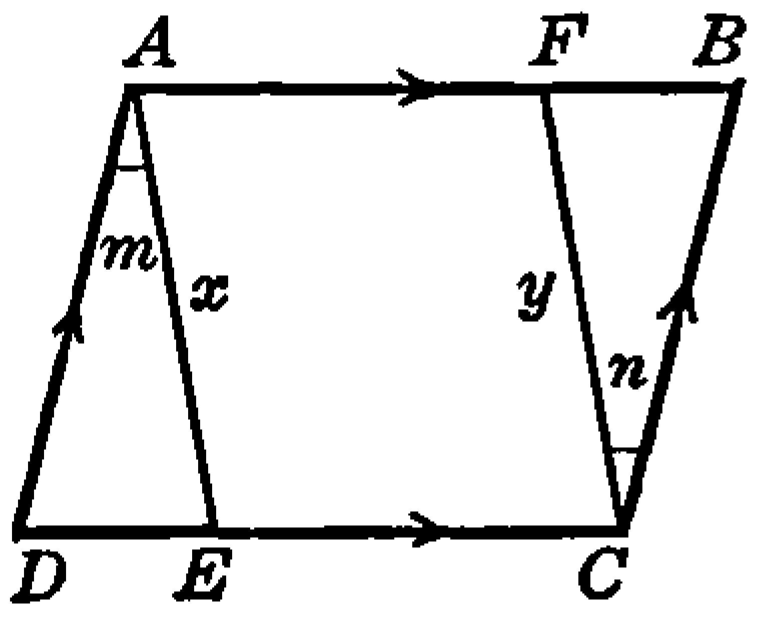
4. (*a*) Construct an isosceles trapezoid.
(*b*) Prove that a point equidistant from the ends of one base of your trapezoid is also equidistant from the ends of the other base.

5. (*a*) Construct a rectangle in which the diagonals form a 60° angle.
(*b*) Prove that each diagonal is twice the shorter side of the rectangle.

6. (*a*) Construct an isosceles triangle. From any point in the base, construct two parallel lines to the legs.
(*b*) Prove that the sum of your line segments within the triangle equals either leg of the triangle.

7. Construct a pentagon in which one angle is 60° and the other four angles are equal to one another.

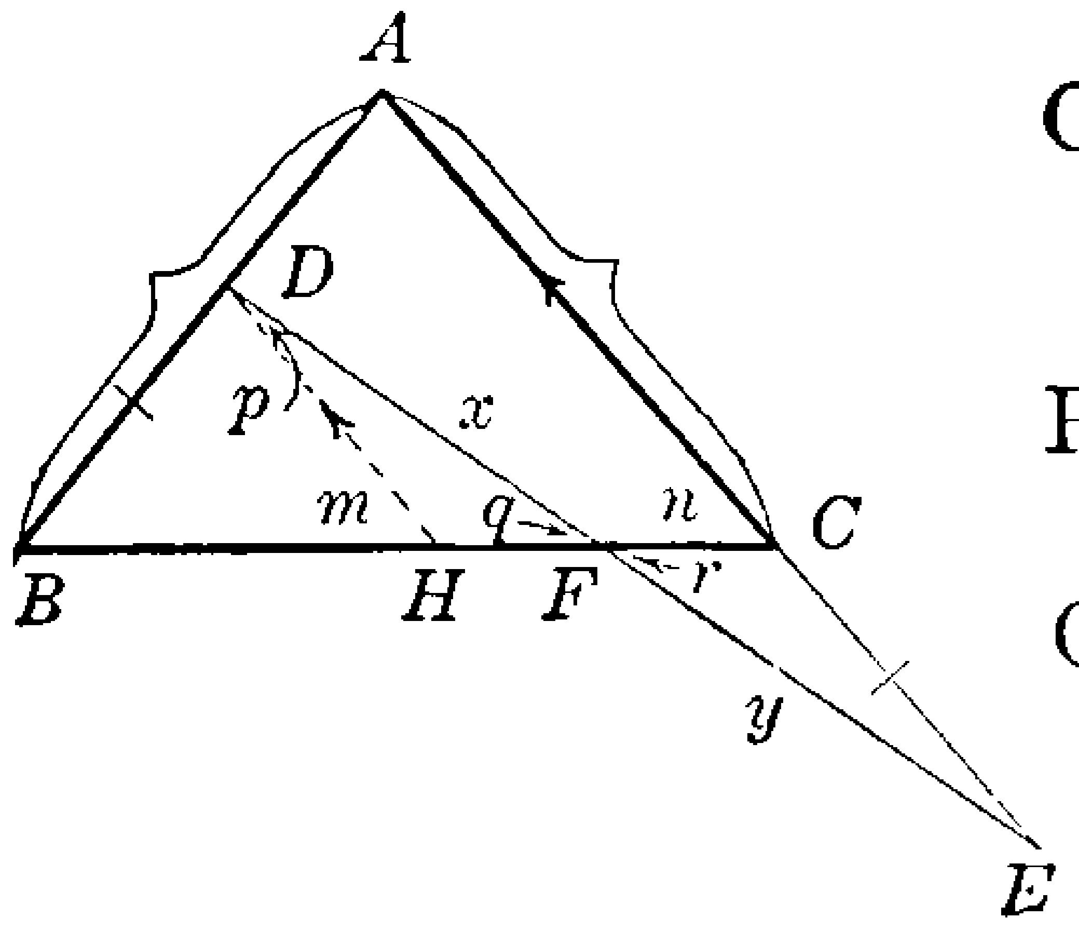
8.



Given:
▱ *ABCD*
$\angle m = \angle n$
Prove:
$AE = FC$

9. Construct a point that is equidistant from the sides of an angle and a given distance from one side of the angle. Discuss the solution.

10. State the 9 reasons for the statements in the proof:



Given:
$AB = AC$
$BD = CE$

Prove:
$DF = FE$

Cons.:
Draw
$DH \parallel AE$.

Statements	*Reasons*
1. $\angle B = \angle n$	1. ? .
2. $\angle m = \angle n$	2. ? .
3. $\angle B = \angle m$	3. ? .
4. $BD = DH$	4. ? .
5. $DH = CE$	5. ? .
6. $\angle p = \angle E$	6. ? .
7. $\angle q = \angle r$	7. ? .
8. $\triangle DHF \sim \triangle CEF$	8. ? .
9. $DF = FE$	9. —?—.

Part Two

Patterns of Thinking

6

Dynamic Thinking

The Circle

ANGULAR MEASURE OF ARCS

Types of Thinking

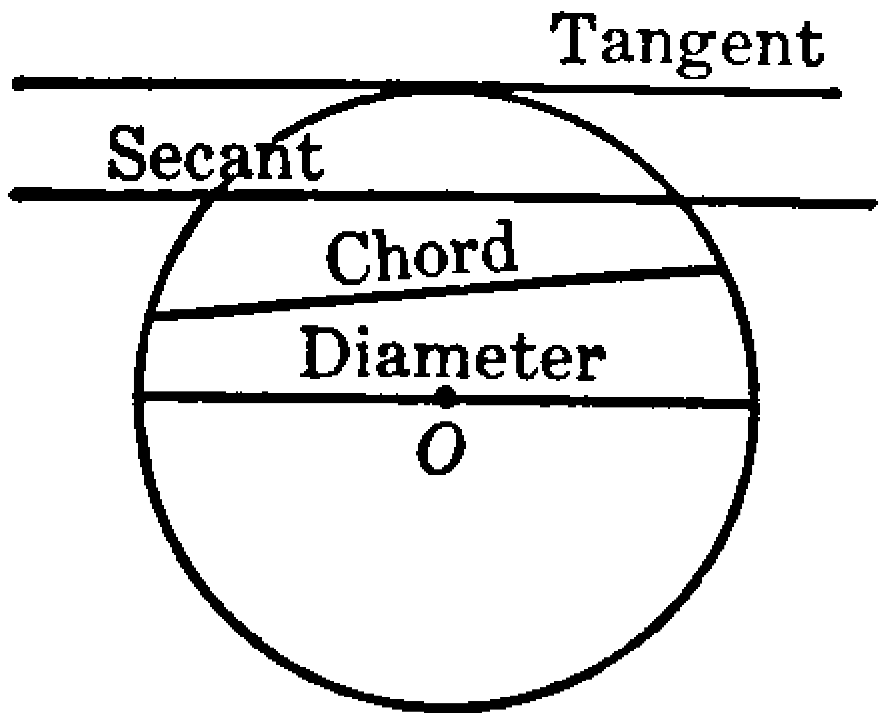
202. In Part One it was shown what kind of argument is accepted as proof in geometry. It was then learned that geometric knowledge is in the form of assumptions and theorems. From such basic assumptions about space as the parallel postulates and the two-point postulate we proved, through reasoning, a number of propositions—some directly and some indirectly. This type of thinking we call *deduction*. Sometimes we call it *postulational* thinking, because it is based on postulates. We saw how *induction*, another type of thinking that depends on experimental evidence, often helps us in solving problems and discovering new theorems. We also discovered that by forming the converse of a proposition, by illustrating it, and by forming more general propositions, we were able to organize and classify our facts. This basic and general type of thinking runs all the way through geometry.

In Part Two we shall study how some special types of thinking best fit the particular things we study. In this chapter we shall use dynamic thinking to study the circle and the various straight-line figures connected with it. To think about a figure dynamically means to study it, not as completed and stationary, but as a figure that is being formed or changed. It is a geometry of figures in motion rather than one of figures at rest. We have at times used this type of thinking previously, but in the study of the circle we shall use it even more. We shall be able to discover and understand many things about a circle by moving straight lines across it. The invention and use of the wheel many years ago was the first step toward producing motion on a large scale. As we merely look at a circle, we often think of it as being drawn by turning a compass, or we imagine the whole circle turning. Both historically and mechanically, therefore, it would seem that the study of the circle is a good place to think of figures in terms of motion.

Moving a Line Across a Circle

203. As a straight line moves across a circle, it first touches the circle at one point, the *point of contact* or the *point of tangency*. In that position we call it a *tangent*. As the line moves on, it cuts the circle in two points and becomes a *secant*. We call the part of the secant within the circle a *chord*. If the chord passes through the center of the circle, we call it a *diameter*.

204. Definition. A tangent to a circle is a straight line that touches the circle in only one point.



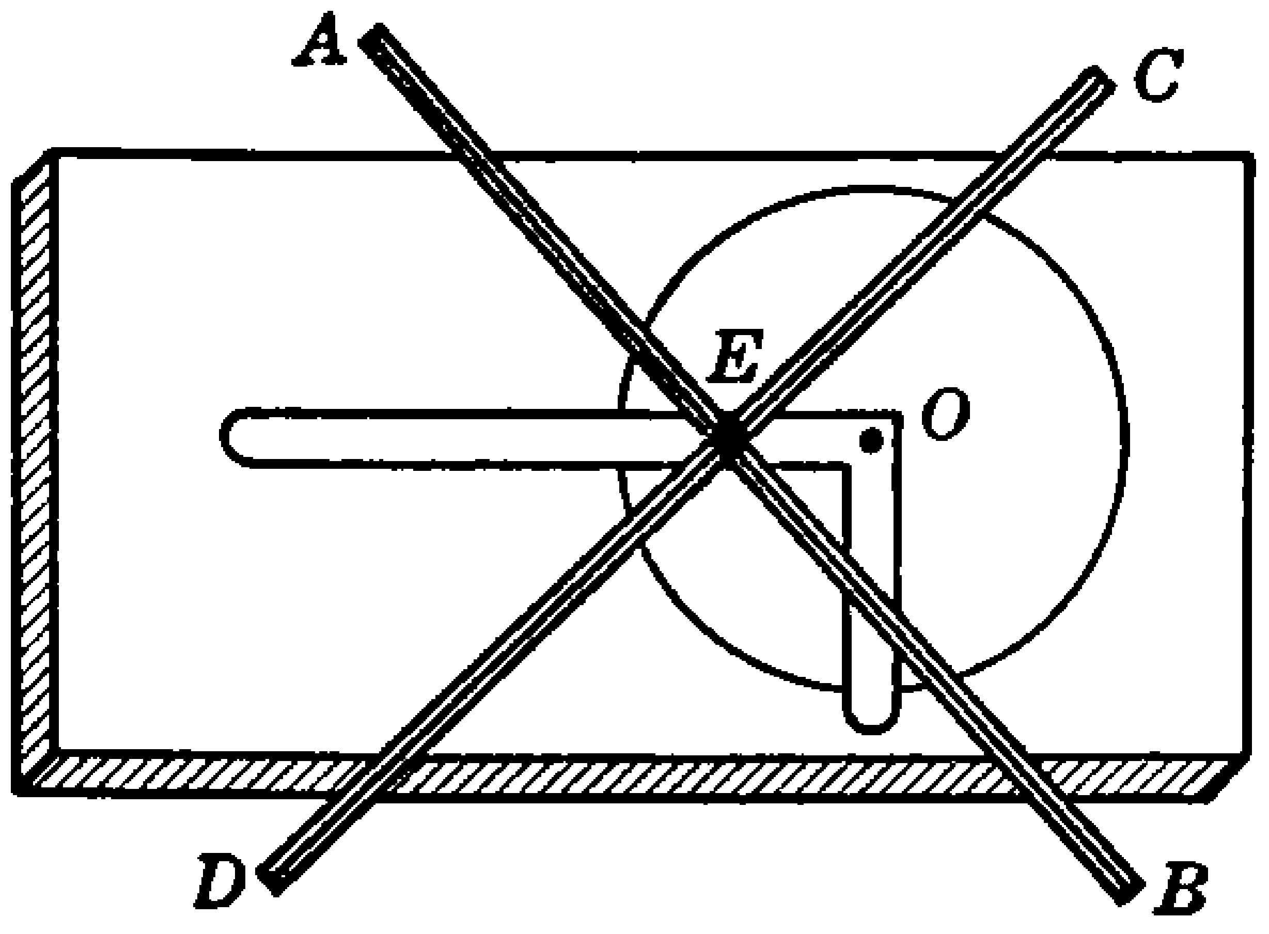
205. Definition. A secant of a circle is a straight line that cuts the circle in two points.

206. Definition. A chord of a circle is a line segment that connects any two points on the circle.

207. Definition. A diameter of a circle is a chord that passes through the center of the circle.

Dynamic Model

208. The model shown in the figure is useful in studying the motion of lines and angles across a circle. You can easily make such a model by joining two rods AB and CD loosely with a small bolt E.



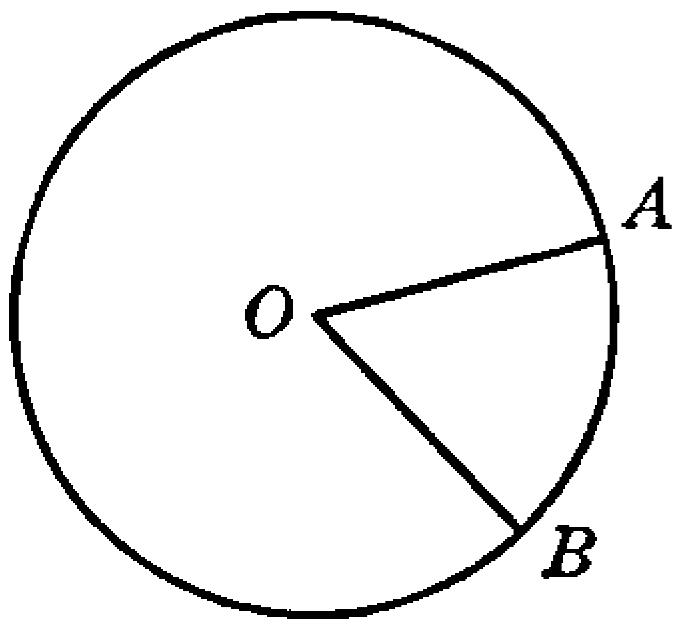
Paint a circle on a wooden board (or cardboard) and cut a right-angle slot in it as pictured. Now mount your rods on the board by inserting the bolt through the slot. Adjust the model so that the bolt moves freely along the slot and so that the angle between the rods may be changed smoothly.

Angular Measure of an Arc

209. When the bolt E in the model is at the center of the circle, we call the angles between the rods *central angles.* In § 41 a degree was defined as an angle that is $\frac{1}{180}$ of a straight angle. The arc that a central angle of one degree intercepts on a circle is called a one-degree arc. An arc is said to contain the same number of degrees as the central angle that intercepts the arc. Another way to say this is that the *degree measure* or *angular measure* of an arc is the degree measure of its central angle.

210. Definition. A central angle is an angle formed by two radii.

211. Definition. The degree measure of an arc is the degree measure of the central angle that intercepts (cuts off) the arc.



$\angle O$ a central $\angle$
$\angle O$ and $\overset{\frown}{AB}$ have the same degree measure, written
$\angle O \overset{\circ}{=} \overset{\frown}{AB}$

212. An arc has also *linear* measure or length. When we think of the length of an arc, we shall compare it with a segment of a *straight* line, but we are not yet ready to do this. In this chapter we are interested only in the degree measure or angular measure of an arc. If two arcs are in the same circle, or in equal circles, and they have equal degree measures, we shall call them equal. Later (§ 364), we shall see that such arcs have also equal lengths.

213. Definition. In the same circle, or in equal circles, equal arcs are arcs that have equal angular measures.

214. Corollary. A diameter bisects a circle.

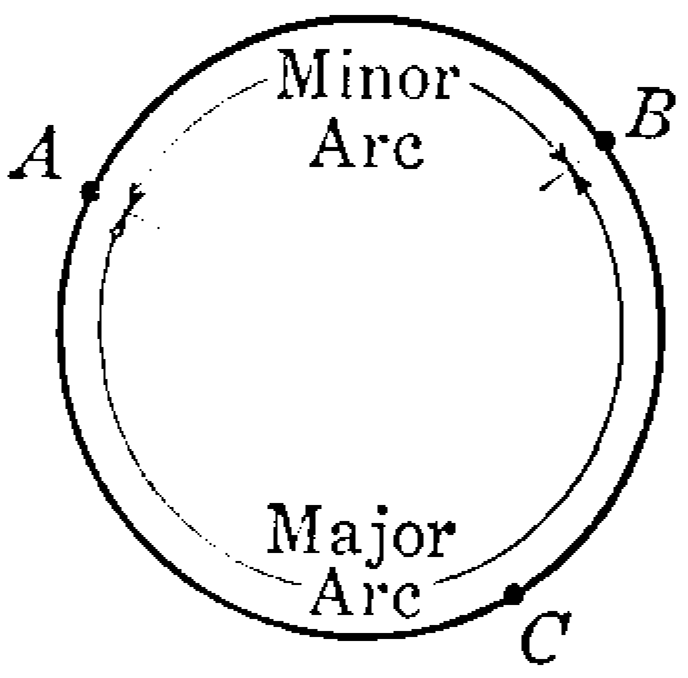
How large are the central angles?

215. Definitions. A semicircle is one-half of a circle.

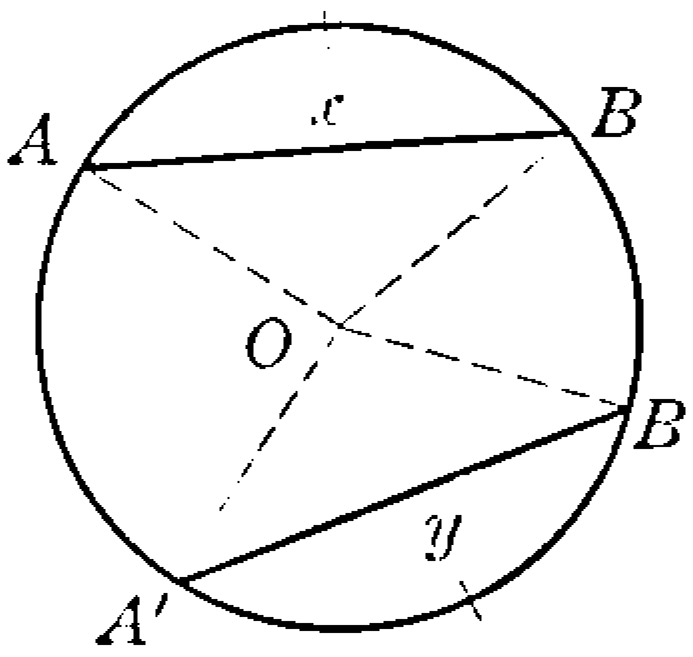
A minor arc is less than a semicircle.

A major arc is greater than a semicircle.

In the figure $\overset{\frown}{AB}$ refers to a minor arc. When we refer to a major arc, we use three letters, as $\overset{\frown}{ACB}$.



216. THEOREM. In the same circle, or in equal circles, if two arcs are equal, their chords are equal.



Given: $\overset{\frown}{AB} = \overset{\frown}{A'B'}$

Prove: $AB = A'B'$

Cons.: Draw radii OA, OB, OA', and OB'.

PROOF

Statements	*Reasons*
1. $OA = OA'$ $OB = OB'$	1. Def. ⊙.
2. $\angle AOB = \angle A'OB'$	2. Def. = arcs (§ 213).
3. $\triangle AOB \cong \triangle A'OB'$	3. s.a.s.
4. $AB = A'B'$	4. Def. $\cong$ $\triangle$.

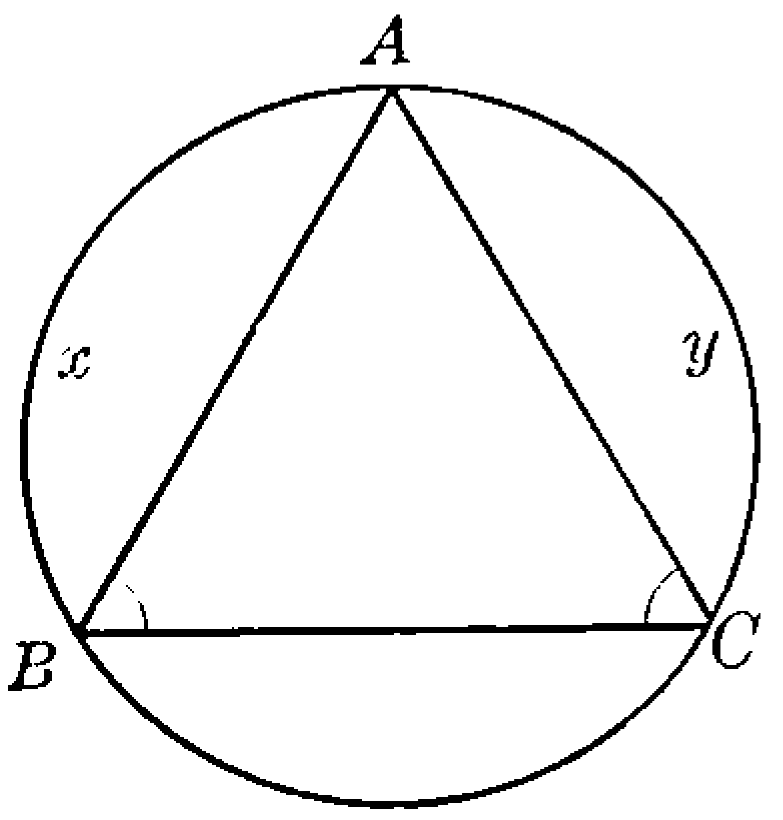
As you have learned, a definition contains a proposition and its converse. State the part of the definition used in each of Steps 1 and 2 above.

217. CONVERSE THEOREM. In the same circle, or in equal circles, if two chords are equal, their arcs are equal.

Draw a figure and write out the proof of this theorem.

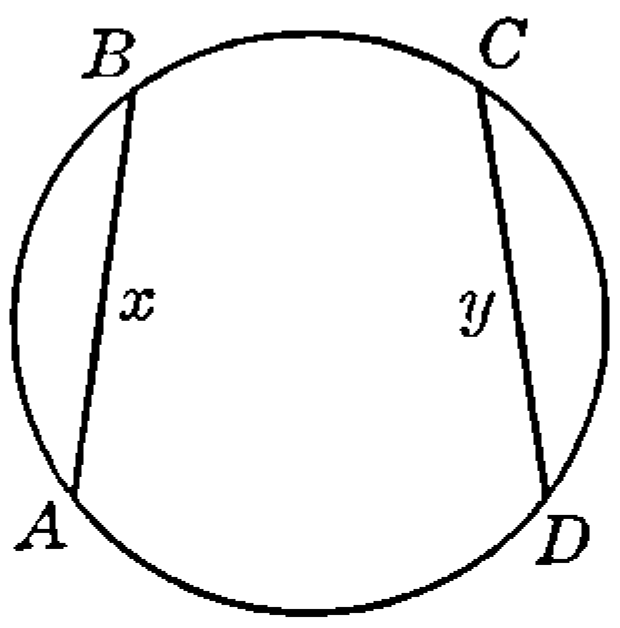
EXERCISES

1.



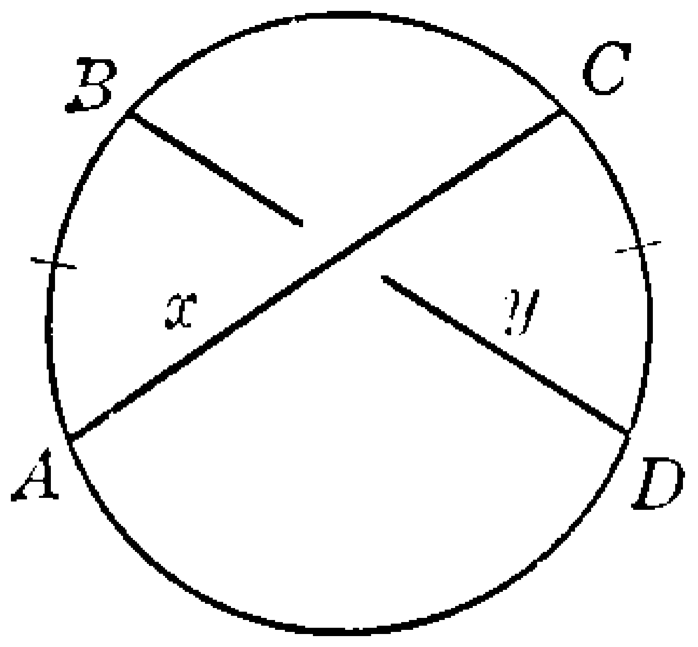
Given:
$\angle B = \angle C$
Prove:
$\overset{\frown}{AB} = \overset{\frown}{AC}$

2.



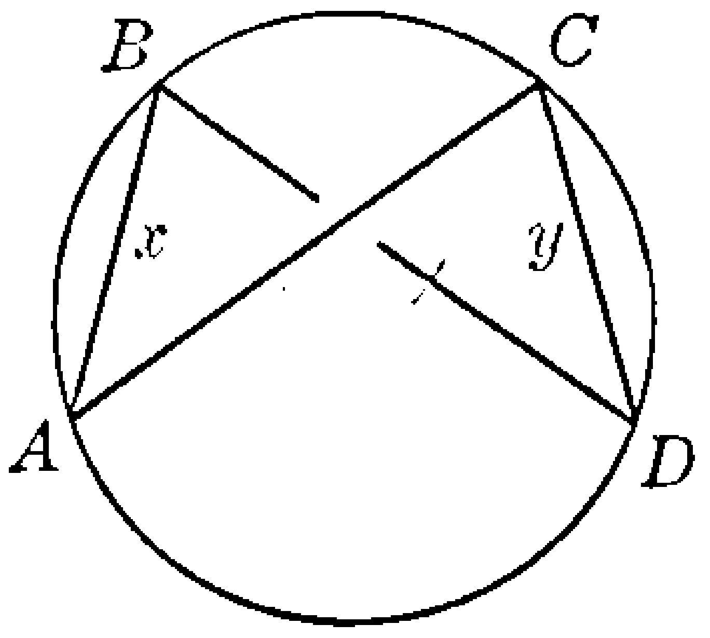
Given:
$\overset{\frown}{ABC} = \overset{\frown}{BCD}$
Prove:
$AB = CD$

3.



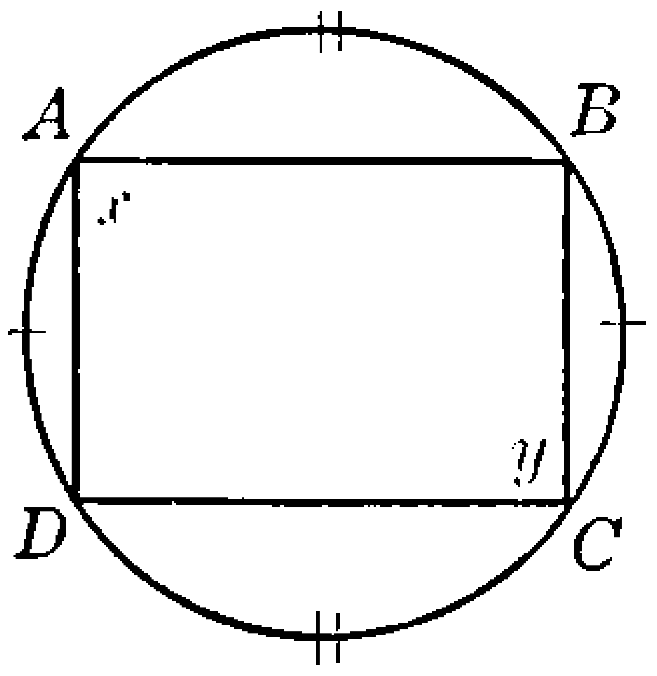
Given:
$\overset{\frown}{AB} = \overset{\frown}{CD}$
Prove:
$AC = BD$

4.



Given:
$AC = BD$
Prove:
$AB = CD$

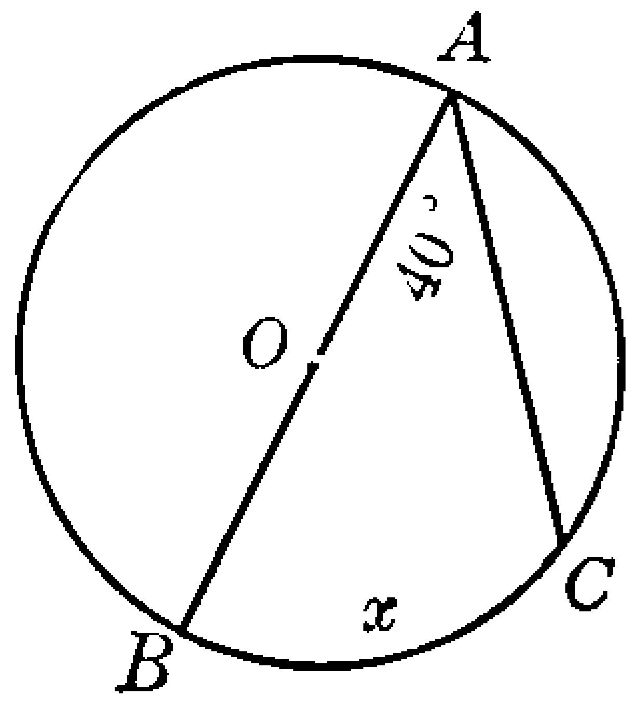
5.



Given:
$\overset{\frown}{AB} = \overset{\frown}{CD}$
$\overset{\frown}{AD} = \overset{\frown}{BC}$
Prove:
$\angle A = \angle C$

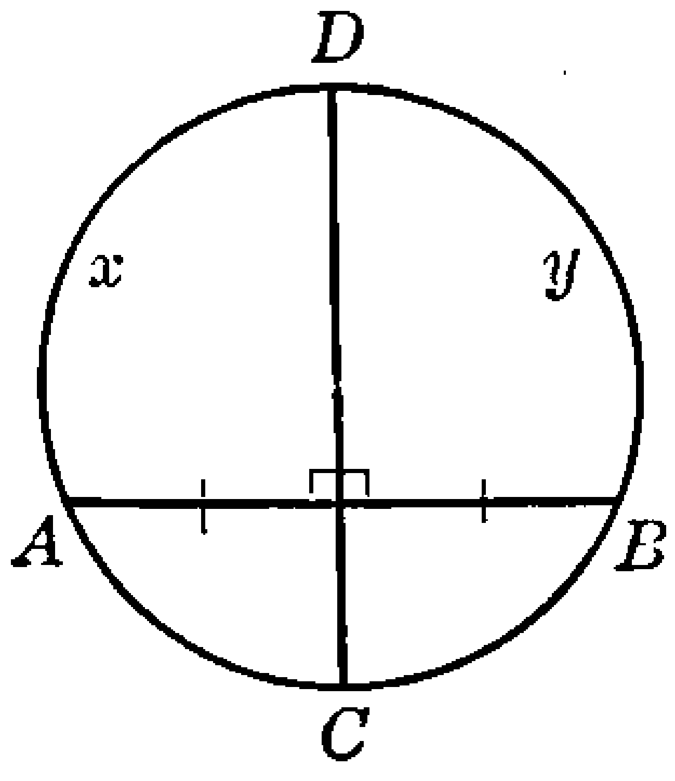
6. In a circle, a certain chord equals the radius of the circle. Find the number of degrees in the minor arc of the chord.

7.



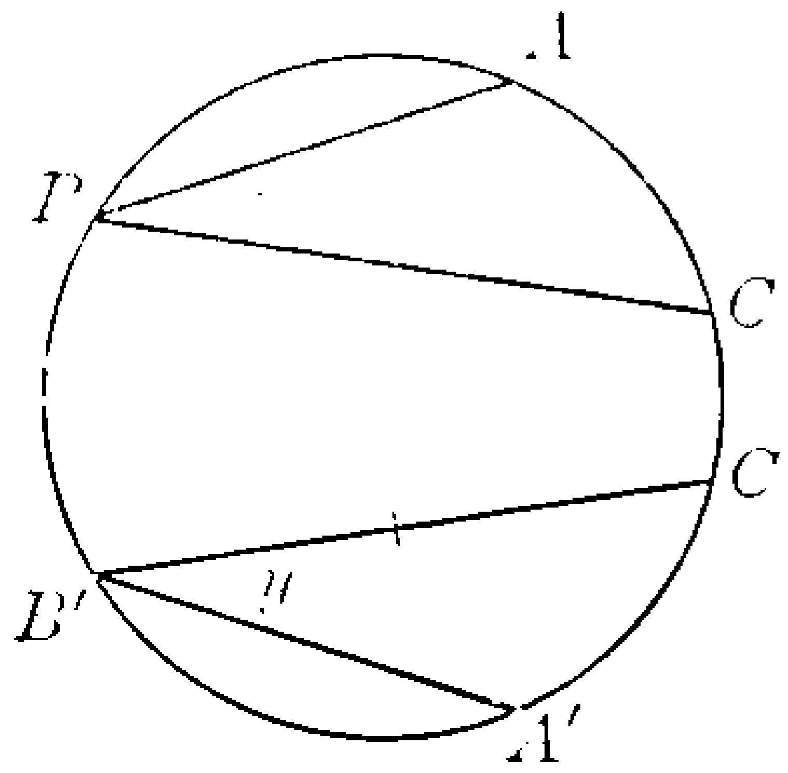
Given:
AB a diam.
of $\odot O$
$\angle A = 40°$
Prove:
$\overset{\frown}{BC} = —?—°$

8.



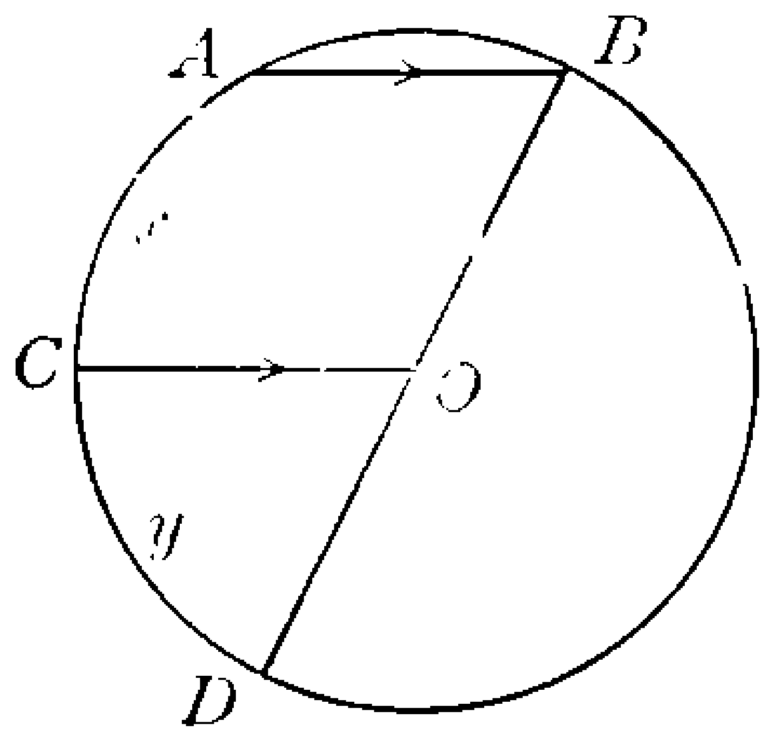
Given:
DC the $\perp$
bisector of AB
Prove:
$\overset{\frown}{AD} = \overset{\frown}{DB}$

9.



Given:
$AB = A'B'$
$BC = B'C'$
Prove:
$\angle B = \angle B'$

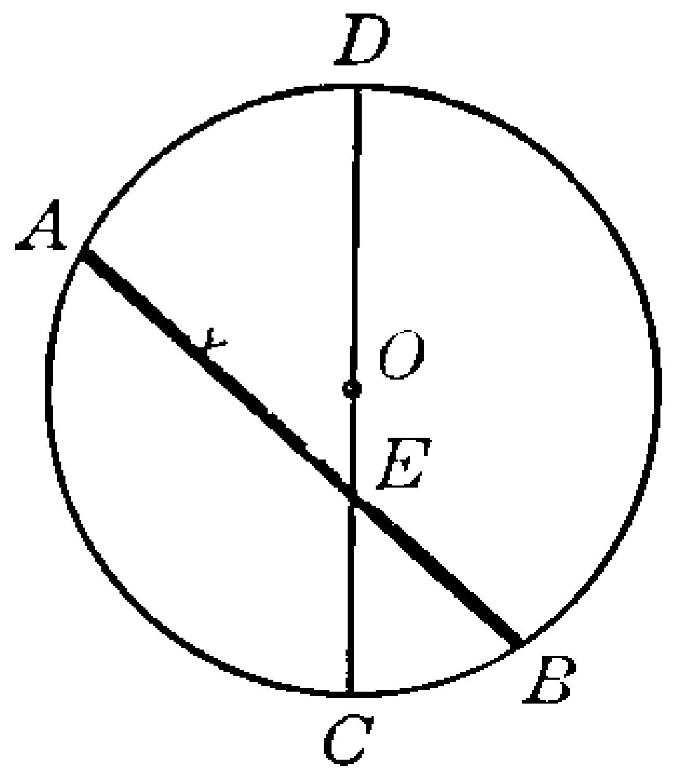
10.



Given:
C the center
of $\odot O$
$AB \parallel CO$
Prove:
$\overset{\frown}{AC} = \overset{\frown}{CD}$

A Diameter and a Chord

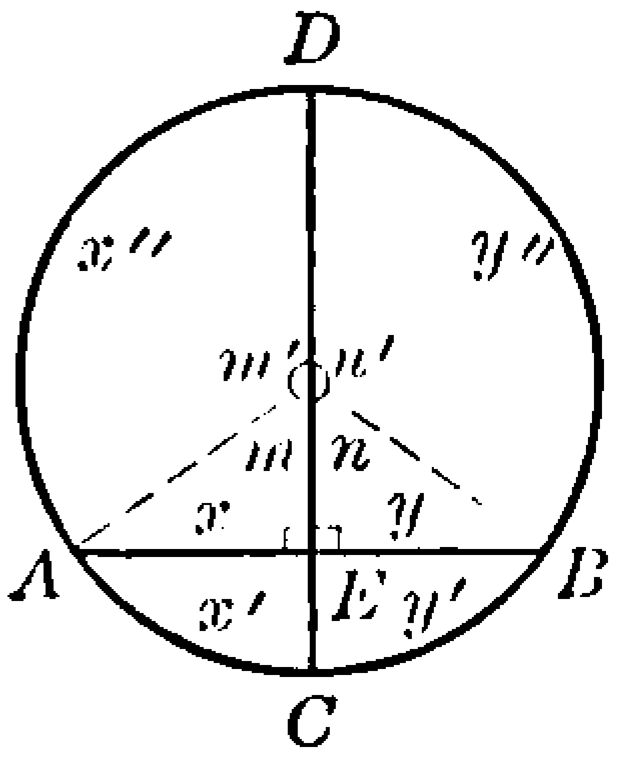
218. Set one rod CD of your dynamic model (§ 208) as a diameter, with the pivot bolt E away from the center. As you turn the other rod AB about E, do the parts AE and EB of the chord change in size? Try to discover in what position AE and EB will become equal. State this as a general proposition.



As you turn the chord AB, do the arcs AC and CB change in size? In what position of the chord will these arcs be equal? Can you say the same about the arcs AD and DB?

These observations lead us to the following fundamental theorem.

219. THEOREM. If a diameter is perpendicular to a chord, it bisects the chord and its arcs.



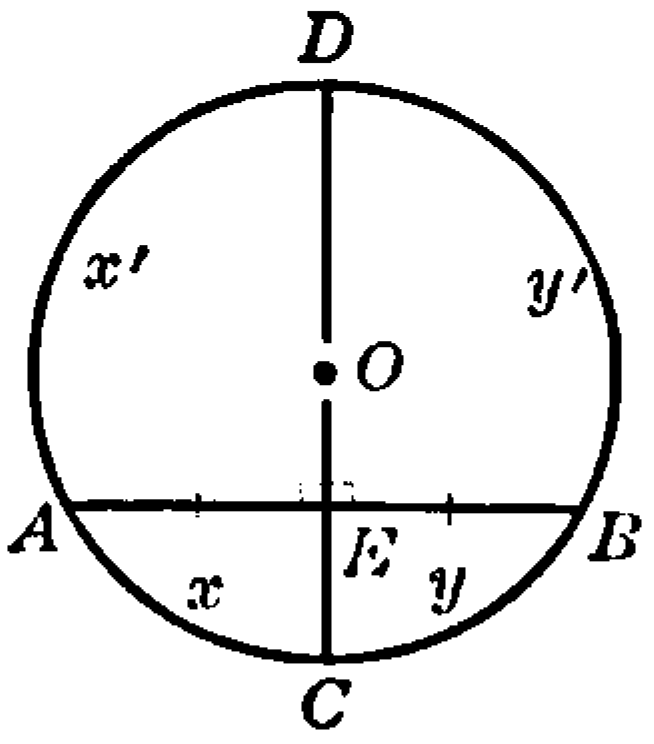
Given: DC a diam. of $\odot O$
$DC \perp AB$
Prove: $AE = EB$
$\overset{\frown}{AC} = \overset{\frown}{CB}$
$\overset{\frown}{AD} = \overset{\frown}{DB}$
Cons.: Draw radii OA and OB.

PROOF

Statements	*Reasons*
1. $OA = OB$	1. Def. $\odot$.
2. $OE \perp AB$	2. Given.
3. $AE = EB$ $\angle m = \angle n$	3. In an isos. $\triangle$, the alt. to the base, the median to the base, the bisector of the vertex $\angle$ are same line segment (§ 138).
4. $\overset{\frown}{AC} = \overset{\frown}{CB}$	4. Def. = arcs (§ 213).
5. $\angle m' = \angle n'$	5. Sups. of = $\measuredangle$ are =.
6. $\overset{\frown}{AD} = \overset{\frown}{DB}$	6. Def. = arcs.

We arrive at a converse of a proposition by interchanging a condition in the hypothesis with a condition in the conclusion (§ 97). In the theorem above, we can do this in many ways. The following is one important converse.

220. CONVERSE THEOREM. The perpendicular bisector of a chord passes through the center of the circle, and bisects the major and minor arcs.

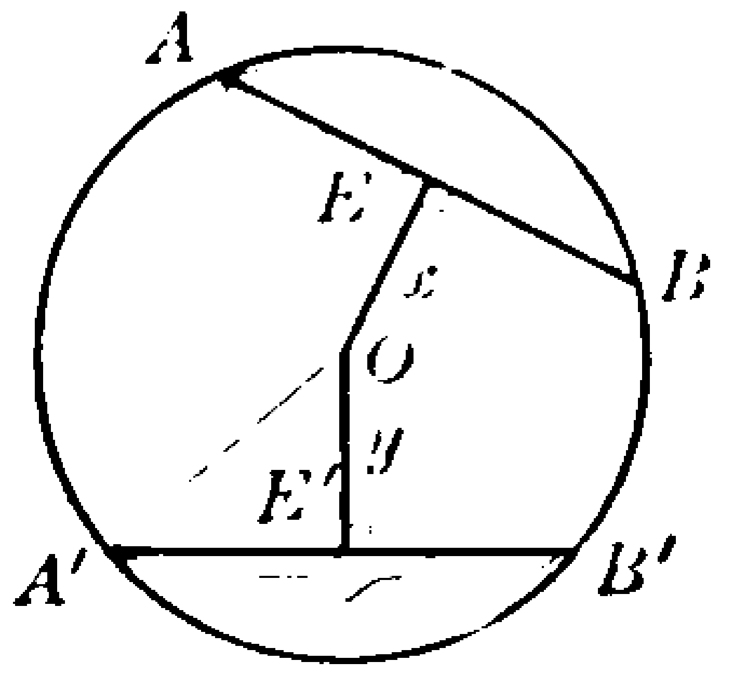


Given: $AE = EB$
$DC \perp AB$
Prove: DC a diam.
$\overset{\frown}{AC} = \overset{\frown}{CB}$
$\overset{\frown}{AD} = \overset{\frown}{DB}$

Write out the proof.
Note that DC is broken near the center O. The reason for this is that DC is not given as a diameter.

We shall now compare two chords in the same circle with their distances from the center. Recall from § 162 that the distance from a point to a line has been defined as the length of the perpendicular from the point to the line.

221. THEOREM. **In the same circle, or in equal circles, if two chords are equal, they are equally distant from the center.**



Given: In ⊙O
Chord AB = chord $A'B'$
$OE \perp AB$
$OE' \perp A'B'$
Prove: $OE = OE'$
Cons.: Draw OA and OA'.

Write out the proof.

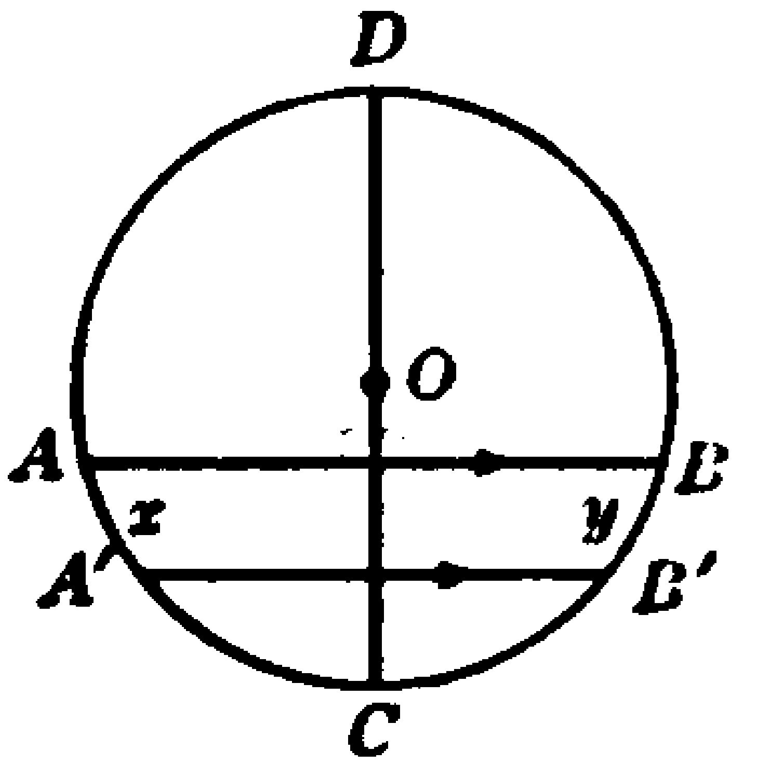
222. CONVERSE THEOREM. **In the same circle, or in equal circles, if two chords are equally distant from the center, they are equal.**

Draw a figure and write out the proof.

EXERCISES

1. (*a*) How many circles can be drawn through two given points? What is the locus of the centers of these circles?
   (*b*) Given the radius, construct a circle that passes through two given points. Discuss the number of possible solutions.
2. Given a circular arc, construct the center of the circle of which the arc is a part.
3. Bisect a given arc.
4. In a given circle, construct a chord that will be bisected by a given point.
5. Prove the following converses of the theorem about a diameter perpendicular to a chord:
   (*a*) If a diameter bisects a chord (not another diameter), then it is perpendicular to it.
   (*b*) If a diameter bisects one arc of a chord, it is perpendicular to the chord.
   (*c*) If a line bisects the major and minor arcs of a chord, then it is perpendicular to the chord.
   (*d*) If a line bisects a chord and one of its arcs, then it is perpendicular to the chord.
   (*e*) If a line is perpendicular to a chord and bisects one of its arcs, then it passes through the center of the circle.
6. Prove that the segments of a chord between two concentric circles are equal. *Hint:* Draw a diameter perpendicular to the chord.
7. A chord is prolonged equal distances in both directions, and the new ends are joined to the center of the circle. Prove that the two arcs within the triangle are equal.
8. Prove: If two equal chords intersect, a diameter through the point of intersection bisects the angle between the chords.
9. Prove: If the center of a circle is on the bisector of an angle, the circle cuts equal chords, if any, from the sides of the angle.

10.

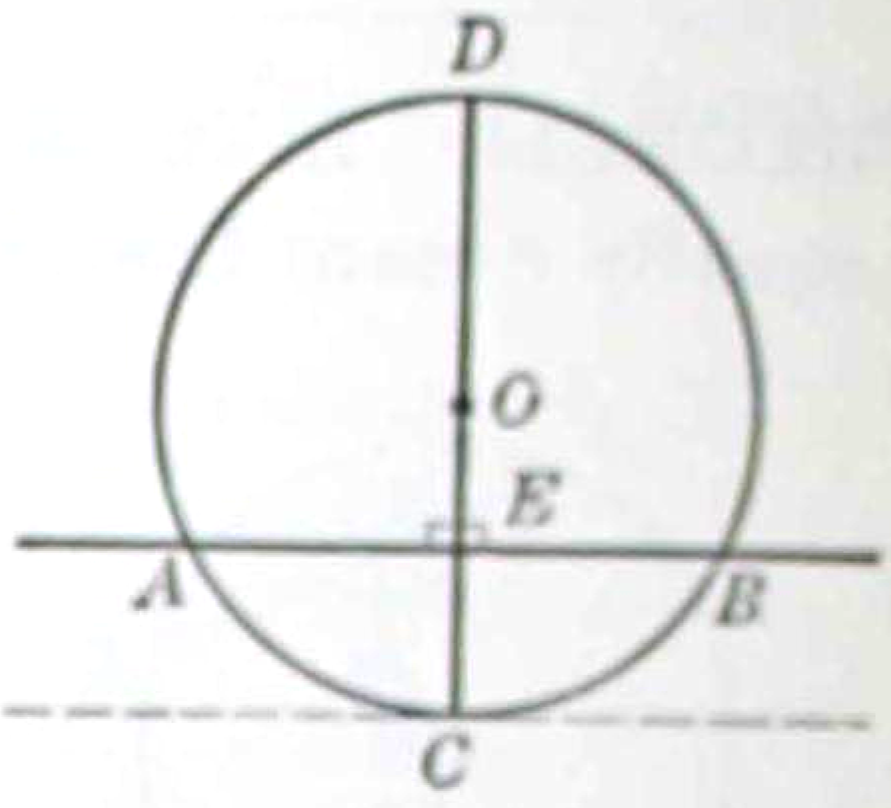


Given:
$AB \parallel A'B'$
Diam. $DC \perp AB$
Prove:
$\widehat{AA'} = \widehat{BB'}$

A Line through One End of a Diameter

223. Set the rods of your dynamic model (§ 208) in the position to illustrate Theorem § 219. Now move the bolt E to the point C on the circle. As the line segment AE becomes zero, why does EB also become zero? What happens to the points A and B? What becomes of the secant through A and B?

This leads us to the following theorem which we shall prove indirectly (§ 99).



We shall show that if the proposition were false it would contradict our theorem about a diameter perpendicular to a chord.

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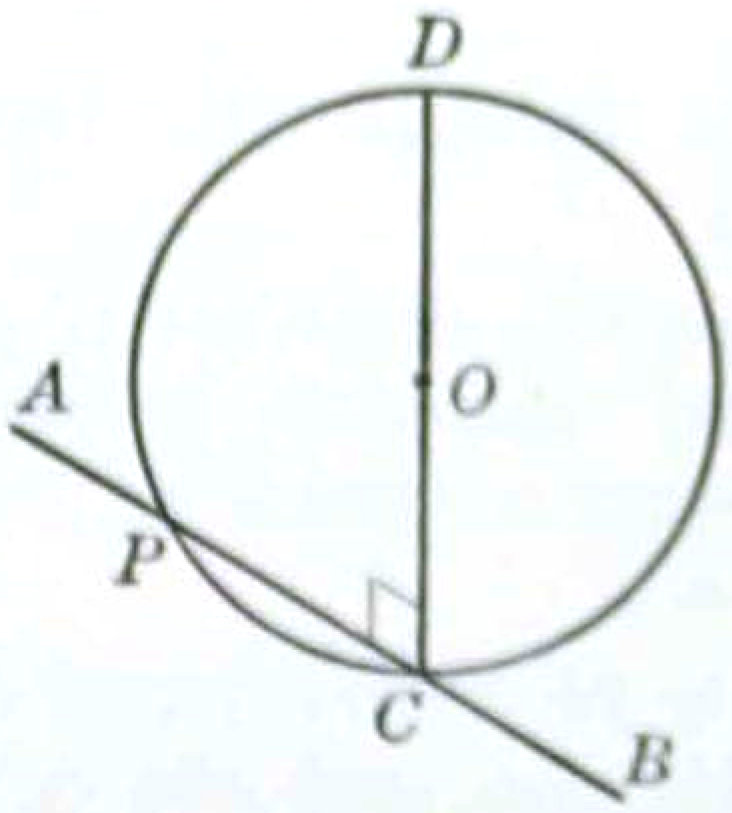
224. THEOREM. A line perpendicular to a diameter at one end is tangent to the circle.



Given: $AB \perp$ diam. CD at C

Prove: AB tan. to $\odot O$

INDIRECT PROOF

1. Either AB is tangent to $\odot O$ or not
2. *Assume* that AB is not tangent to $\odot O$, then it will cut the circle in some other point P
3. This contradicts the theorem that a diameter perpendicular to a chord (chord PC) bisects it
4. Therefore, AB is tangent to $\odot O$

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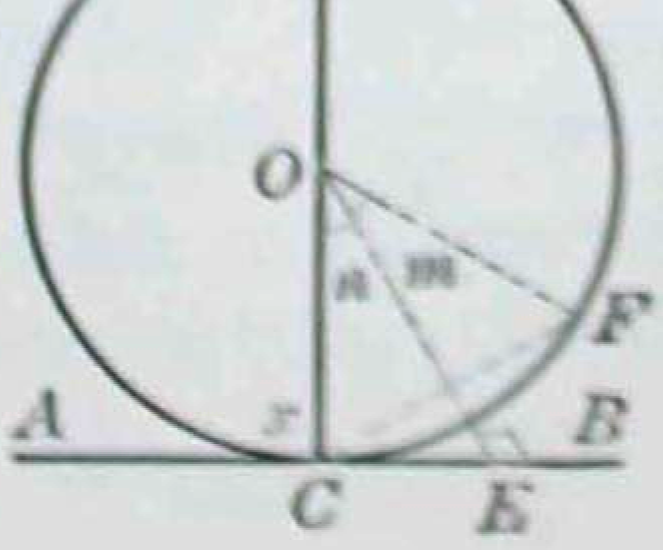
225. CONVERSE THEOREM 1. If a line is tangent to a circle at the end of a diameter, it is perpendicular to the diameter.



Given: AB tan. to $\odot O$ at C
DC a diam.

Prove: $DC \perp AB$

INDIRECT PROOF

1. Either $DC \perp AB$ or not	
2. *Assume* that DC is not perpendicular to AB, then we can draw $OE \perp AB$	
3. Draw radius OF making $\angle m = \angle n$ and chord CF.	
4. $\triangle OCF$ isos.	4. Def. ⊙.
5. $OE \perp CF$	5. In isos. △, the bisector of the vertex ∠ is ⊥ to the base (§ 138).
6. Now tan. AB and chord CF, two separate lines through the same pt. C, are both ⊥ to OE—which is impossible	6. Through a given pt., only one ⊥ can be drawn to a given line (§ 100).
7. Therefore, $DC \perp AB$	

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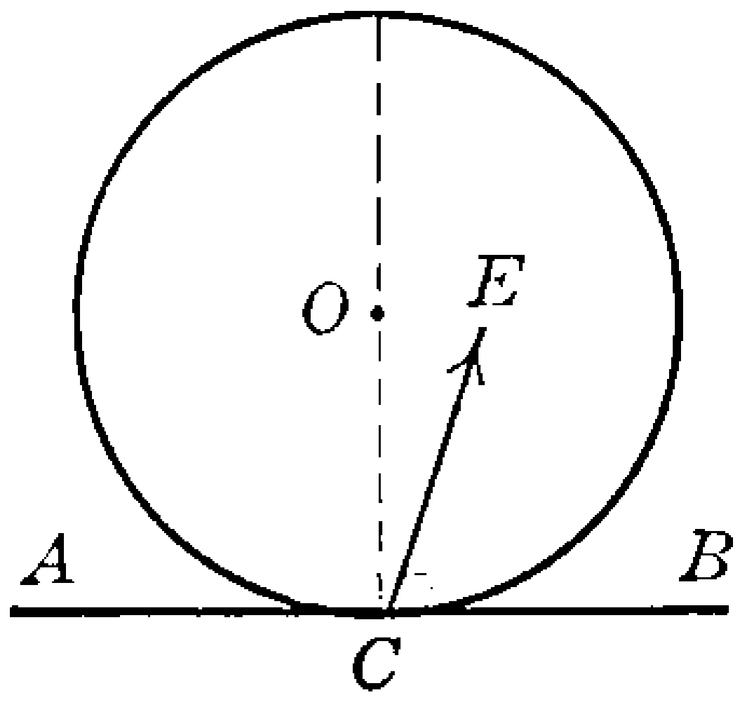
226. CONVERSE THEOREM 2. A perpendicular to a tangent at the point of contact passes through the center of the circle.



Given: AB tan. to ⊙O at C
$CE \perp AB$ at C
Prove: CE passes through O

INDIRECT PROOF

1. Either CE passes through O or not
2. *Assume* that CE does not pass through O, then we can draw another line through O and C

Complete the proof.

---

227. CONVERSE THEOREM 3. If a diameter is perpendicular to a tangent, it passes through the point of contact.

Draw a figure and prove the theorem indirectly, following the method of the preceding theorem.

EXERCISES

1. (*a*) Prove: If a circle is drawn with one end of the hypotenuse of a right triangle as a center, and the adjacent leg as a radius, then it will be tangent to the other leg.
(*b*) Prove: The tangents drawn to a circle through the ends of a diameter are parallel.

2. (*a*) Prove: If two circles are tangent to the same line, then the radii to the points of contact are parallel.
(*b*) Illustrate (*a*) by drawing the circles in several positions.

3. (*a*) The diameter of a circle is one leg of an isosceles triangle where the angle opposite the diameter is 45°. Prove that the circle is tangent to the other leg of the triangle.
(*b*) One end of a diameter is connected to a point in a tangent through the other end of the diameter in such a way that an isosceles triangle is formed. Prove that the angle be-

tween the tangent and the secant is 45°.

4. From an external point, a tangent is drawn to a circle. Prove that the midpoint of the line segment connecting the external point with the center of the circle is equidistant from the center and the point of contact.

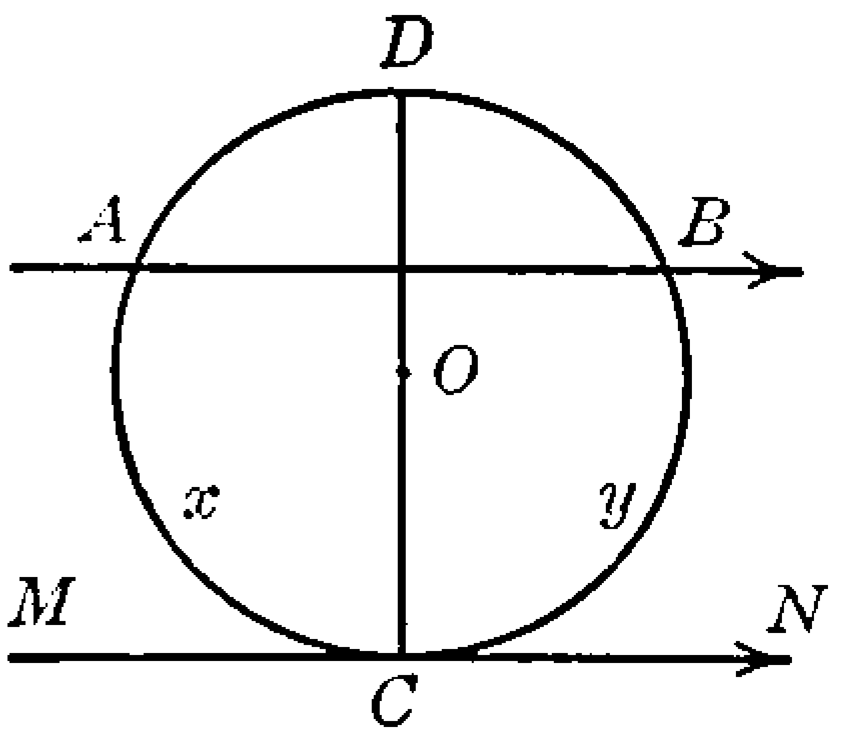
5. A quadrilateral is formed by two tangents to a circle and the radii to the points of contact. Prove that the angle between the tangents is the supplement of the angle between the radii.

6. A line segment is tangent to a circle with its midpoint as the point of contact. The ends of the line segment are joined to the center of the circle. Prove that the arc within the triangle is bisected by the point of contact.

7. In two concentric circles, a chord of the larger circle is tangent to the smaller circle. Prove that the chord is bisected by the point of contact.

8. If a tangent makes an angle of 30° with a chord through the point of contact, then the chord equals the radius.

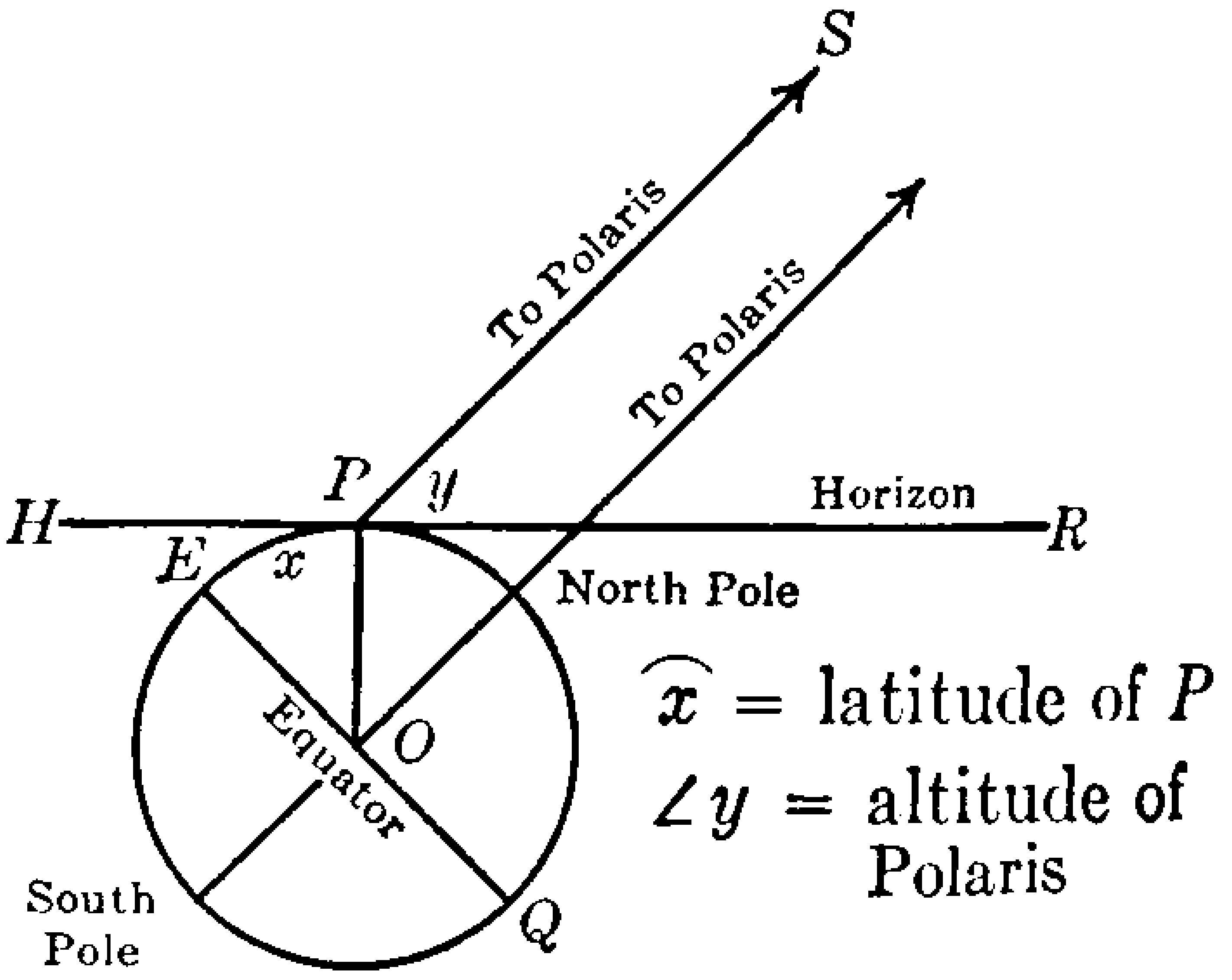
9.



Given:
MN tan. to $\odot O$ at C
$AB \parallel MN$
DC a diam.
Prove:
$\overset{\frown}{AC} = \overset{\frown}{CB}$

10. (*a*) State in words the proposition proved in Ex. 9. The diameter acts only as an auxiliary line in the proof. Do not mention the diameter in your proposition.
    (*b*) Copy the figure in Ex. 9. On this figure, omit the diameter DC but draw another secant parallel to AB cutting the circle at A' and B'. Using the proposition you stated in (*a*), prove $\overset{\frown}{AA'} = \overset{\frown}{BB'}$.
    (*c*) State one general proposition that covers all the cases in (*a*) and (*b*).

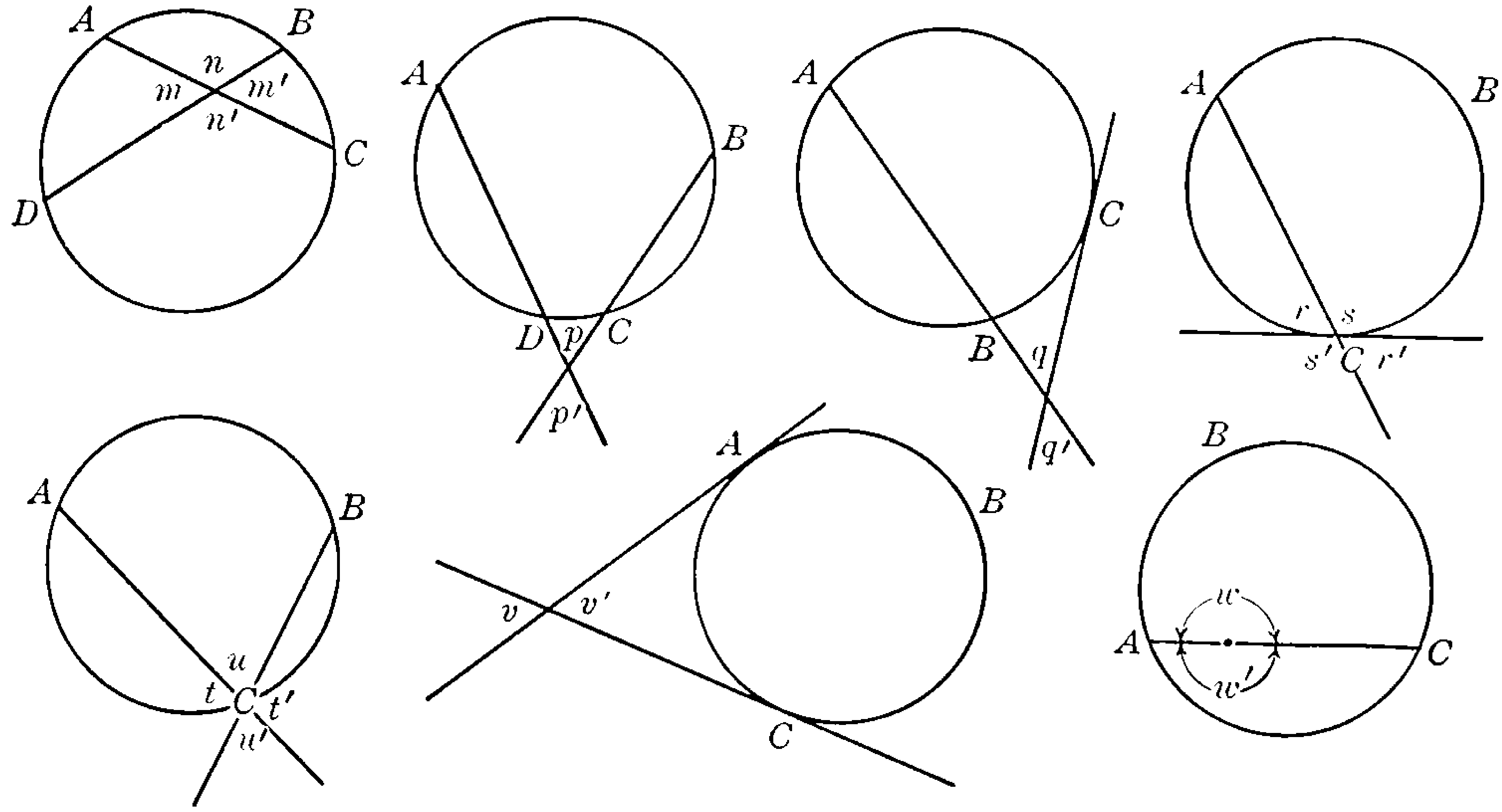
11. The axis of the earth, if prolonged far enough, would almost pierce the North Star, Polaris. If an observer wishes to find his latitude EP (the angular distance from the equator), he need only measure the angle that the line of sight to Polaris makes with the horizon HR. This angle is called the *altitude* of Polaris. Prove that this angle SPR equals $\overset{\frown}{EP}$ in degrees.



Rotating a Straight Line

228. If a straight line is turned about a point from one position to another, it moves over a pair of vertical angles. If at the same time the line also moves over certain arcs of a circle, we think of these arcs as being intercepted by the vertical angles and belonging to them. We will soon learn that there is an important relationship between the angles and the arcs that are traced by rotating a straight line.

In the following diagrams, $\measuredangle m$ and m', $\measuredangle n$ and n', and so on, are pairs of vertical angles. Name the arcs over which a line moves while turning through each pair of vertical angles. You can determine this by placing your pencil on one side of the angles and turning it about the vertex through the vertical angles. Or you can use your dynamic model (§ 208).



Translating a Line

229. Definition. To translate a line means to move it so that it remains parallel to its original position.

Now let us prove that if we translate a line across a circle, it moves over equal arcs. Like most proofs in geometry, this proof will have to do with a stationary and completed figure. In other words, we shall prove that the arcs on a circle between two stationary positions of a translated secant are equal.

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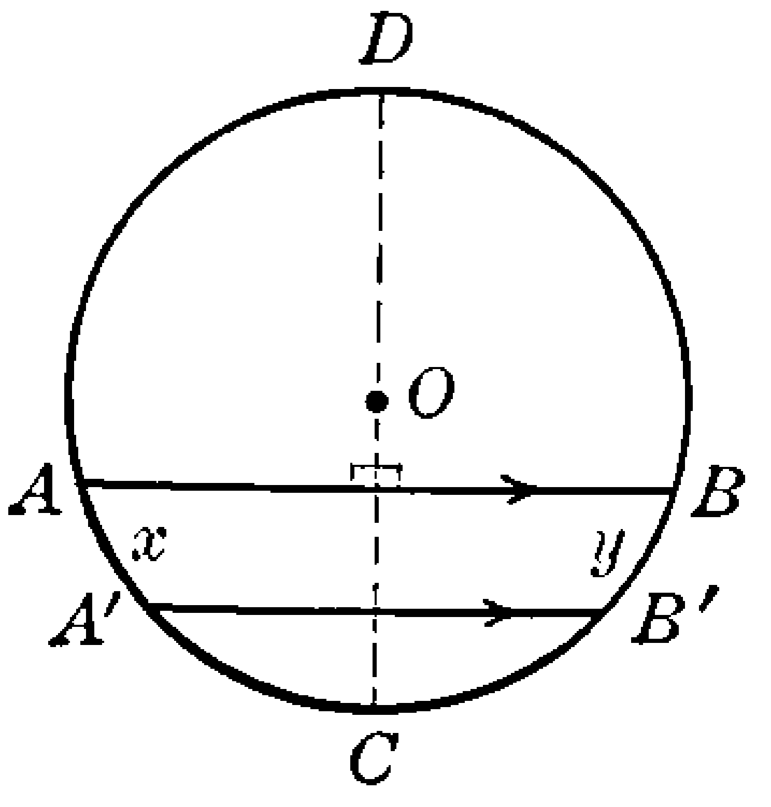
230. THEOREM. If two parallel lines cut a circle, they intercept equal arcs.



Given: $AB \parallel A'B'$

Prove: $\overset{\frown}{AA'} = \overset{\frown}{BB'}$

Cons.: Draw diam. $DC \perp AB$.

PROOF

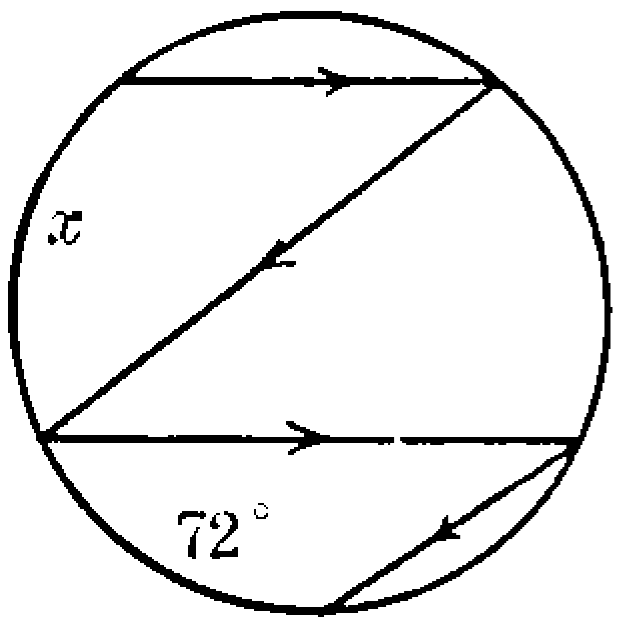
Statements	*Reasons*
1. $DC \perp A'B'$	1. A line $\perp$ to one of 2 $\parallel$s is $\perp$ to the other.
2. $\overset{\frown}{AC} = \overset{\frown}{BC}$ $\overset{\frown}{A'C} = \overset{\frown}{B'C}$	2. A diam. $\perp$ to a chord bisects the chord and its arcs.
3. $\overset{\frown}{AA'} = \overset{\frown}{BB'}$	3. Subt. Ax.

Draw figures and prove that this theorem is also valid if one or both of the parallels are tangents.

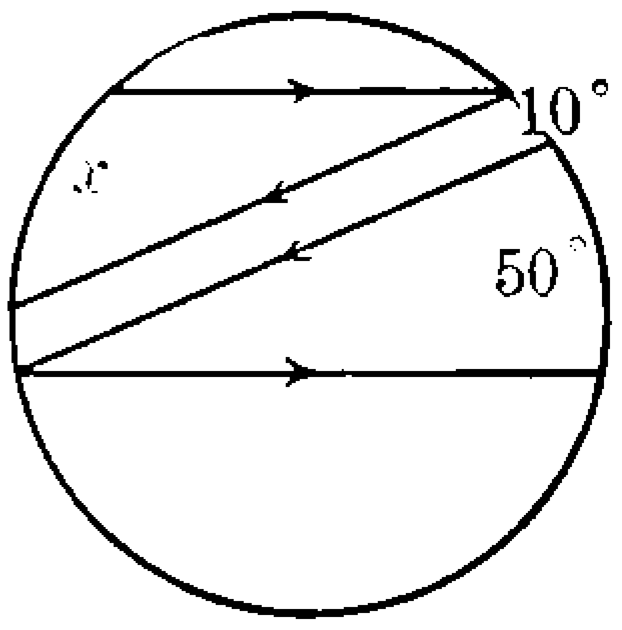
EXERCISES

In the following figures, the lines are parallel as shown by the arrows, and the point O is the center of the circle. Find the number of degrees in the arc x or the angle x.

1.



2.



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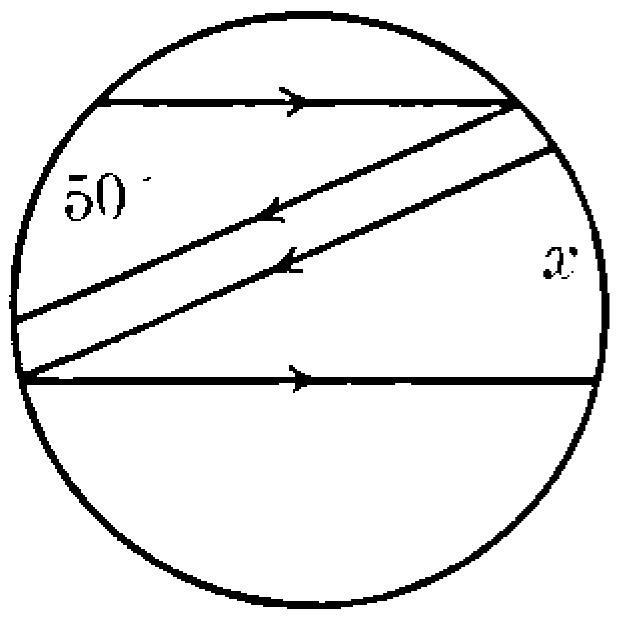

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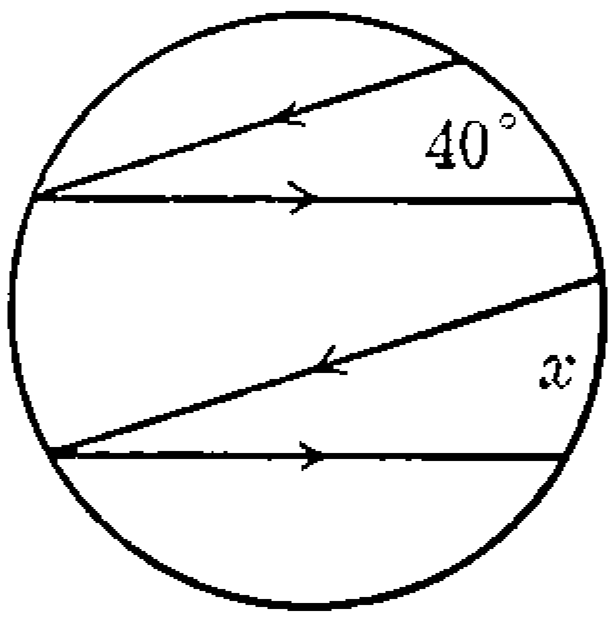

Help: Let the small arc = any no. of degrees — say 10°.

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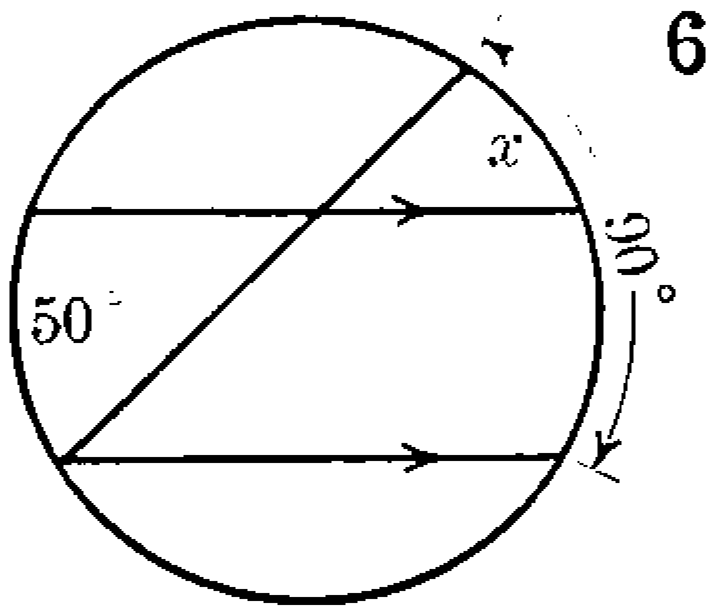

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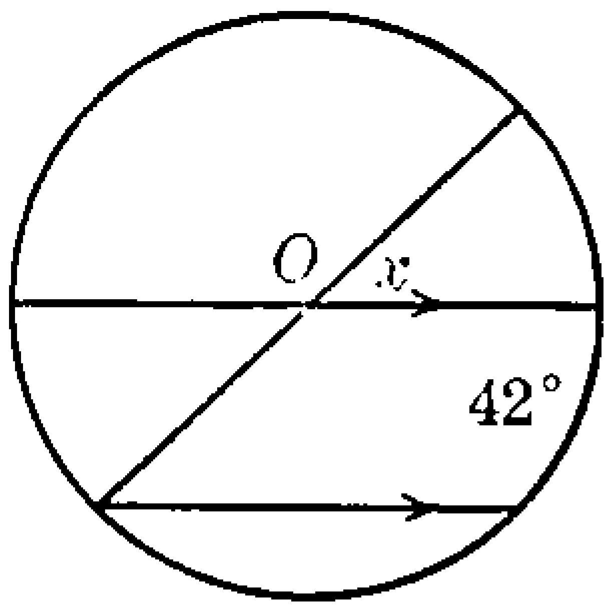

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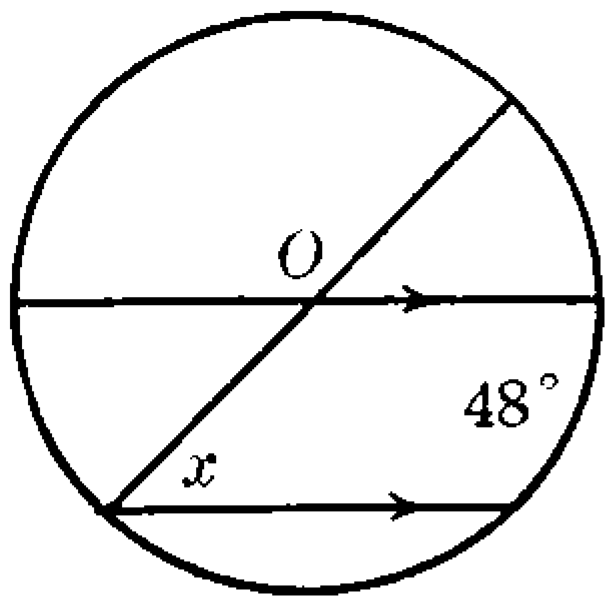

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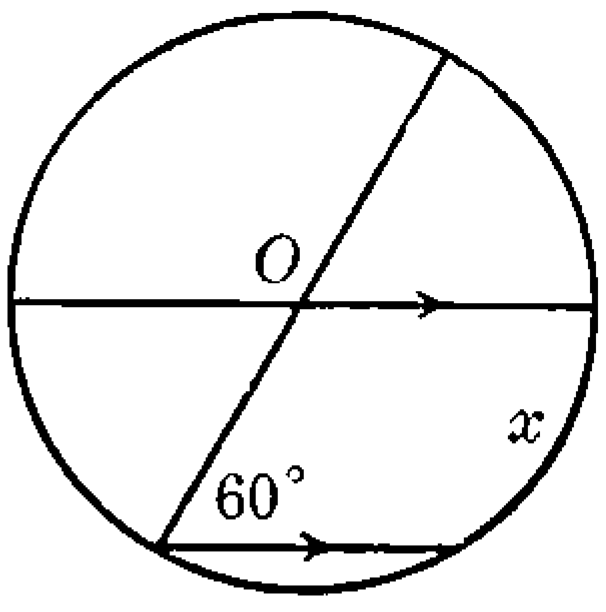

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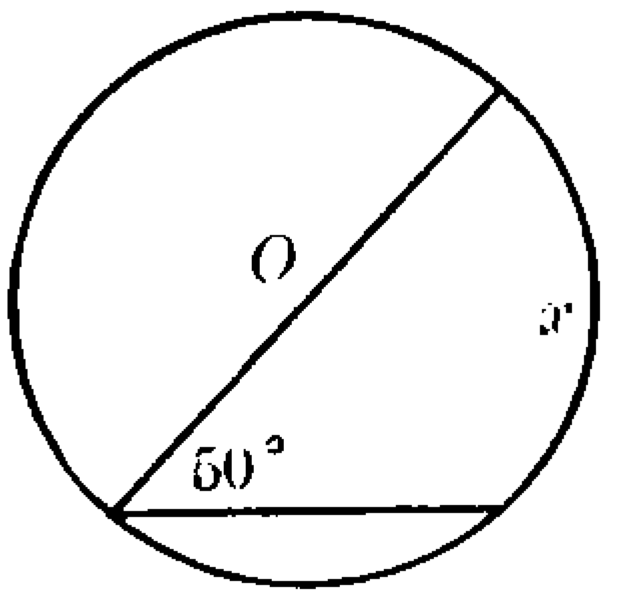

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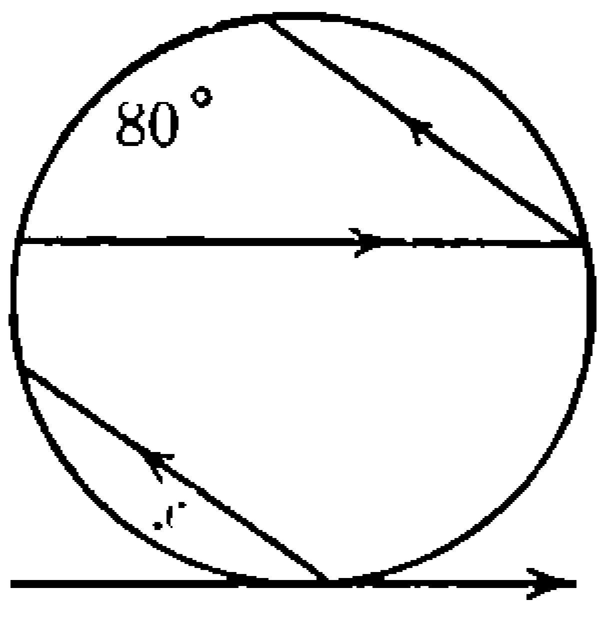

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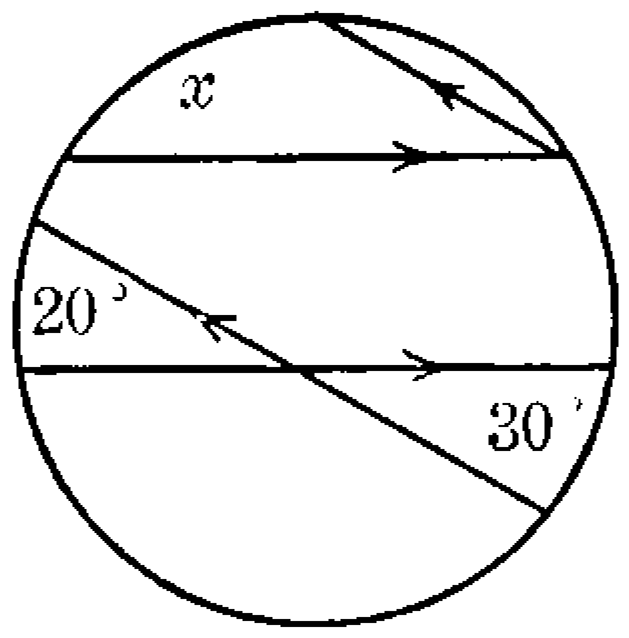

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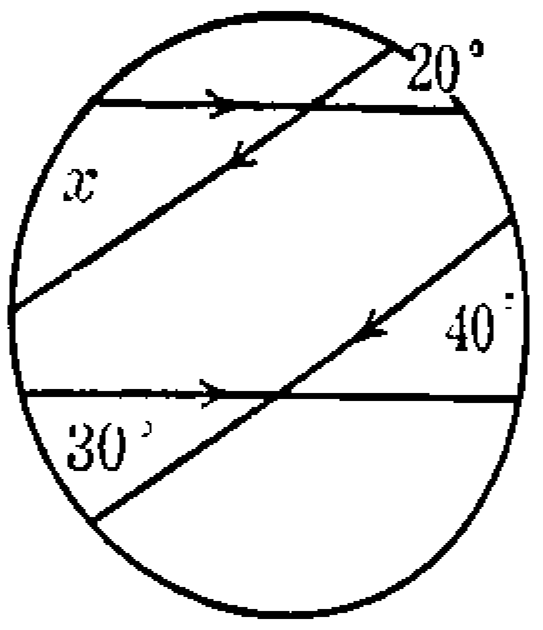
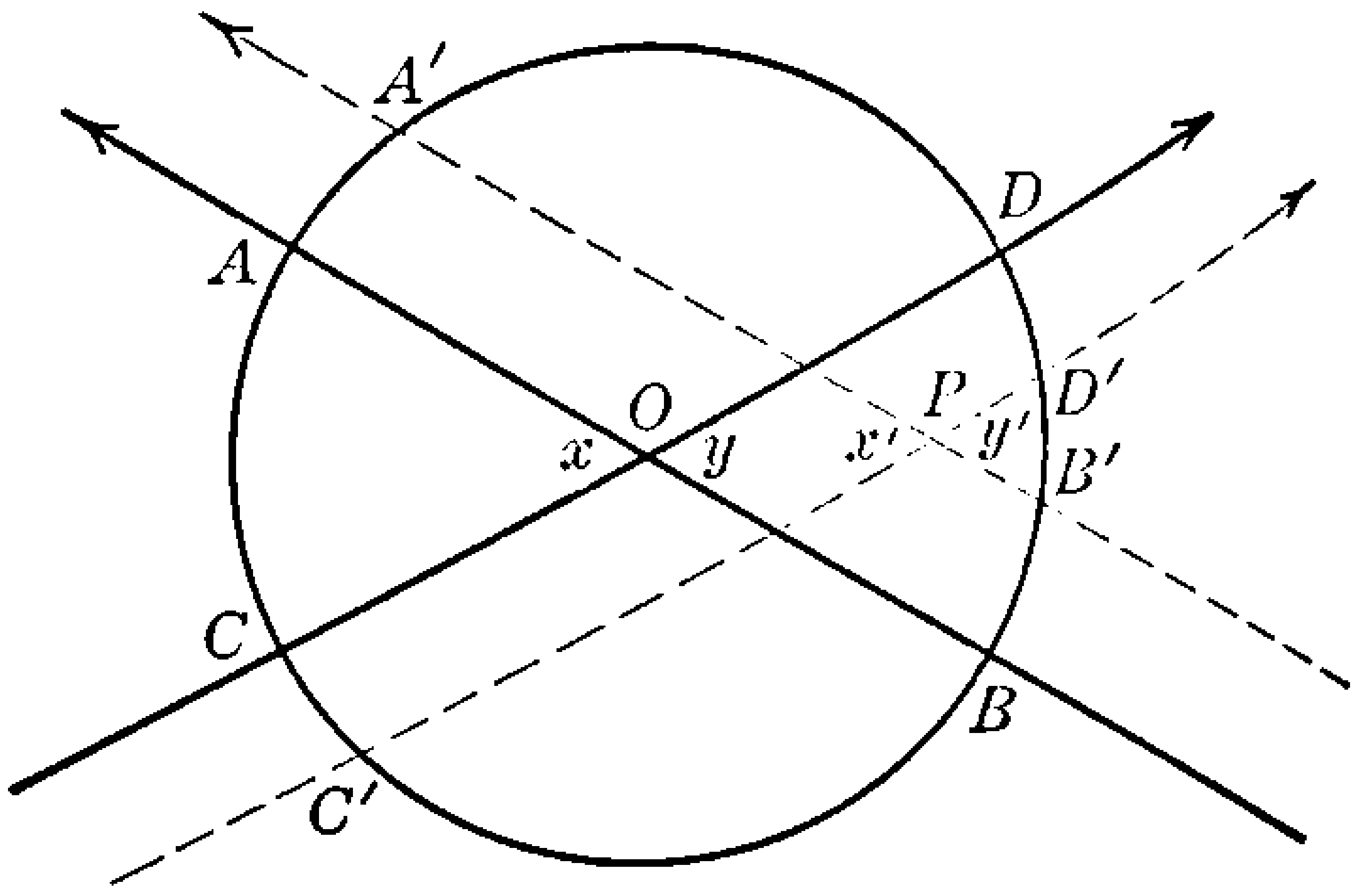


Vertical Angles In a Circle

231. From the definition of the angular measure of an arc, it is seen that the sum of a pair of vertical angles x and y at the center of a circle equals the sum of the degrees in the arcs AC and DB. As the sides of the angles are translated away from the center, so that the vertex falls at P within the angle y, the angles remain the same size because they are opposite angles of a parallelogram. However, the arcs intercepted by the angles have changed. The arc AC has increased by AA' and CC', whereas the arc DB has decreased by DD' and BB'.



Since a translated secant moves over equal arcs, we know that the increase in arc AC is exactly canceled out by the decrease in arc DB. Therefore, the sum of the new arcs equals the sum of the old arcs, and $\angle x' + \angle y' \overset{\circ}{=} \widehat{A'C'} + \widehat{D'B'}$.

If each member of this equation is subtracted from 360°, the sum of the other pair of vertical angles $A'PD'$ and

$C'PB'$ also equals the sum of their arcs $A'D'$ and $C'B'$.

We can also see that as the point P moves farther from the center of the circle, the decreasing arc DB becomes zero when P is on the circle; and, continuing to decrease, it will become negative when the vertex P is outside the circle.

We have, therefore, a simple way to measure the arcs that are intercepted by vertical angles. We think of the sides of the angles as having been translated from the center of the circle. In that case, the sum of the angles was equal to the sum of the arcs. This fact remains true anywhere in the circle because the translated secants, in moving over equal arcs, will cause one arc to lose whatever the other arc has gained. Finally, if the vertex is outside the circle, we merely think of the *history* of the angles in terms of motion; that is, as the vertex *came out* of the circle, one arc must have decreased through zero and has become negative.

The following theorem summarizes what we have discovered by moving a pair of vertical angles across a circle. We call the sum of the arcs an *algebraic* sum, because one of the arcs may be negative.

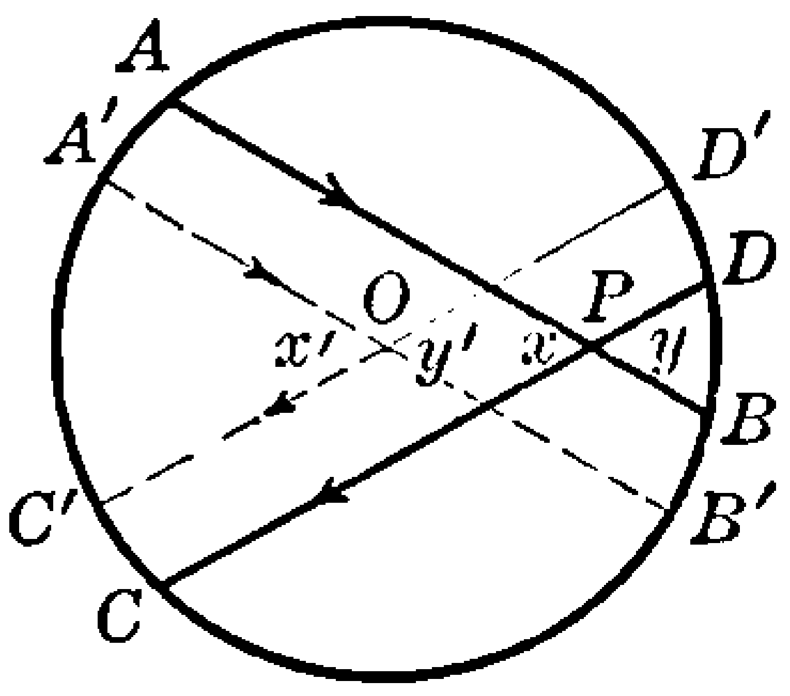
---

232. THEOREM. If the sides of two vertical angles cut or touch a circle, then the sum of the angles equals the algebraic sum of the arc degrees that the angles intercept—the smaller arc being negative if the vertex is outside the circle.

STATIC PROOF (OPTIONAL)

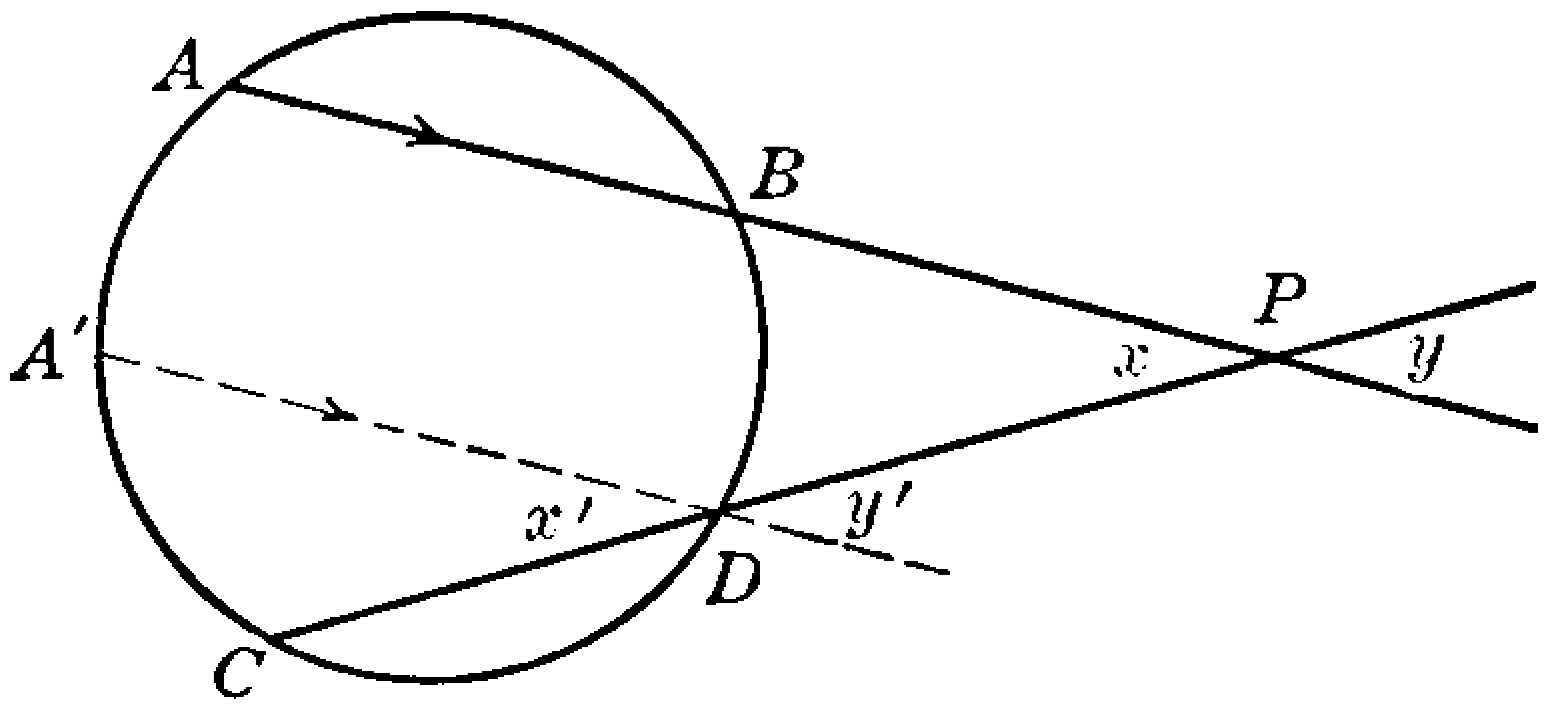
In § 231 you have a dynamic proof of the theorem giving the relation between vertical angles and their arcs on a circle. The proof covers all cases and helps you to visualize these cases as one whole unit. If you wish to write out a static proof of this important theorem, the following outline will help you.

Case I. The vertex is inside (or on) the circle.



$$\begin{aligned} x &= x' \overset{\circ}{=} \overset{\frown}{A'C'} = \overset{\frown}{AC} - \overset{\frown}{AA'} - \overset{\frown}{CC'} \\ y &= y' \overset{\circ}{=} \overset{\frown}{D'B'} = \overset{\frown}{DB} + \overset{\frown}{BB'} + \overset{\frown}{DD'} \\ \hline x + y &\overset{\circ}{=} \qquad\qquad \overset{\frown}{AC} + \overset{\frown}{DB} \end{aligned}$$

Case II. The vertex is outside the circle.

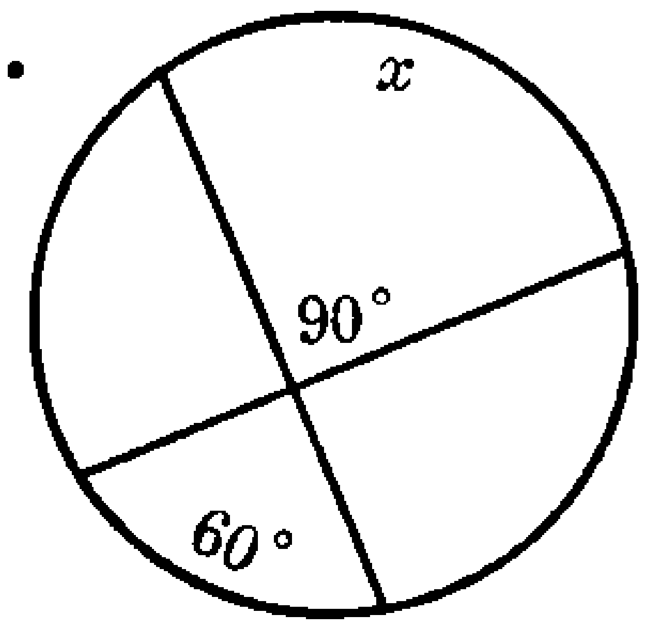


$$\begin{aligned} x + y &= x' + y' \overset{\circ}{=} \overset{\frown}{A'C} \text{ (Case I.)} \\ &= \overset{\frown}{AC} - \overset{\frown}{AA'} = \overset{\frown}{AC} - \overset{\frown}{BD} \end{aligned}$$

EXERCISES

Find the number of degrees in the angle x or the arc x in each diagram.

1.



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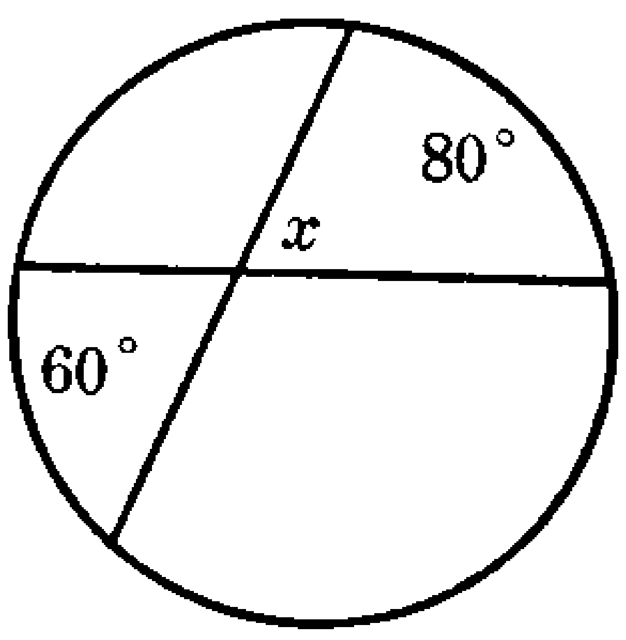

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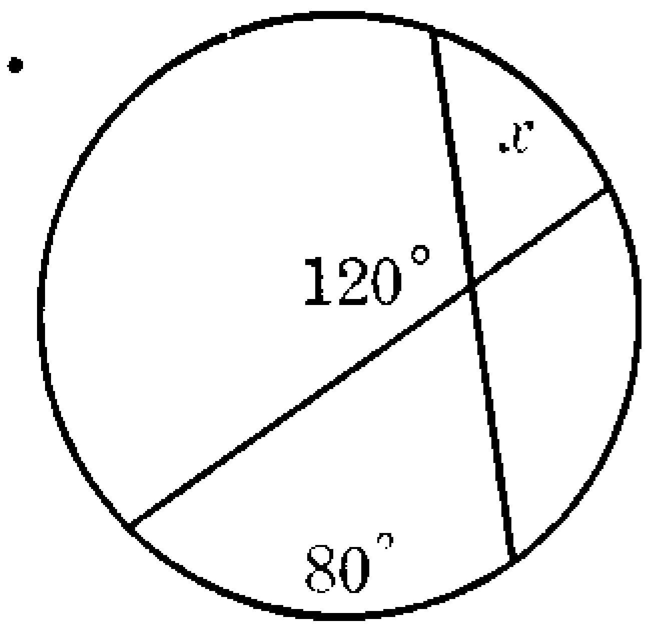

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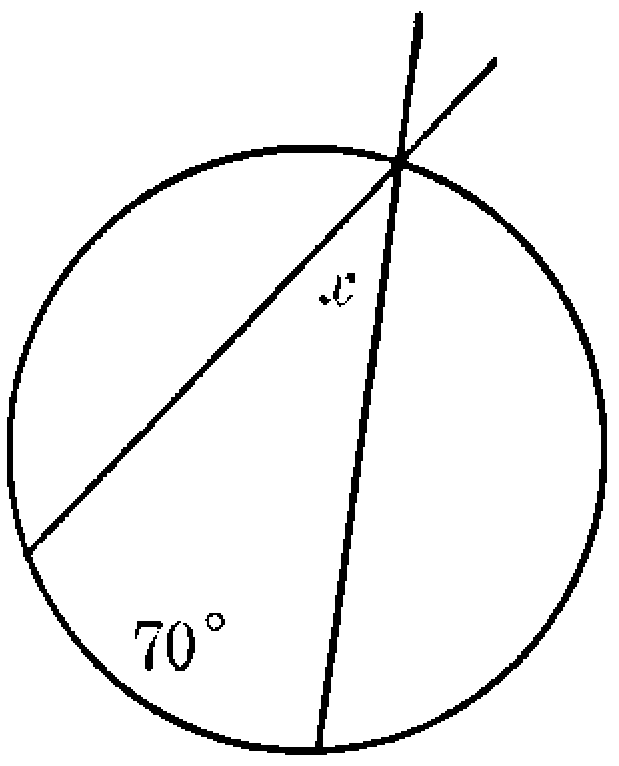

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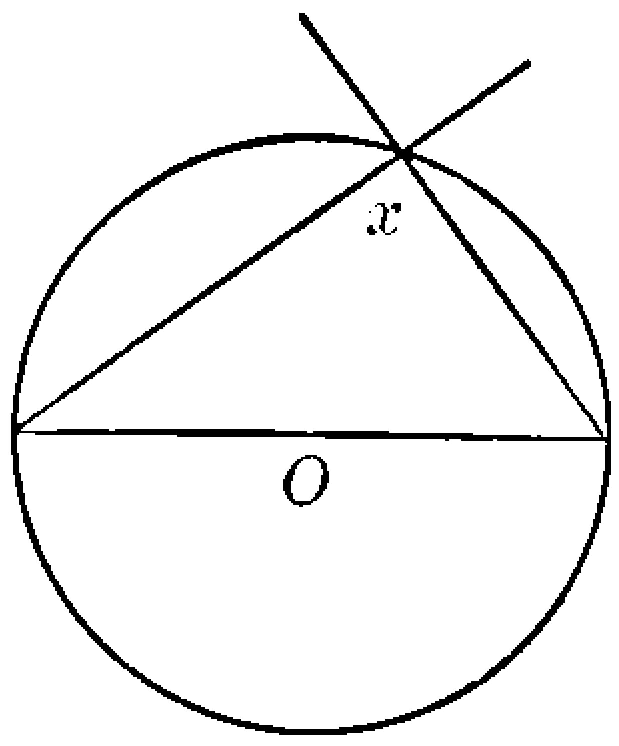

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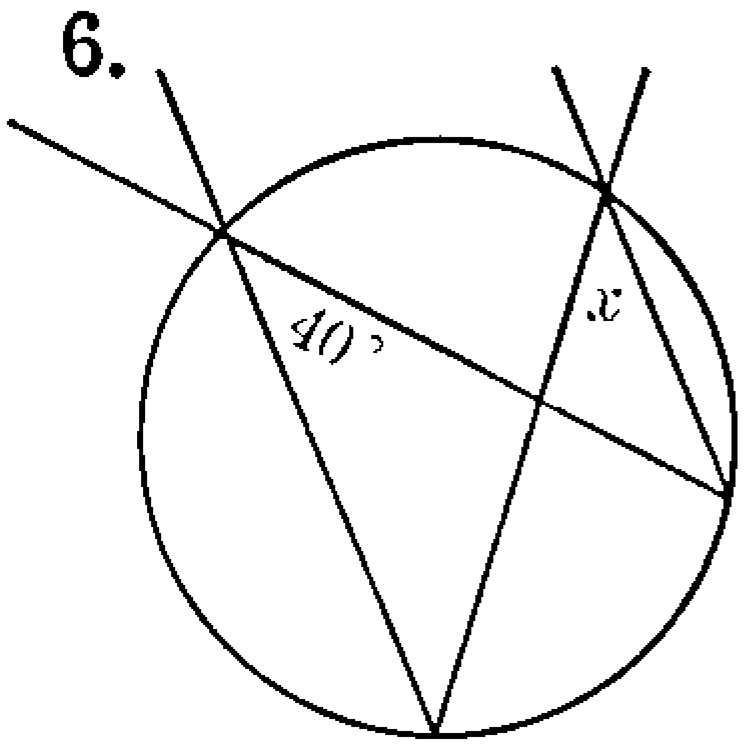

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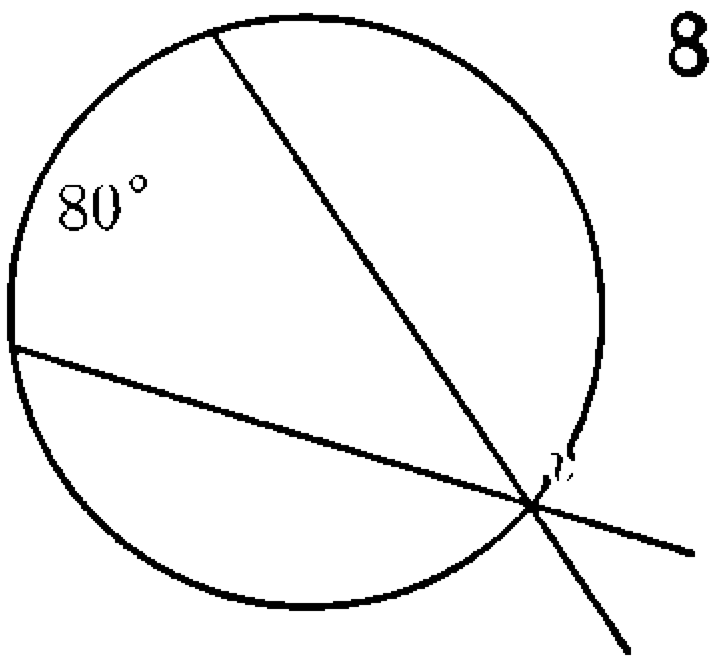

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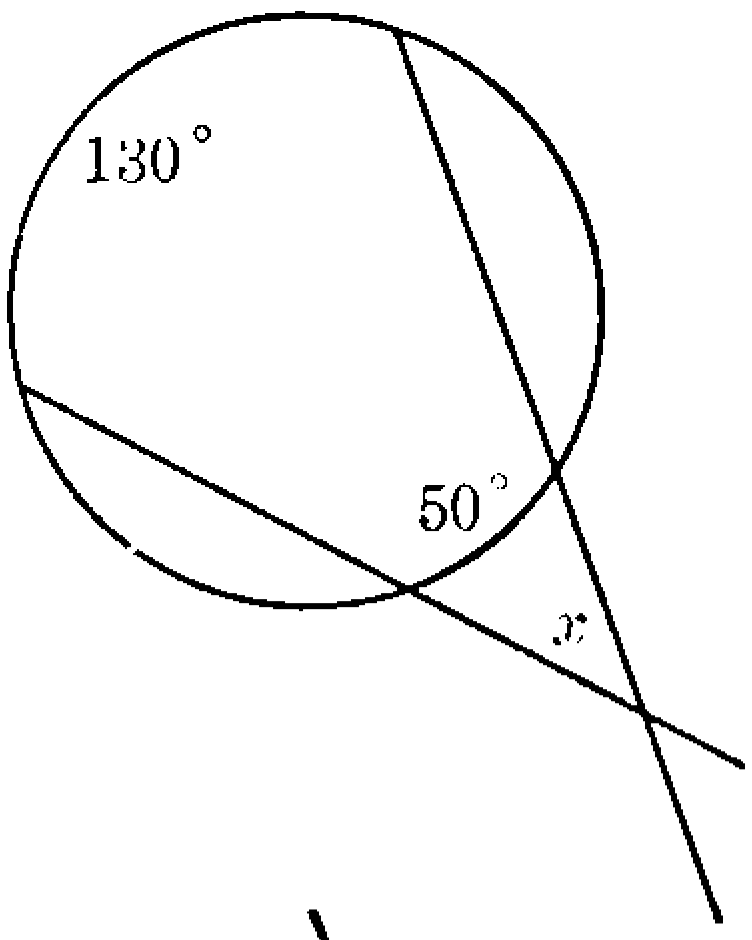

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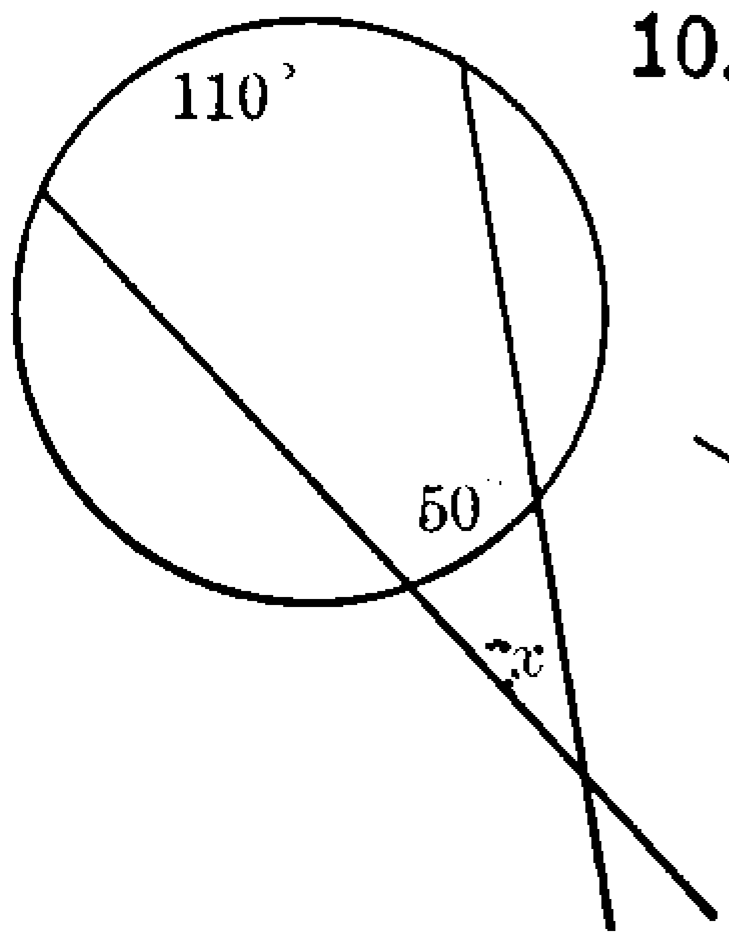

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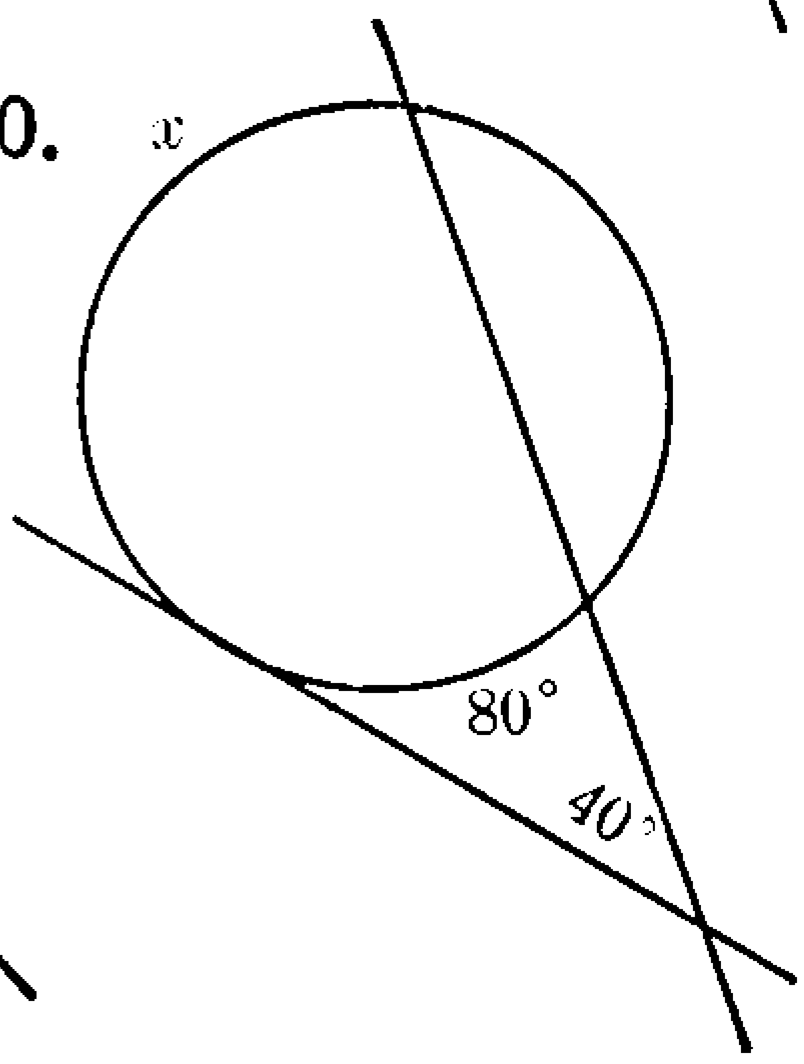

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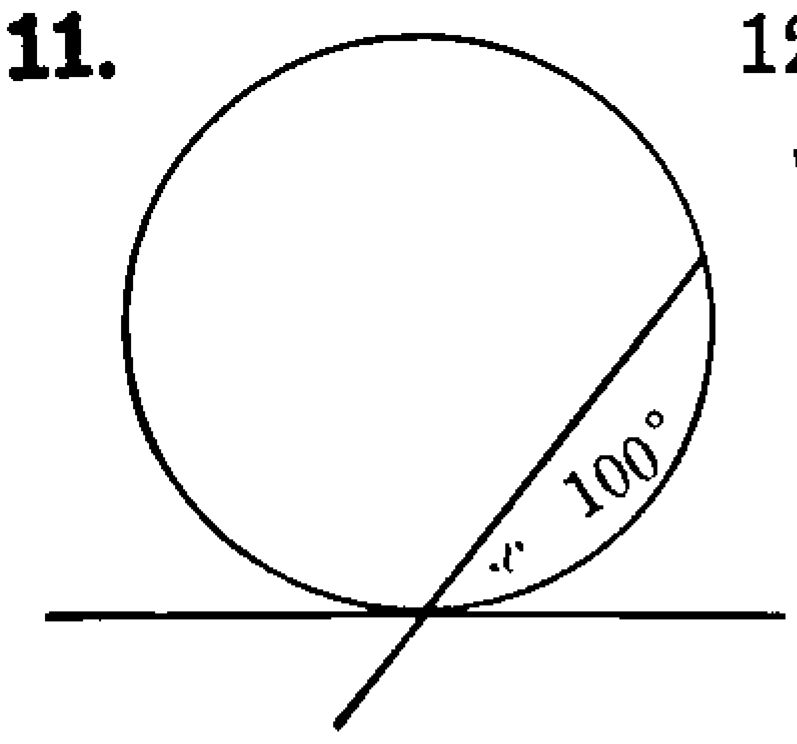

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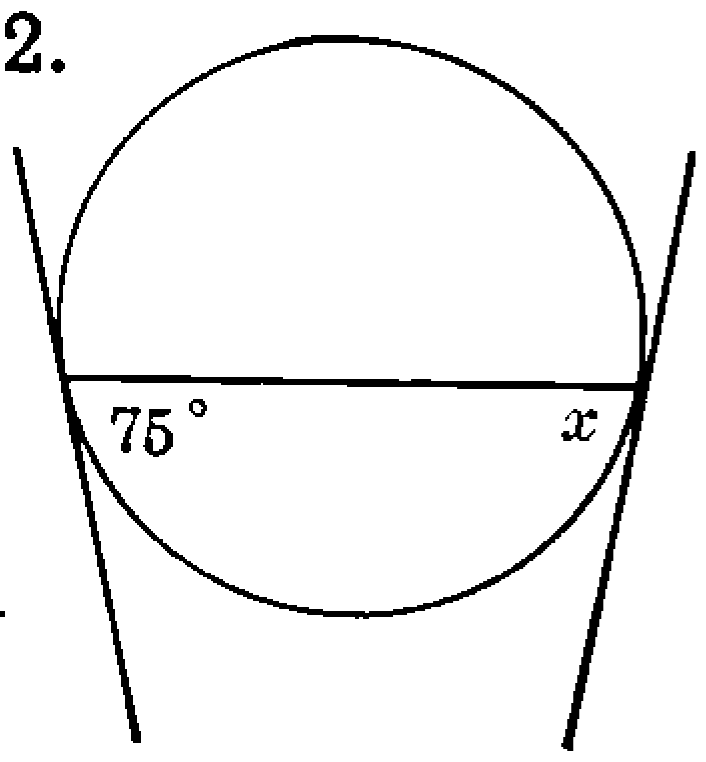

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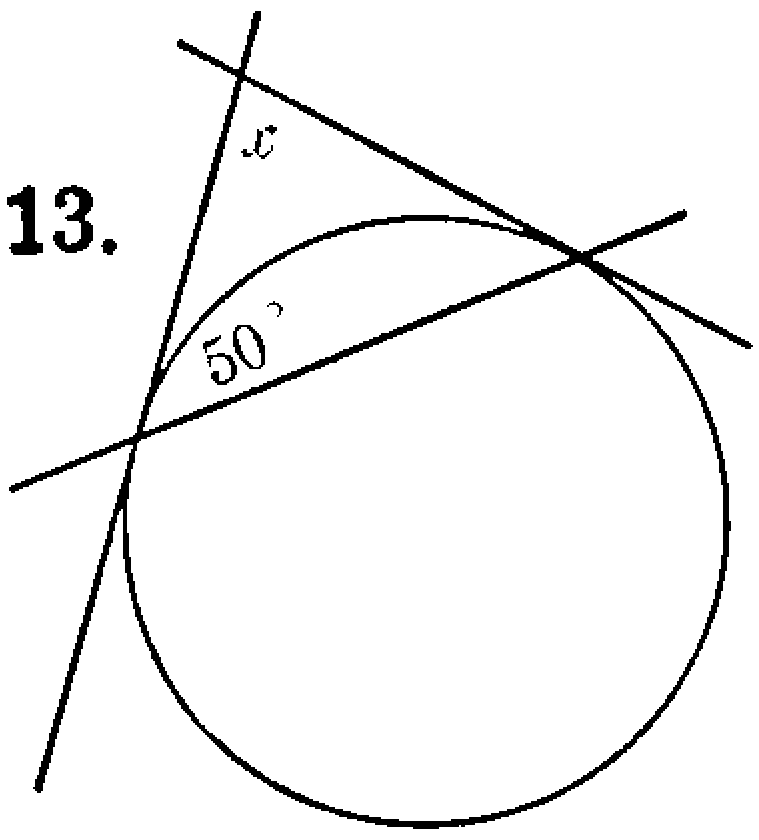

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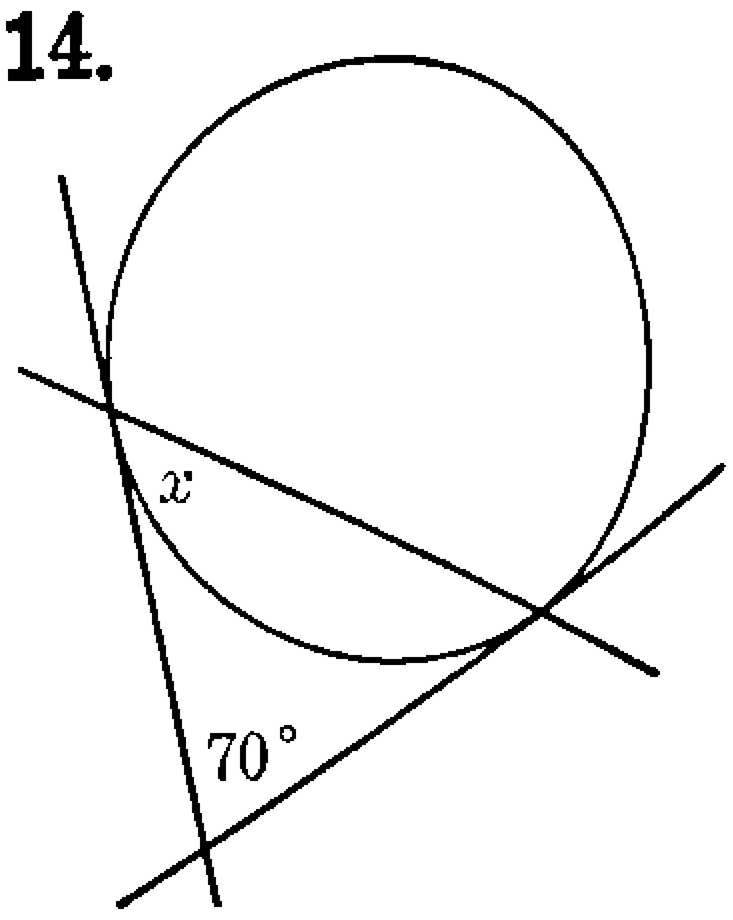

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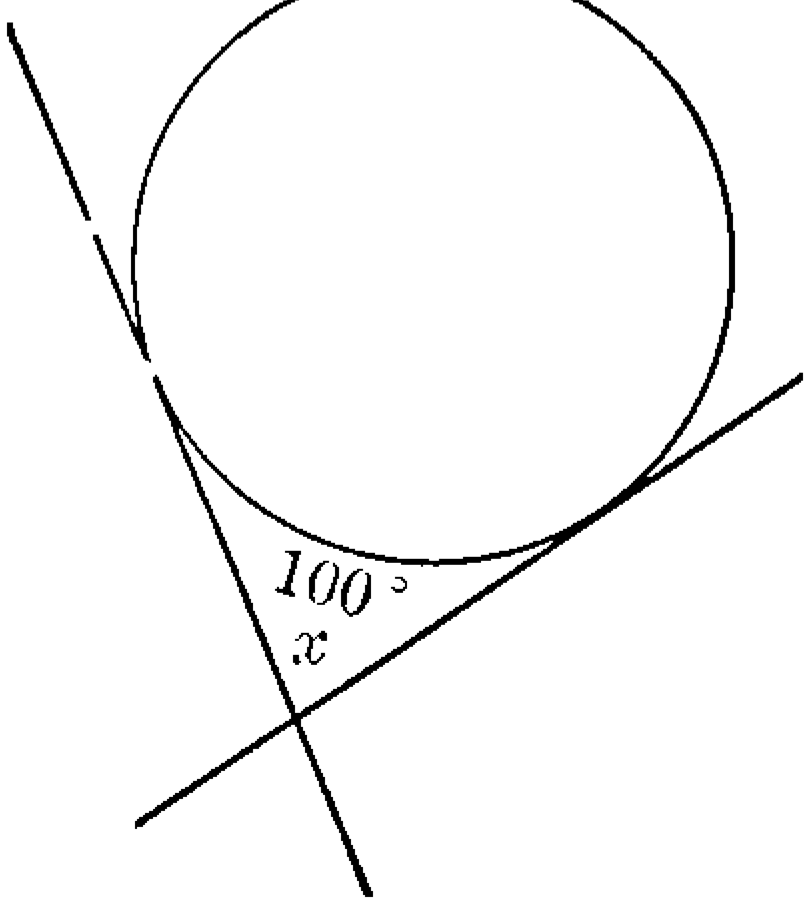

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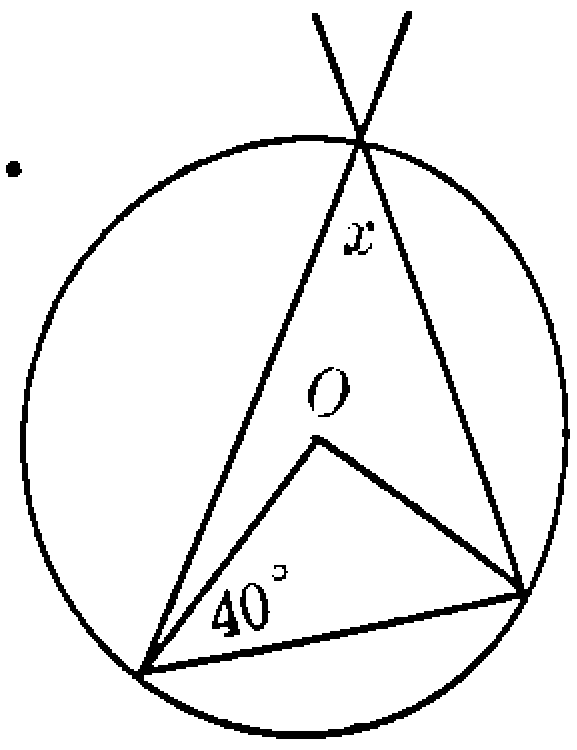

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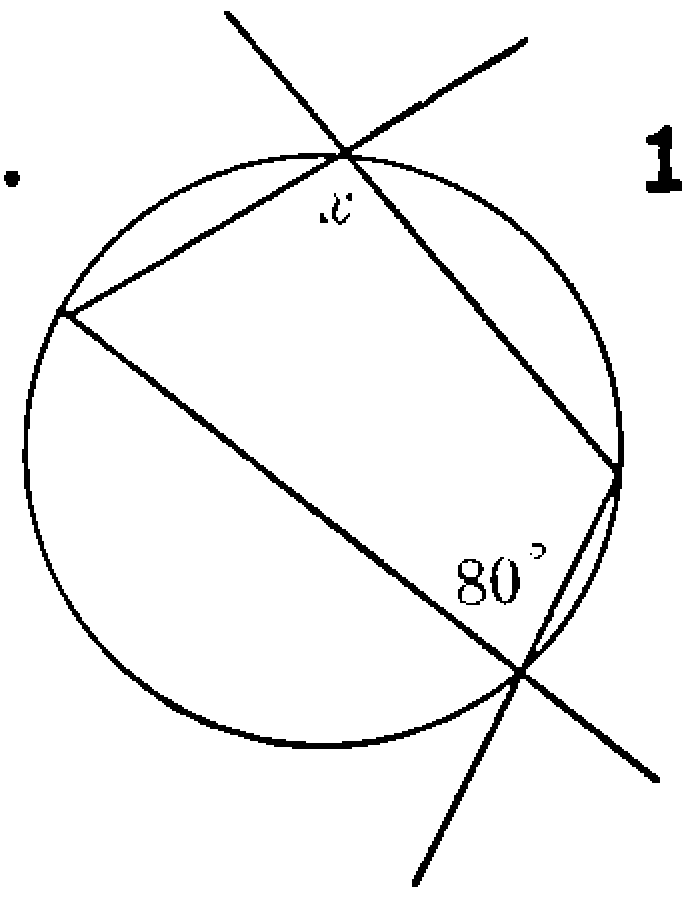

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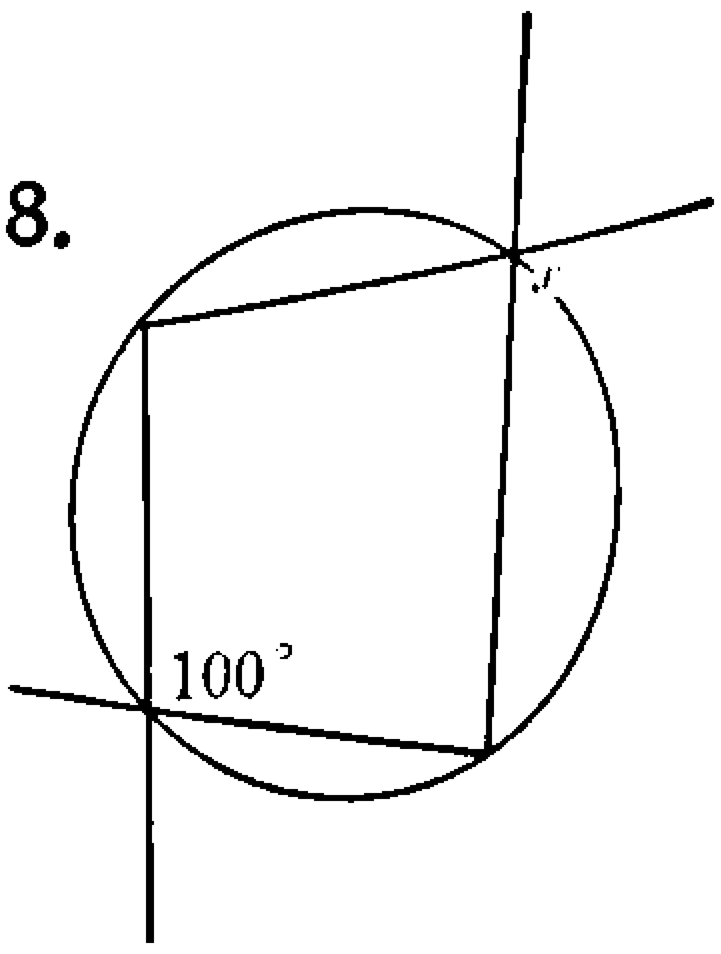
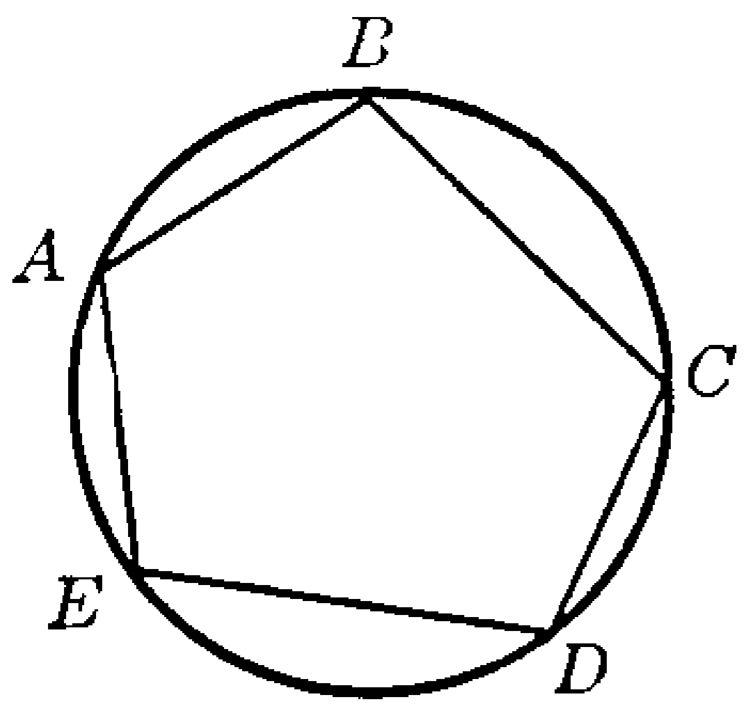


19. In each of the preceding 18 exercises, replace the given numerical value by y. If two numerical values are given, replace the smaller value by y and the larger value by z. Find the simplest equation connecting your letters, and solve for x in terms of the other letter or letters. Thus in Ex. 1, replace 60° by y and 90° by z. Then $x + y = z + z$, and $x = 2z - y$.

20. Complete the following statement: If the vertices of a quadrilateral are on a circle, then the opposite angles of the quadrilateral are —?—

Inscribed Polygons and Angles

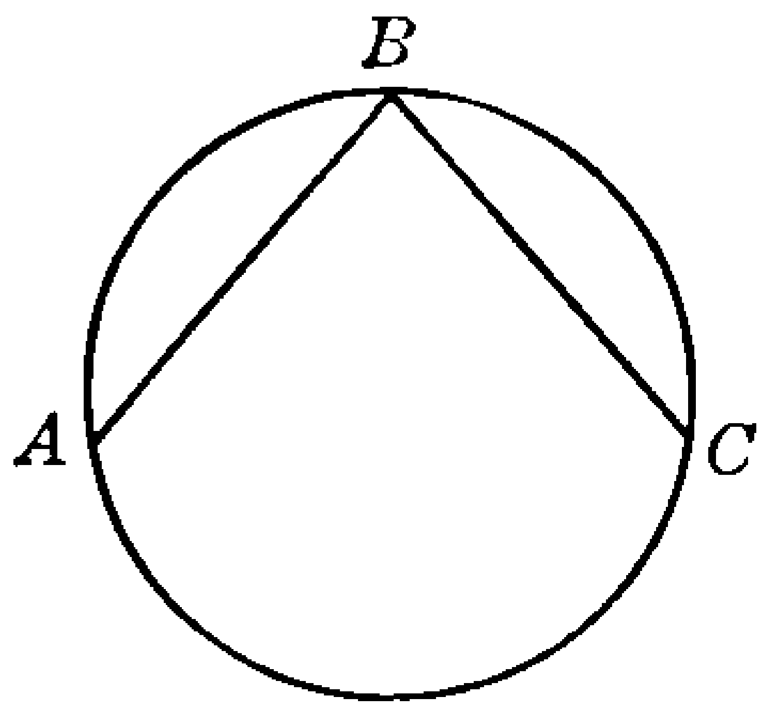
233. Definition. A polygon is inscribed in a circle if all its vertices are on the circle.



234. Definition. A circle is circumscribed about a polygon if the polygon is inscribed in the circle.

The angles of an inscribed polygon are called inscribed angles.

235. Definition. An inscribed angle is an angle that is formed by two chords meeting on a circle.



$\angle B$ an inscribed angle.

$\angle B$ intercepts $\overset{\frown}{AC}$.

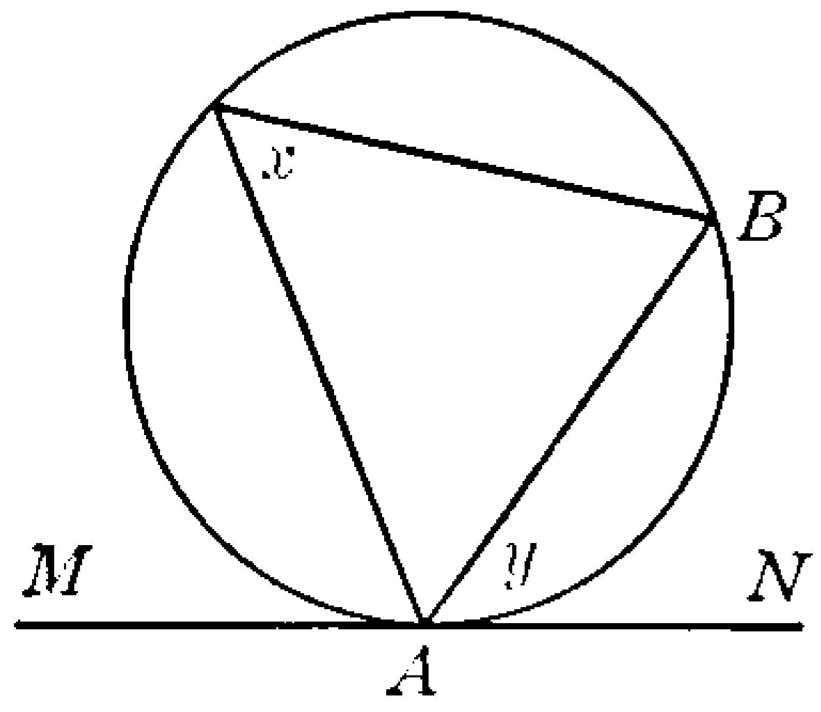
$\angle B$ inscribed in $\overset{\frown}{ABC}$.

236. Definition. An inscribed angle is said to be inscribed in the part of the circle that it does not intercept.

237. Corollary. If the sides of an angle cut or touch a circle, the angle equals half the algebraic sum of the arc degrees between the sides and between the extensions of the sides of the angle—one arc being negative if the vertex is outside the circle (§ 232).

(Abbreviation: An angle equals the average of its arcs.)

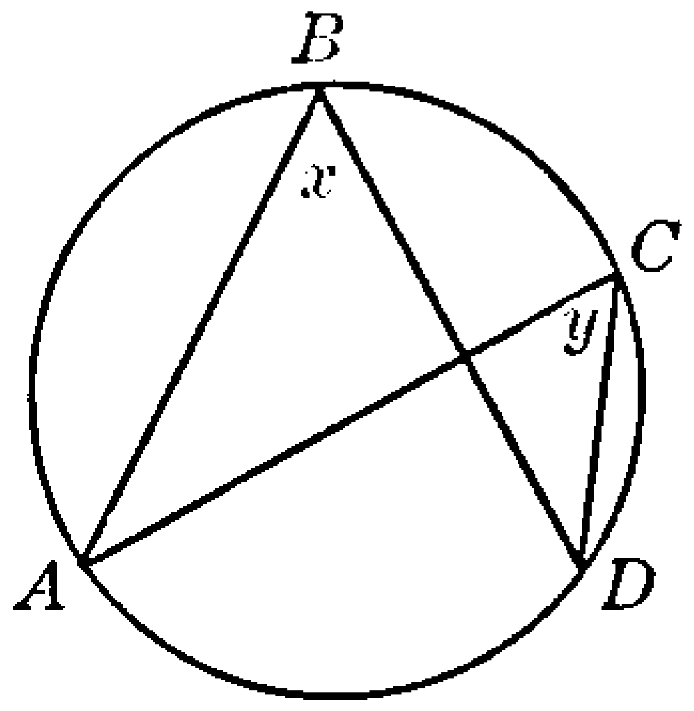
238. Corollary. An inscribed angle, or an angle between a tangent and a chord to the point of contact, equals half the intercepted arc in degrees.



$\angle x \overset{\circ}{=} \frac{1}{2}\overset{\frown}{AB}$

$\angle y \overset{\circ}{=} \frac{1}{2}\overset{\frown}{AB}$

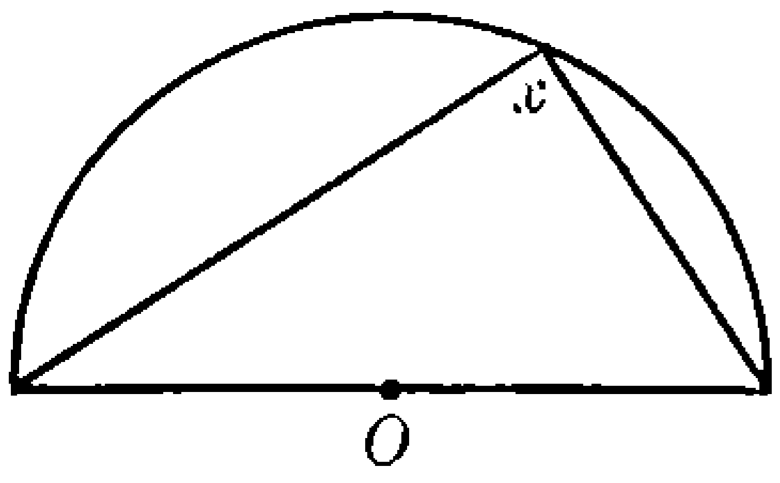
239. Corollary. Angles inscribed in the same arc are equal.



$\angle x$ and $\angle y$ both inscribed in $\overset{\frown}{ABCD}$.

$\angle x = \angle y$

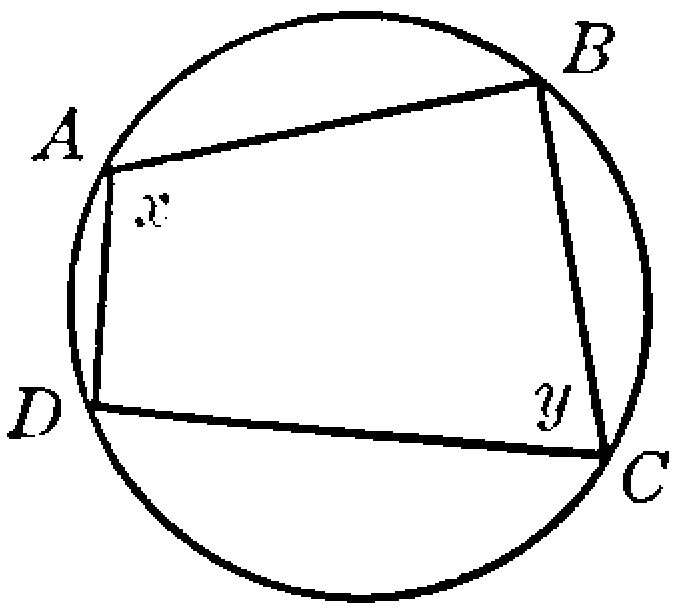
240. Corollary. An angle inscribed in a semicircle is a right angle.



$\angle x$ is inscribed in a semicircle.

$\angle x = 90°$

241. THEOREM. **The opposite angles of an inscribed quadrilateral are supplementary.**



Given: Inscribed quad. $ABCD$

Prove: $\angle A + \angle C = 180°$

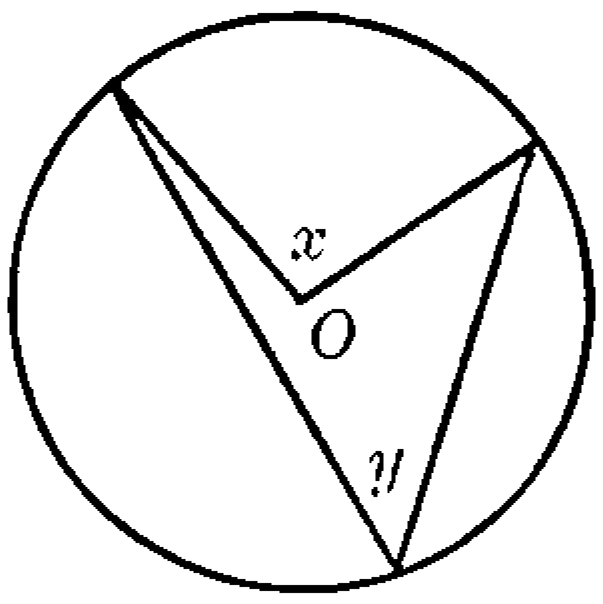
$\angle B + \angle D = 180°$

PROOF

Statements	*Reasons*
1. $\angle A \overset{\circ}{=} \frac{1}{2}\overset{\frown}{BCD}$ $\angle C \overset{\circ}{=} \frac{1}{2}\overset{\frown}{BAD}$	1. Inscribed $\angle \overset{\circ}{=} \frac{1}{2}$ its arc (§ 238).
2. $\angle A + \angle C \overset{\circ}{=} \frac{1}{2}(\overset{\frown}{BCD} + \overset{\frown}{BAD})$ $= \frac{1}{2} \odot ABCDA$ $= \frac{1}{2} \cdot 360° = 180°$ In the same way, $\angle B + \angle D = 180°$	2. Add. Ax.

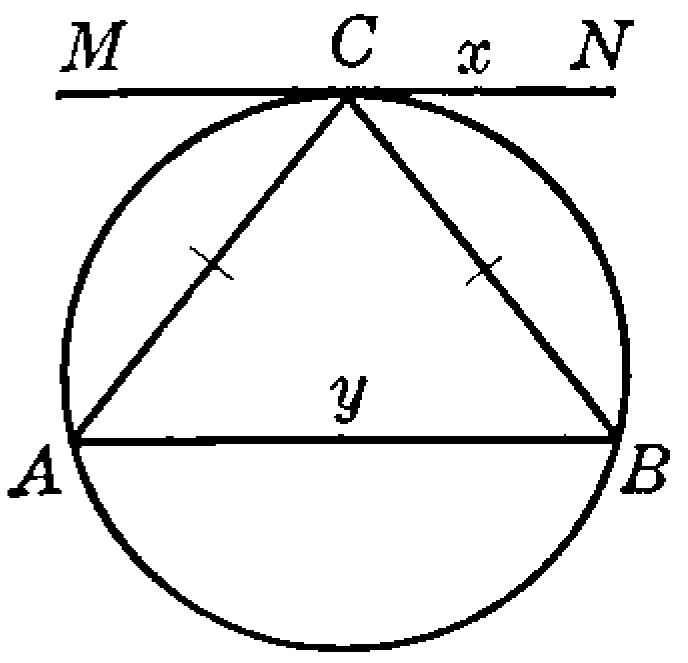
EXERCISES

1. (*a*) What is the relationship between the angles x and y in the figure?



(*b*) Why do the bisectors of all the angles inscribed in the same arc meet in a point?

2.



Given:

MN **tan. to** $\odot$ **at** C

$AC = CB$

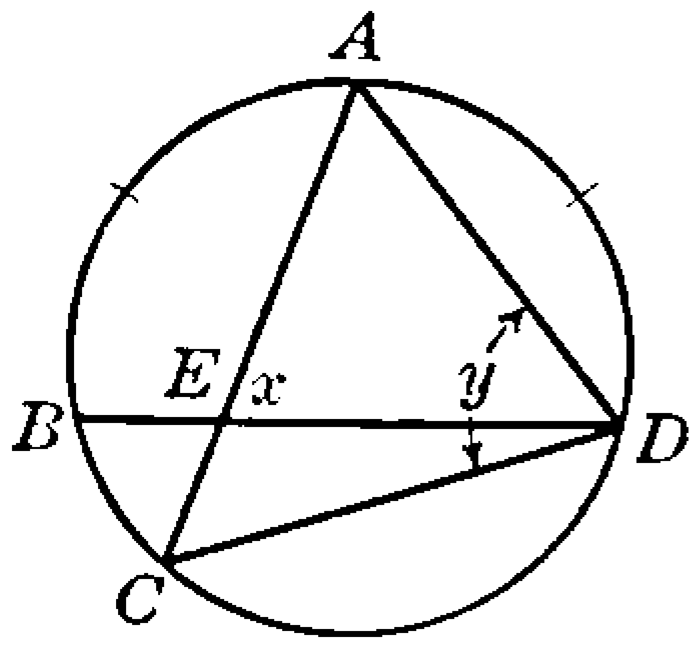
Prove:

$MN \parallel AB$

Write a general word statement of this proposition.

3. Prove: If a quadrilateral with two equal adjacent sides is inscribed in a circle, the diagonal through the angle between the two equal sides bisects the opposite angle of the quadrilateral.

4.



Given:

$\overset{\frown}{AB} = \overset{\frown}{AD}$

Prove:

$\angle AED = \angle ADC$

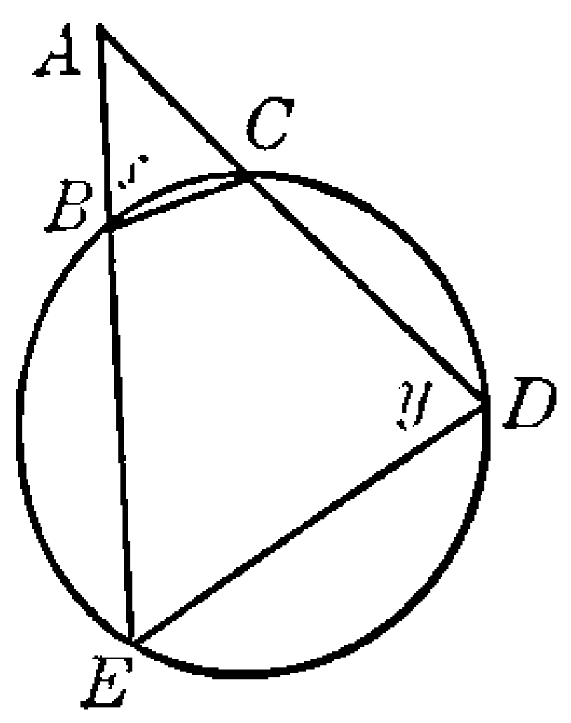
5. Prove: A trapezoid inscribed in a circle is isosceles.

6. Prove: (*a*) A parallelogram inscribed in a circle is a rectangle.

(*b*) An equilateral quadrilateral inscribed in a circle is a square.

(*c*) If two consecutive angles of an inscribed quadrilateral are equal, the quadrilateral is either an isosceles trapezoid or a rectangle.

7.

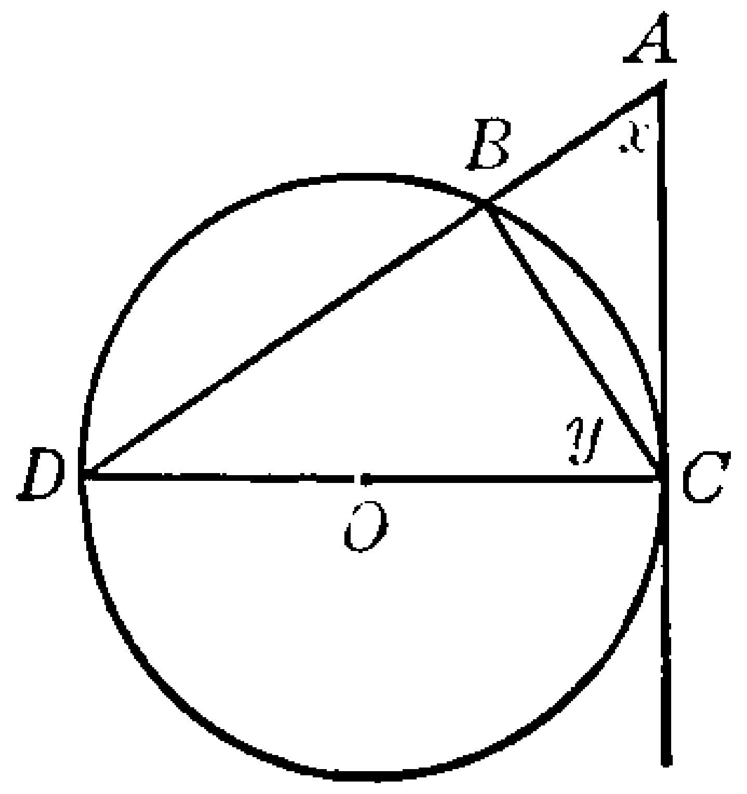


Given:
B, C, D and E
on the circle
Prove:
$\angle x = \angle y$

8. Prove: Two chords perpendicular to a third chord at its ends are equal.

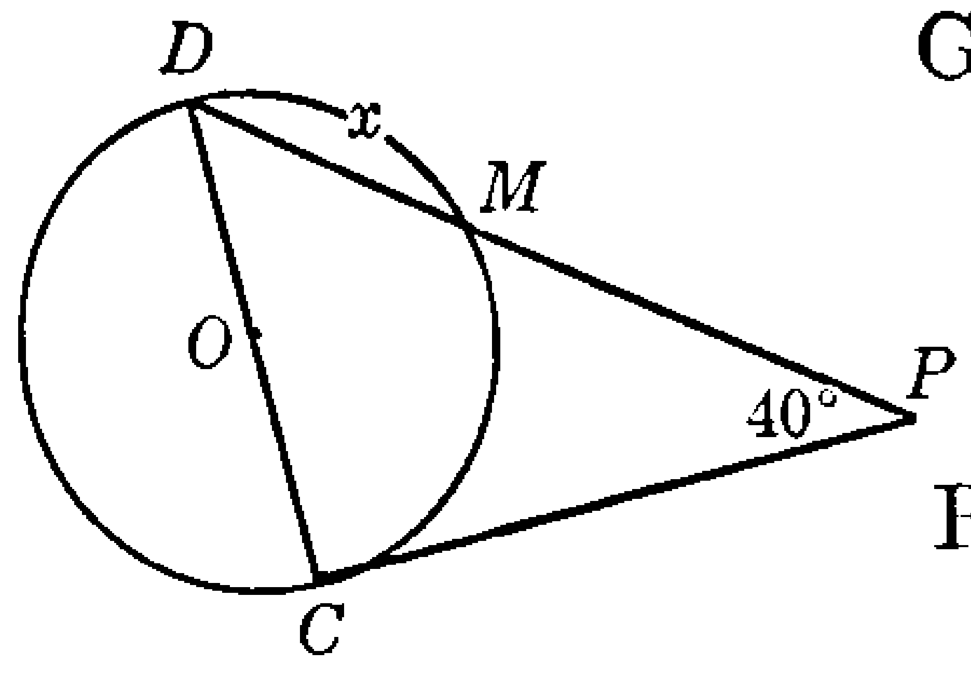
9. Prove: If two lines that intercept equal arcs on a circle do not meet inside the circle, then they do not meet at all.

10.



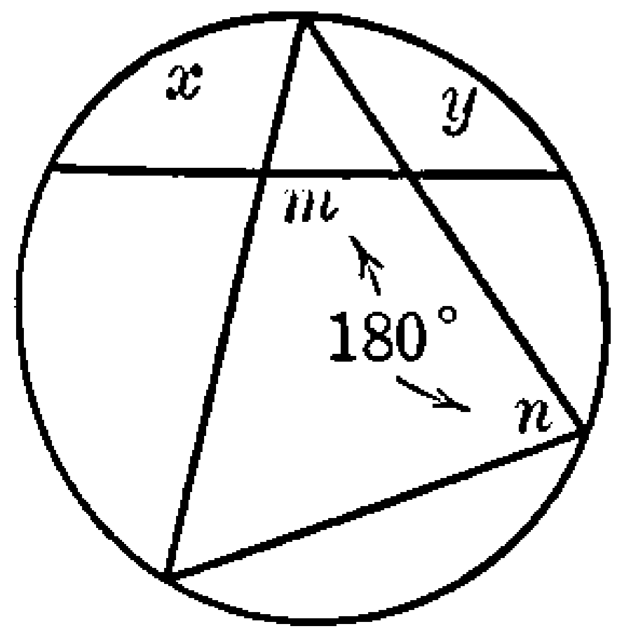
Given:
AC tan.
to $\odot O$ at C
DC a diam.
Prove:
$\angle x = \angle y$

11.



Given:
PC tan. to
$\odot O$ at C
DC a diam.
$\angle P = 40°$
Req.:
To find $\widehat{DM}$

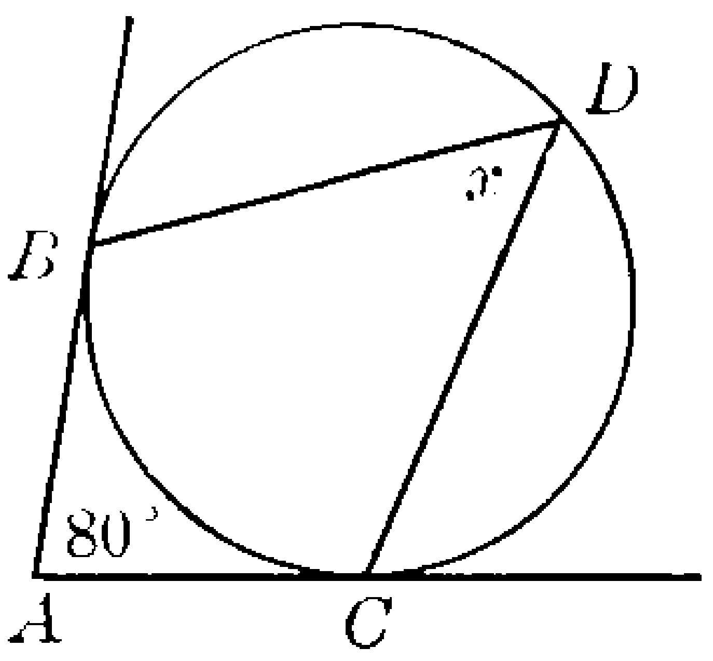
12.



Given:
$\angle m + \angle n = 180°$
Prove:
$\widehat{x} = \widehat{y}$

Hint: What angle in your figure equals $\angle n$? Express the measure of this angle in terms of its arcs.

13. (*a*)



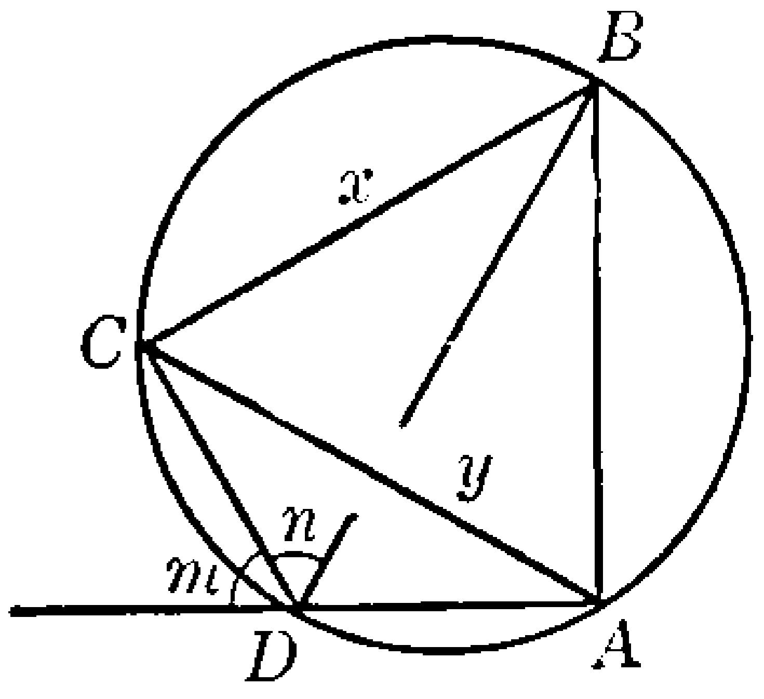
Given:
AB and AC
tan. to $\odot O$
at B and C
$\angle A = 80°$
Req.:
To find $\angle D$

(*b*) In part (*a*), let $\angle A = y$. Find a relationship between x and y.

14. If one side of an equilateral triangle is a diameter of a circle, then the circle bisects the other two sides.

15. From two vertices of an acute inscribed triangle, altitudes are drawn and extended to cut off an arc on a circle. Prove that this arc is bisected by the third vertex of the triangle.

16.

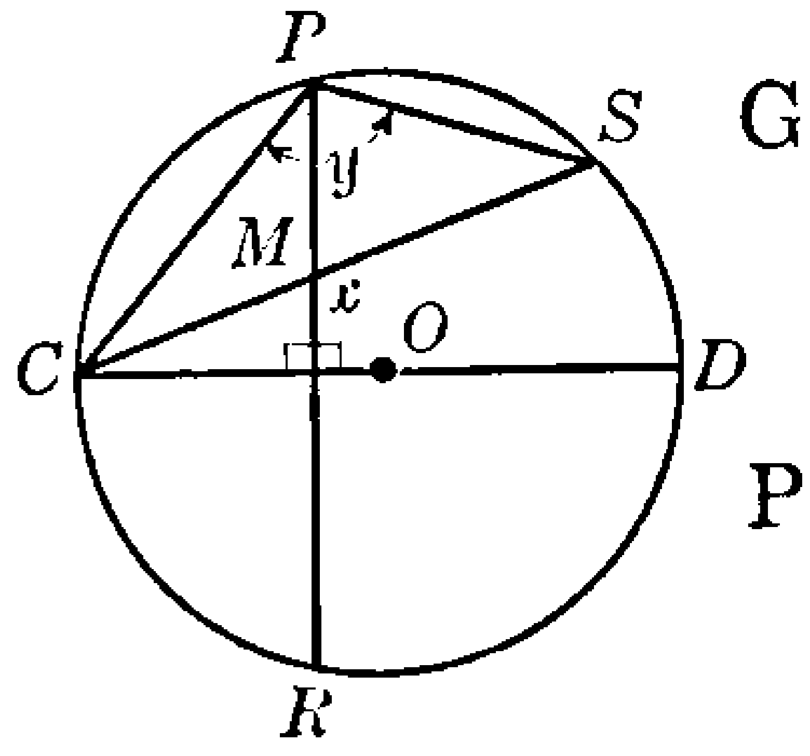


Given:
$ABCD$ an
inscribed quad.
with AD
prolonged
$\angle m = \angle n$
Prove:
$\triangle ABC$ isos.

SUPERIOR WORK

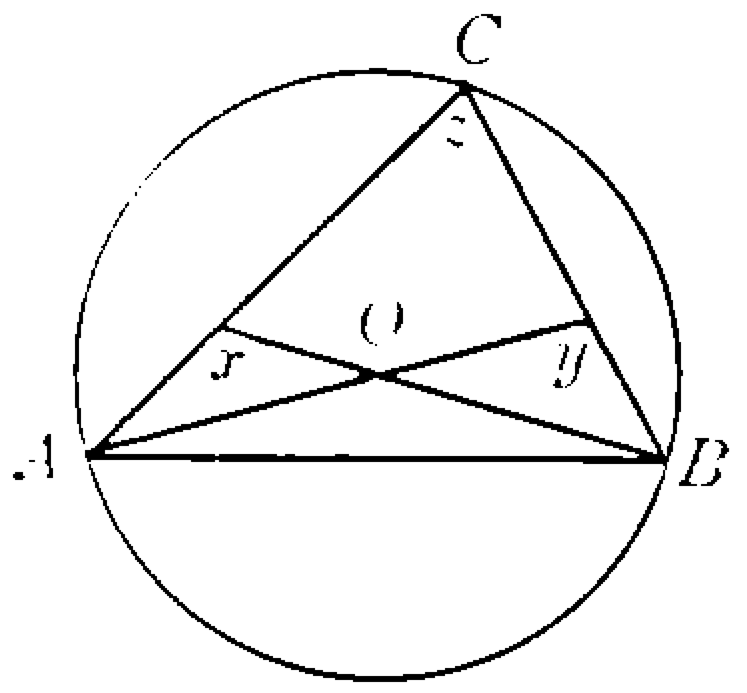
S1. From the same point on a semicircle, two chords are drawn so that the arc of the shorter chord is one-third the arc of the longer chord. A radius intersecting the longer chord is drawn to the end of the shorter chord. Prove that an isosceles triangle is formed.

S2.



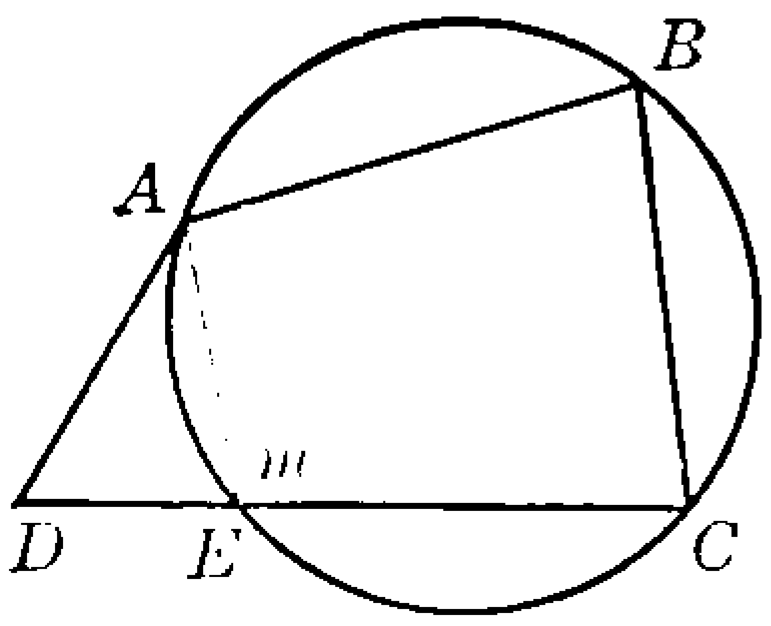
Given:
Diam. CD
$PR \perp CD$
Prove:
$\angle RMS = \angle CPS$

S3.



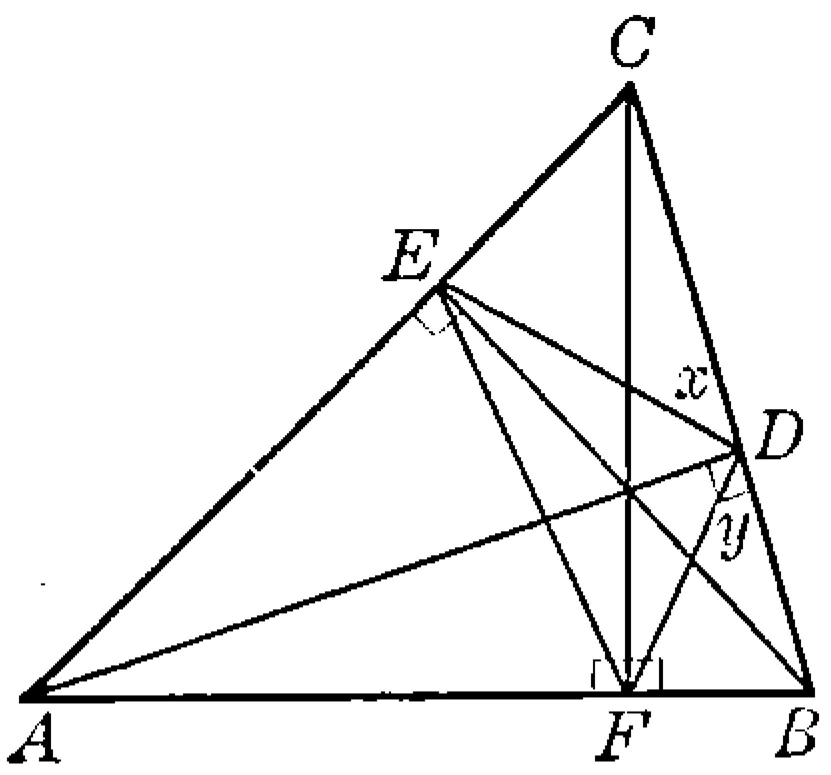
Given:
$\triangle ABC$ inscribed
in $\odot O$
Prove:
$\angle x + \angle y = 3 \cdot \angle z$

S4. Prove this theorem. If the opposite angles of a quadrilateral are supplementary, then a circle may be circumscribed about the quadrilateral. (Converse of § 241.)



Draw a circle passing through the vertices A, B, and C. By indirect proof, show that D cannot be off the circle. Prove that $\angle m$ and $\angle D$ are each supplementary to $\angle B$ and are therefore equal to each other. This will contradict the theorem about an exterior angle of a triangle.

S5. (*a*) Prove that two vertices of a triangle and the feet of the altitudes from these vertices lie on a circle; that is, a circle can be drawn through A, B, D, and E.



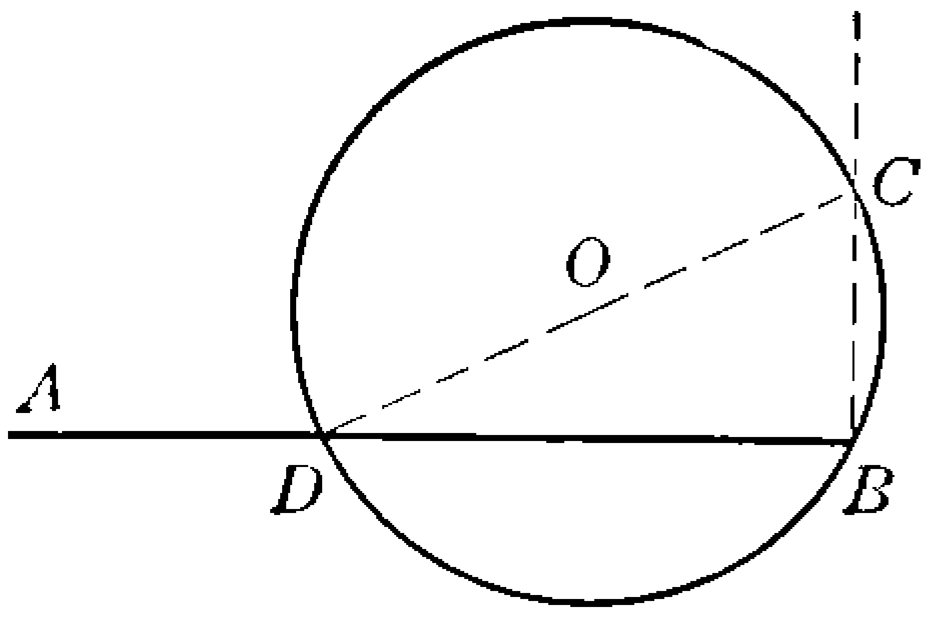
(*b*) In the figure, $\triangle DEF$ is called the pedal triangle of $\triangle ABC$ because it is formed by joining the feet of the altitudes. Prove that the altitudes of an acute triangle bisect the angles of the pedal triangle.

Hint: From (*a*), you can prove $\angle x = \angle A$. Similarly, prove $\angle y = \angle A$.

(*c*) Why do the altitudes of the triangle meet in a point?

Construction Problems

242. PROBLEM. Construct a perpendicular to a given line segment at one end, without prolonging the segment.



Given: Line segment AB
Req.: To cons. $CB \perp AB$

Construction

1. With any convenient pt. O as the center, and OB as a radius, draw a circle cutting AB at D.
2. Draw diam. DC.
3. Draw CB.

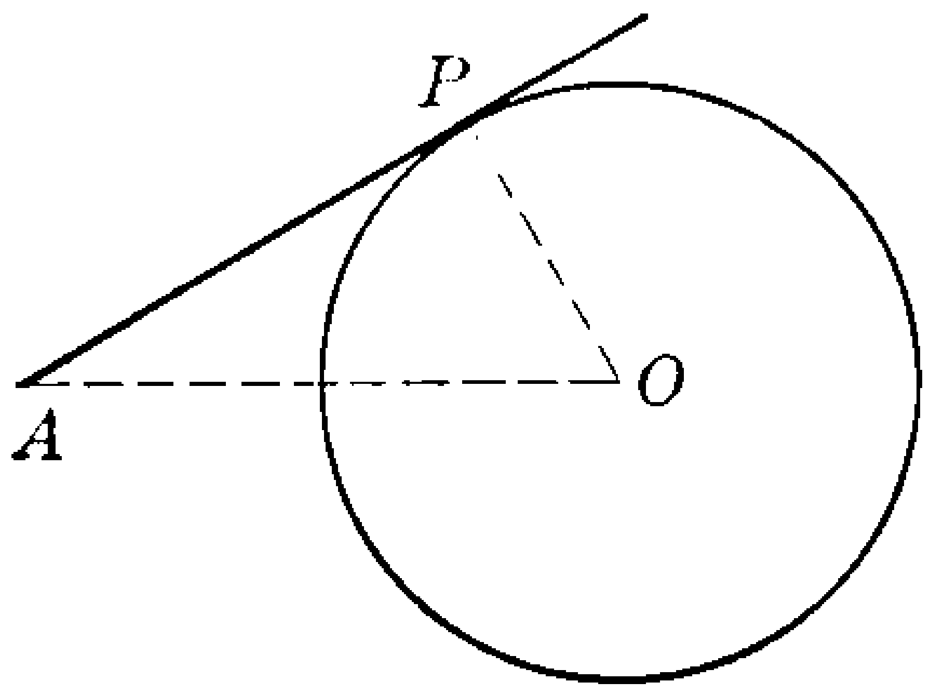
Conclusion: $CB \perp AB$.

Make this construction and write out the proof.

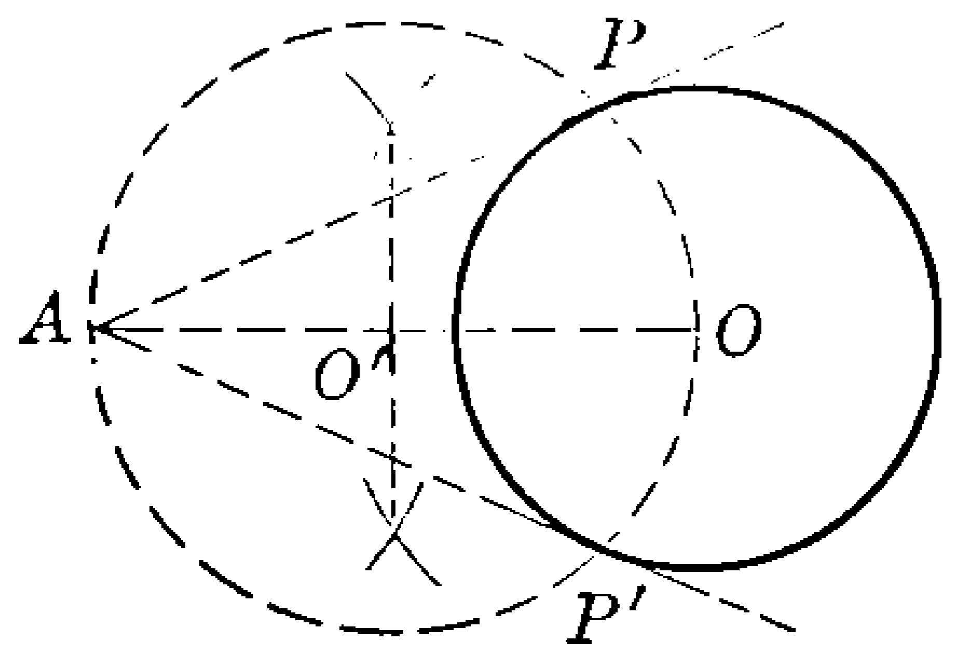
243. PROBLEM. Through a given point on a given circle, construct a tangent to the circle.

Draw a diameter to the given point, and then construct a line perpendicular to the diameter through the given point. If the method of the preceding problem for constructing a perpendicular is used, it will not be necessary to extend the diameter.

244. PROBLEM. Through a given external point, construct a tangent to a given circle.



Analysis. Assume that AP is the required tangent from the given point A to the circle O. Since APO is a right angle, it can be inscribed in a semicircle with AO as the diameter. The intersection of this semicircle with the given circle locates the point P.



Given: $\odot O$ and pt. A outside

Req.: To cons. a tan. to $\odot O$ from pt. A

Construction

1. Draw AO.
2. Cons. O', the midpt. of AO.
3. With AO as diam., draw $\odot O'$, cutting $\odot O$ at P and P'.
4. Draw AP and AP'.

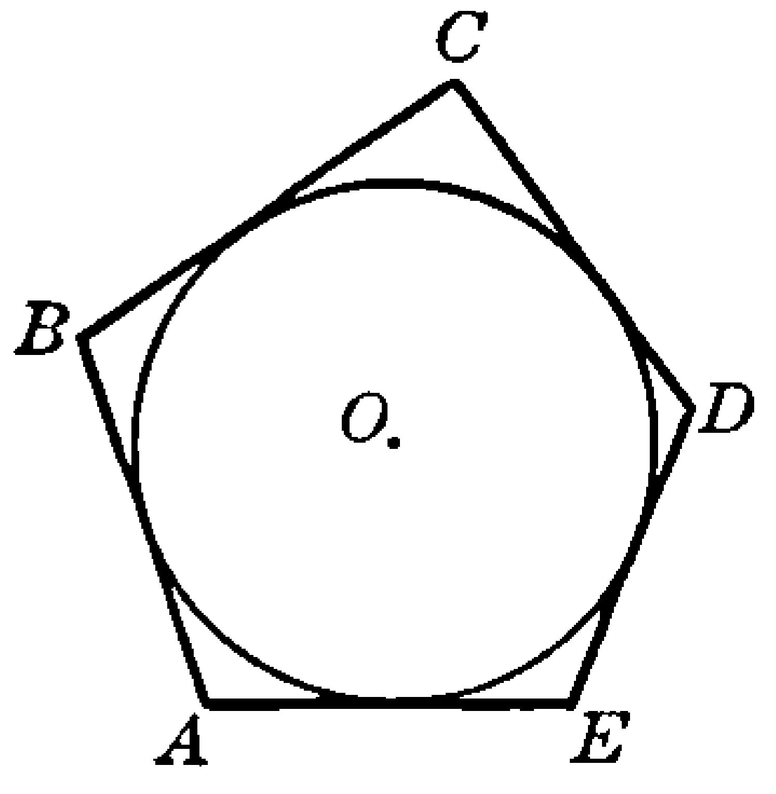
Conclusion: AP or AP' is the req. tan.

Make this construction and write out the proof.

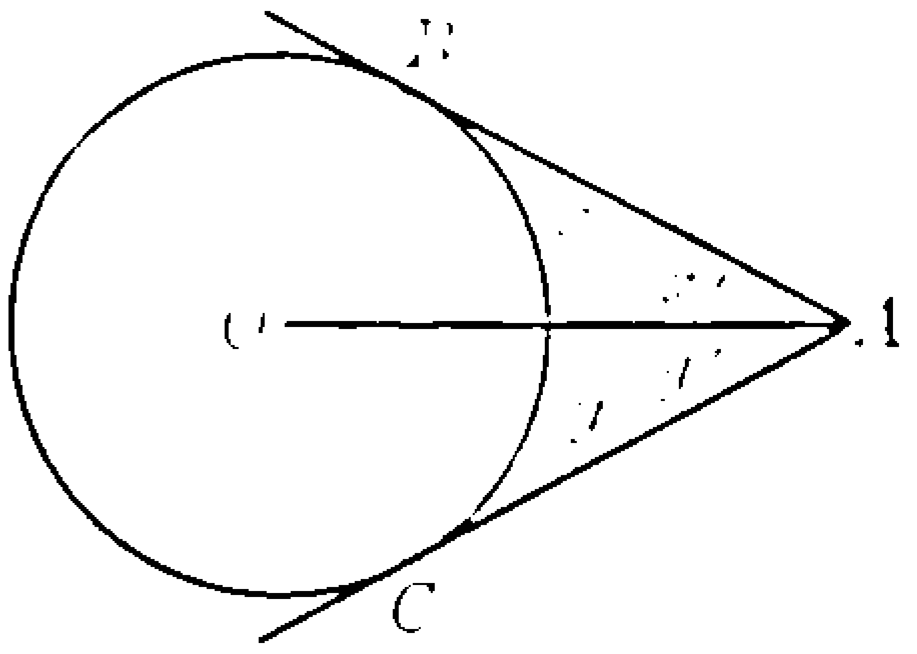
Inscribed Circles

245. Definition. A circle is inscribed in a polygon if it is tangent to all the sides of the polygon.

246. Definition. A polygon is circumscribed about a circle if the circle is inscribed in the polygon.



247. THEOREM. The tangents to a circle from an external point are equal and make equal angles with the line joining that point to the center of the circle.



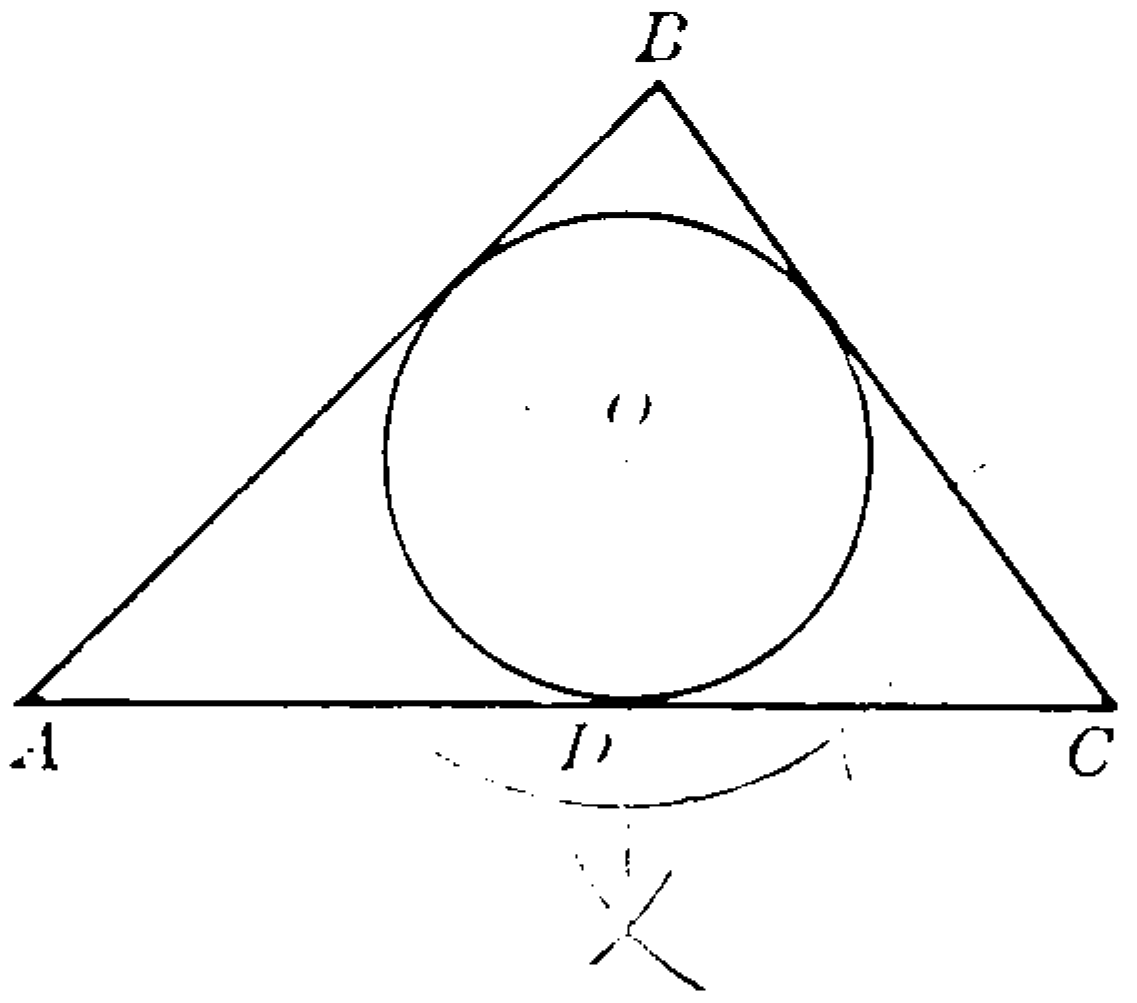
Given: AB and AC tan. to $\odot O$ at B and C

Prove: $x = y$

$\angle x' = \angle y'$

Draw the radii OB and OC and write out the proof.

248. PROBLEM. Inscribe a circle in a given triangle.



Given: $\triangle ABC$

Req.: To inscribe a $\odot$ in $\triangle ABC$

Construction

1. Cons. the bisectors of $\angle A$ and $\angle C$, meeting at O.
2. Cons. $OD \perp AC$.
3. With O as the center and OD as a radius, draw a circle.

Conclusion: $\odot O$ is inscribed in $\triangle ABC$.

PROOF

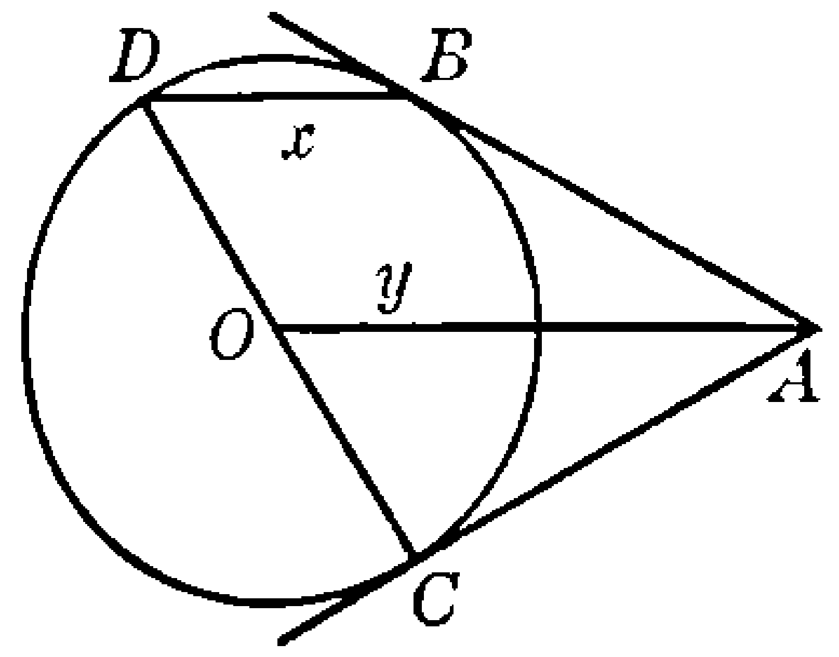
Statements	*Reasons*
1. O equidistant from AB, AC, and BC	1. Bisector of $\angle$ is locus of pt. equidistant from sides of the $\angle$ (§ 166).
2. AC, AB, and BC tan. to $\odot O$	2. A $\perp$ to a diam. (or radius) at one end is tan. to the $\odot$ (§ 224).

EXERCISES

1. Prove: If two parallel tangents are cut by a third tangent to the same circle, and the points of intersection are joined to the center, then a right triangle is formed.

2. Prove: The midpoint of an arc is equidistant from its chord and the tangent at one end of the chord.

3.

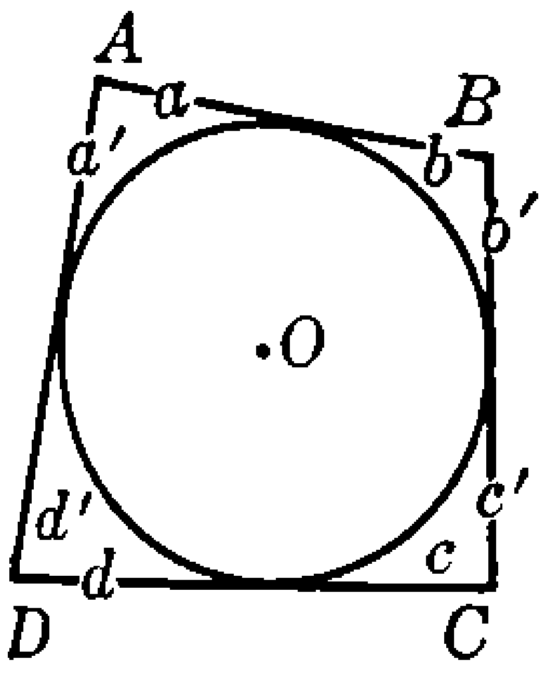


Given:
AB and AC
tan. to $\odot O$
Prove:
$DB \parallel OA$

4. (a) Construct a triangle with two angles 30° and 60°, and inscribe a circle in it.
   (b) Find the number of degrees in each arc between the points of tangency in (a).

5. Construct a circle tangent to three sides of a given quadrilateral.

6. Prove the following proposition:
Theorem. The sum of two opposite sides of a quadrilateral in which a circle is inscribed equals the sum of the other two sides.



Given:
$\odot O$ inscribed
in quad. $ABCD$
Prove:
$AB + CD = BC + AD$

PROOF

Statements	*Reasons*
1. $a = a'$ $b = b'$ $c = c'$ $d = d'$	1. Why?

Complete the proof.

SUPERIOR WORK

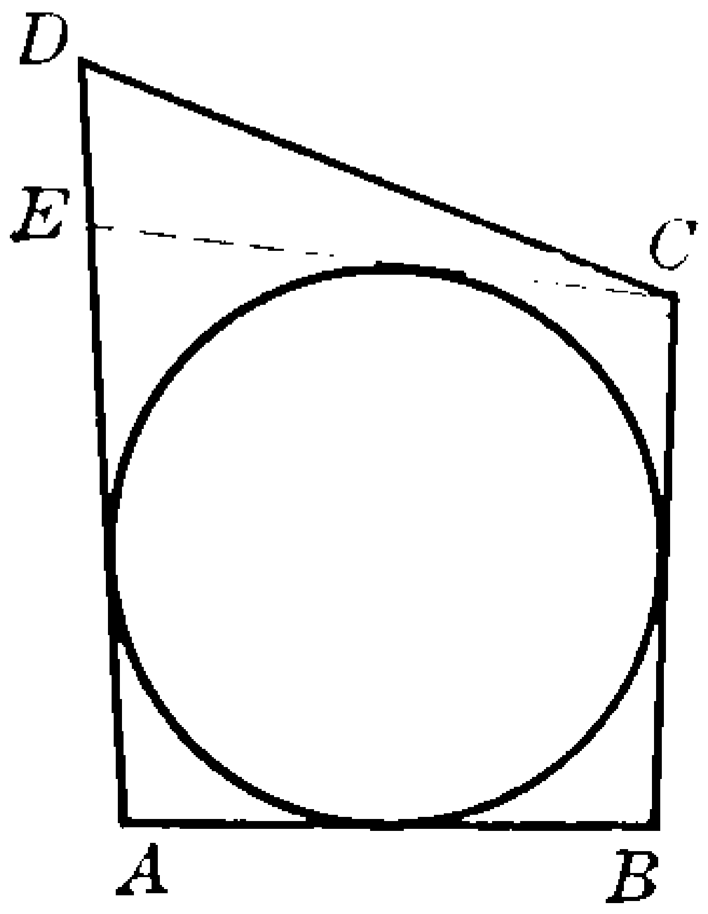
S1. State and prove the converse of the theorem in Ex. 6 above. Use the indirect proof. Draw a circle tangent to three sides of the quadrilateral. Now, assuming that DC is not tangent to the circle, draw CE which is tangent.

From the original theorem,

$$AB + CE = AE + BC$$

It is now given $AB + CD = AD + BC$

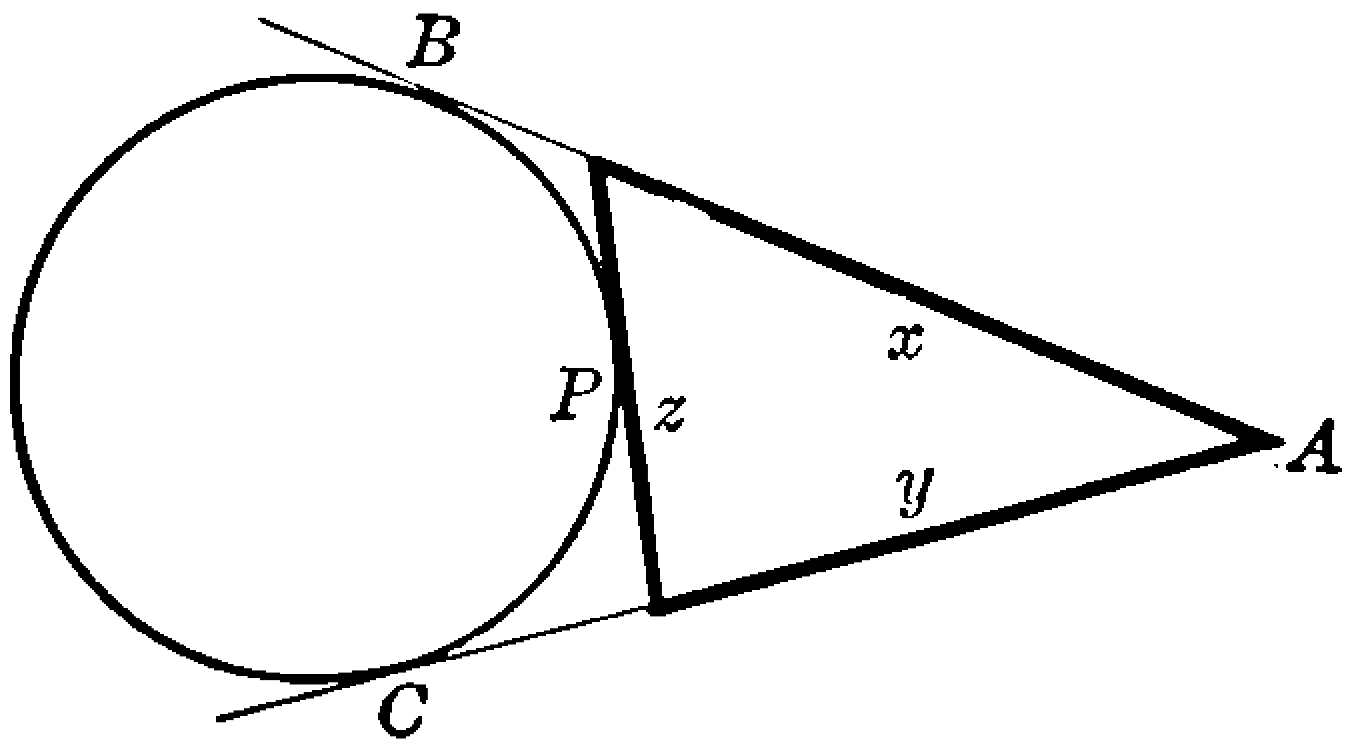
Subtract these two equations and show that an impossible relationship concerning the sides of $\triangle DEC$ would result.



Tell which of the following special quadrilaterals can have inscribed circles: a rhombus, a square, a parallelogram, an isosceles trapezoid, and a kite (a quadrilateral formed by the legs of two isosceles triangles on each side of a common base).

S2. If two stationary tangents from a common point are cut by a movable tangent to the same circle, then the perimeter of the triangle thus formed is constant. (P is any point on arc BC.)

Prove: $x + y + z = k$



S3. In the figure above, the circle is said to be an *escribed* circle or an excircle of the triangle.

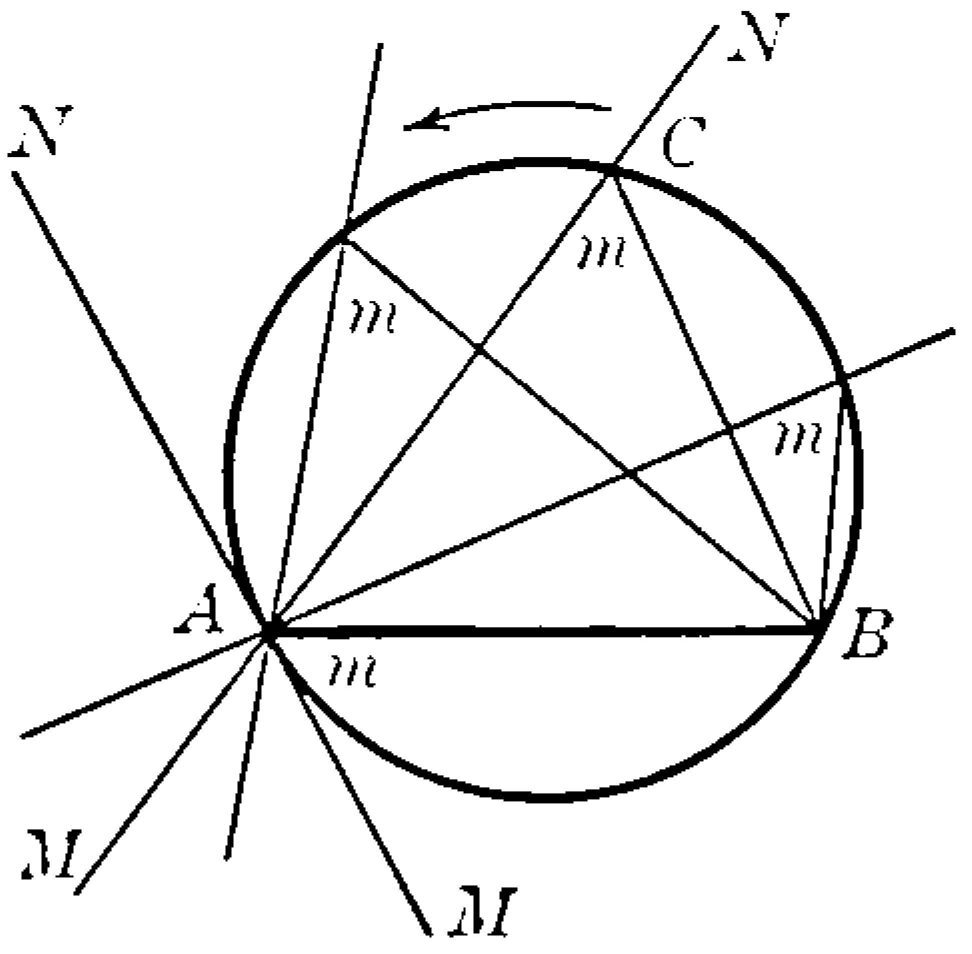
Definition. An escribed circle of a triangle is a circle that is tangent to one side of the triangle and to the prolongations of the other two sides.

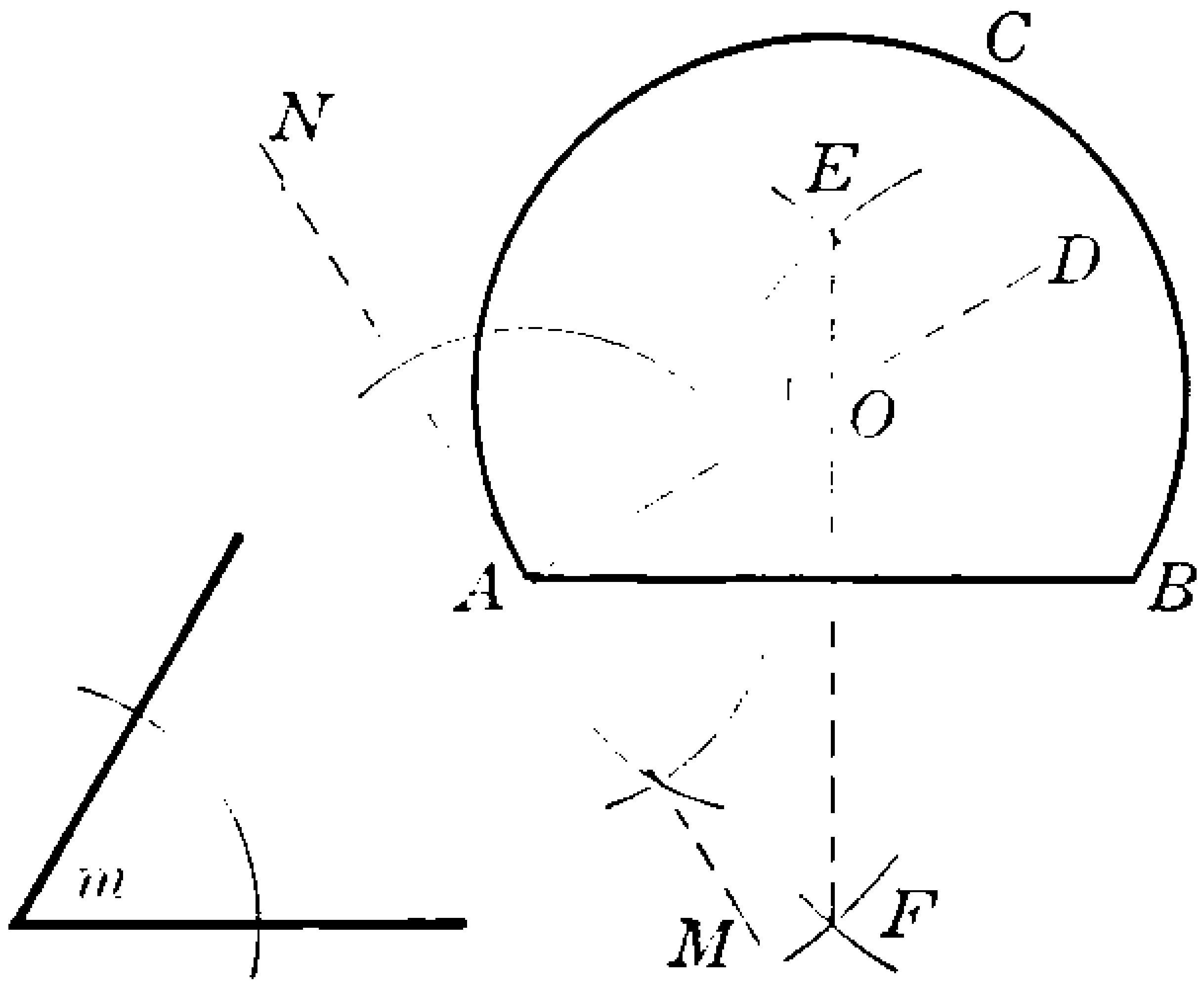
(a) On what lines is the center of an escribed circle?

(b) Construct two escribed circles of a given triangle.

(c) Prove that the line connecting the centers of your escribed circles passes through a vertex of the triangle.

S249. **PROBLEM. Having given its chord, construct an arc in which a given angle may be inscribed.**





Analysis. Assume that AB is the given chord and $\overset{\frown}{ACB}$ is the required arc. As the point C moves over $\overset{\frown}{BCA}$, MN rotates about point A and the given $\angle m$ remains the same size. When C reaches the point A, MN becomes a tangent, and the inscribed $\angle m$ becomes $\angle MAB$.

We therefore begin our construction by drawing tangent MN, making $\angle MAB$ equal to the given $\angle m$. The center of the circle must then be on the perpendicular bisector of the chord AB, and also on the perpendicular to tangent MN at the point of contact A.

Given: $\angle m$ and chord AB

Req.: To construct an arc in which $\angle m$ may be inscribed, having AB as its chord

Construction

1. Cons. $\angle MAB = \angle m$.
2. Cons. $AD \perp MN$, and EF the $\perp$ bisector of AB, the two meeting at O.
3. With O as the center and OA as a radius, draw $\overset{\frown}{ACB}$.

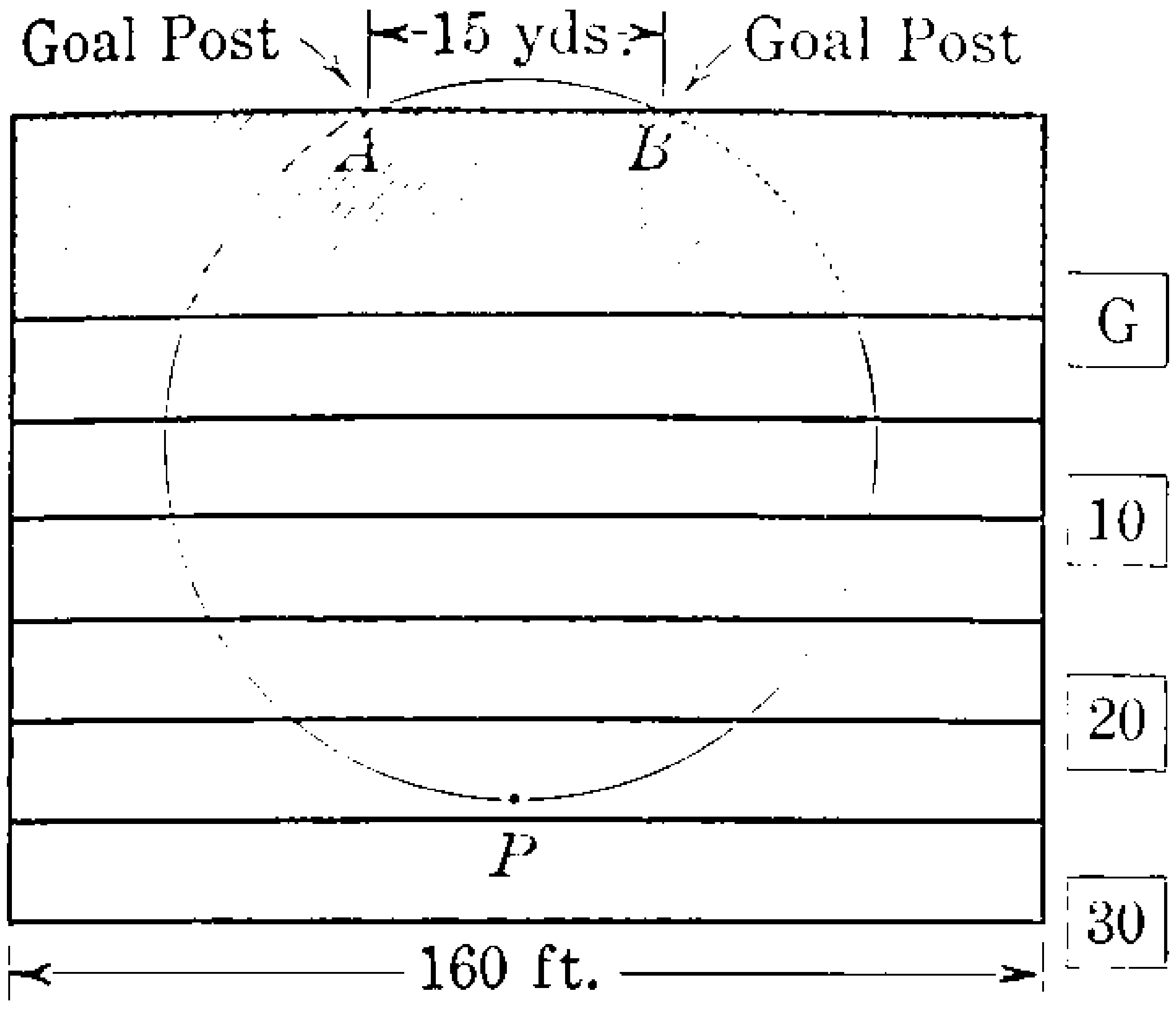
Conclusion: $\overset{\frown}{ACB}$ is the req. arc.

PROOF

Statements	*Reasons*
1. $\angle ACB = \angle MAB$	1. An inscribed $\angle$ or an $\angle$ formed by a tan. and a chord to pt. of contact $\overset{\circ}{=} \frac{1}{2}$ the intercepted arc (§ 238).
2. $\angle MAB = \angle m$	2. Cons.
3. $\angle ACB = \angle m$	3. Subs. Ax.

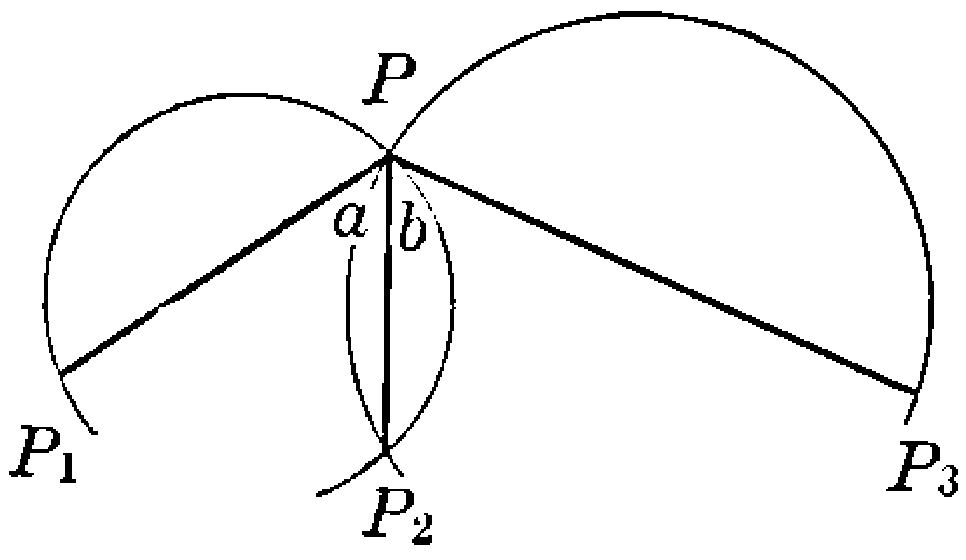
S4. On a given chord, construct an arc in which a 45° angle may be inscribed.

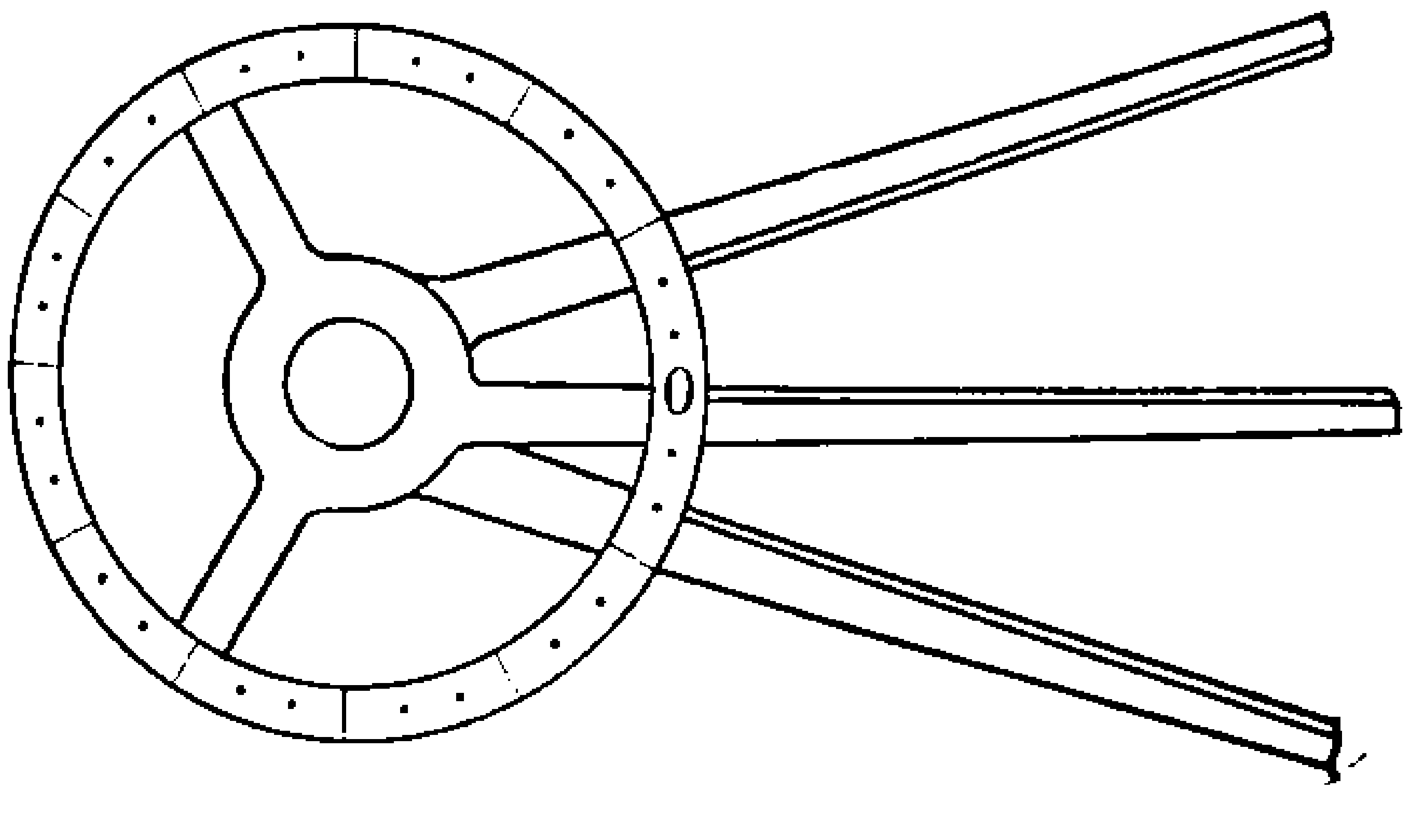
S5. You are a quarterback on a football team. You have a player who is a reliable goal kicker from the middle of the 24-yard line. Draw to scale a plan of a portion of the football field. On it locate a circle showing the locus of other points from which a field goal would be reliable. How close to the sideline on the 20-yard stripe would you call for a field goal?



The same principle is used in navigation. If the part APB of the circle is dangerous to sail, how can a mariner determine whether he is in or out of the danger zone by measuring the "danger angle" APB?

S6. An aviator at P measures $\angle a$ and $\angle b$, using three points P_1, P_2, and P_3, which he identifies on his chart. Make the construction, showing how he could locate his unknown position P on the chart. (This is known as the three-point problem.)





Help: Three points P_1, P_2, and P_3 are given; also $\angle a$ and $\angle b$. Construct the locus of P so that $\angle P_1PP_2 = \angle a$; and the locus of P so that $\angle P_2PP_3 = \angle b$. (See § S249.)

The diagram shows a "three-arm" protractor. By setting the measured angles a and b, the required point can be located quickly. Explain how.

By bolting together three thin sticks, show how the three-arm protractor is used.

S7. *Definition.* Either pair of opposite sides of an inscribed quadrilateral are said to be antiparallel with respect to the other two sides.

In the figure on top of page 154, AB and CD are antiparallel with respect to AD and BC; AD and BC are antiparallel with respect to AB and CD.

(a) Prove: If the bisector of a given angle cuts two lines making the interior angles on the same side of the transversal (the bisector) equal, then the two lines are antiparallel with respect to the sides of the given angle.

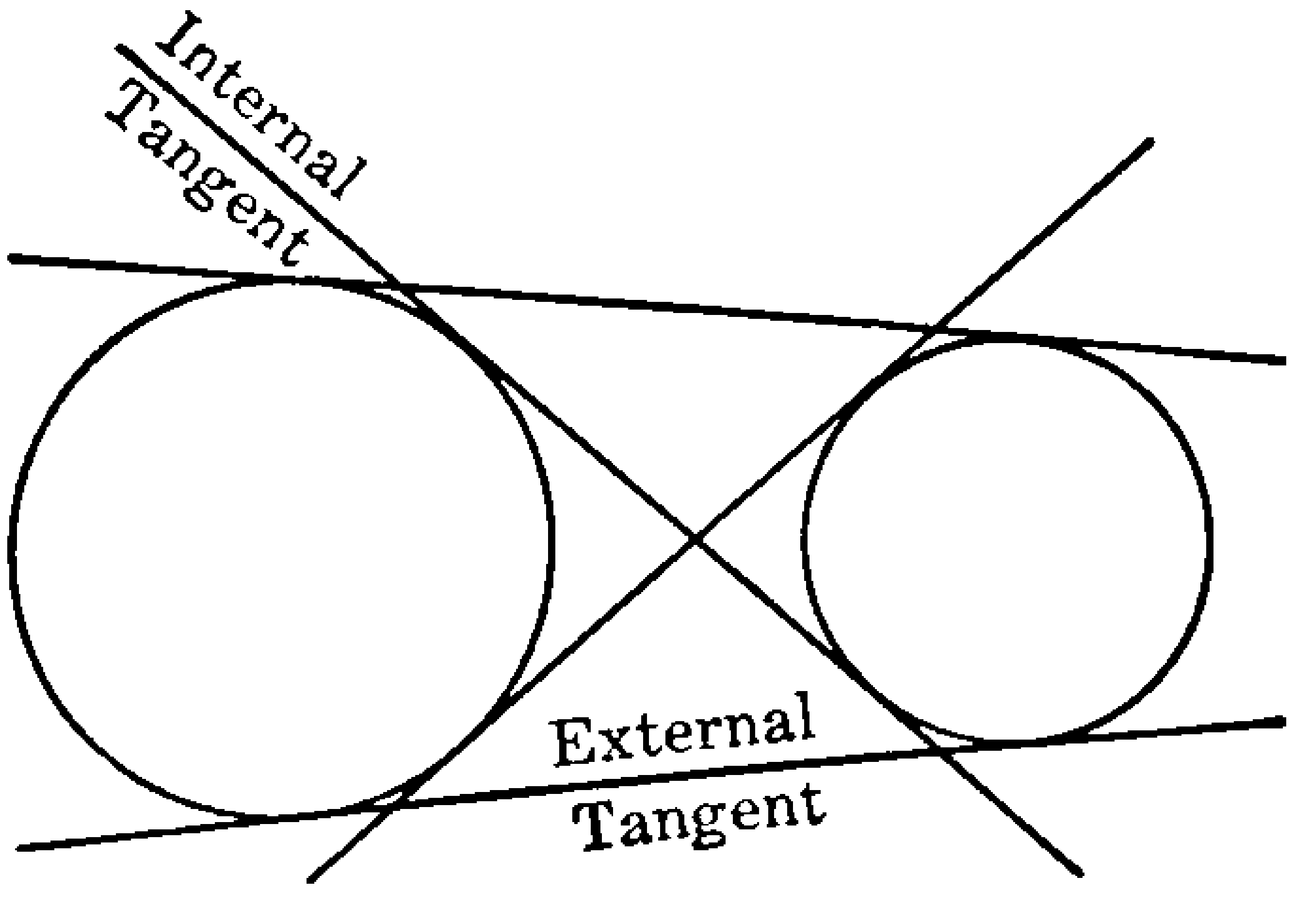
(b) In your figure for (a), there are two triangles that have the given angle in common. Imagine that one of these triangles is turned through 180° about the angle bisector as an axis. Draw the new figure and prove that two lines which were originally antiparallel have now become parallel. This is why the term antiparallel is used.

Note that there are no *absolutely* antiparallel lines. They must be related to other lines so that we may know what rotation will make them parallel.

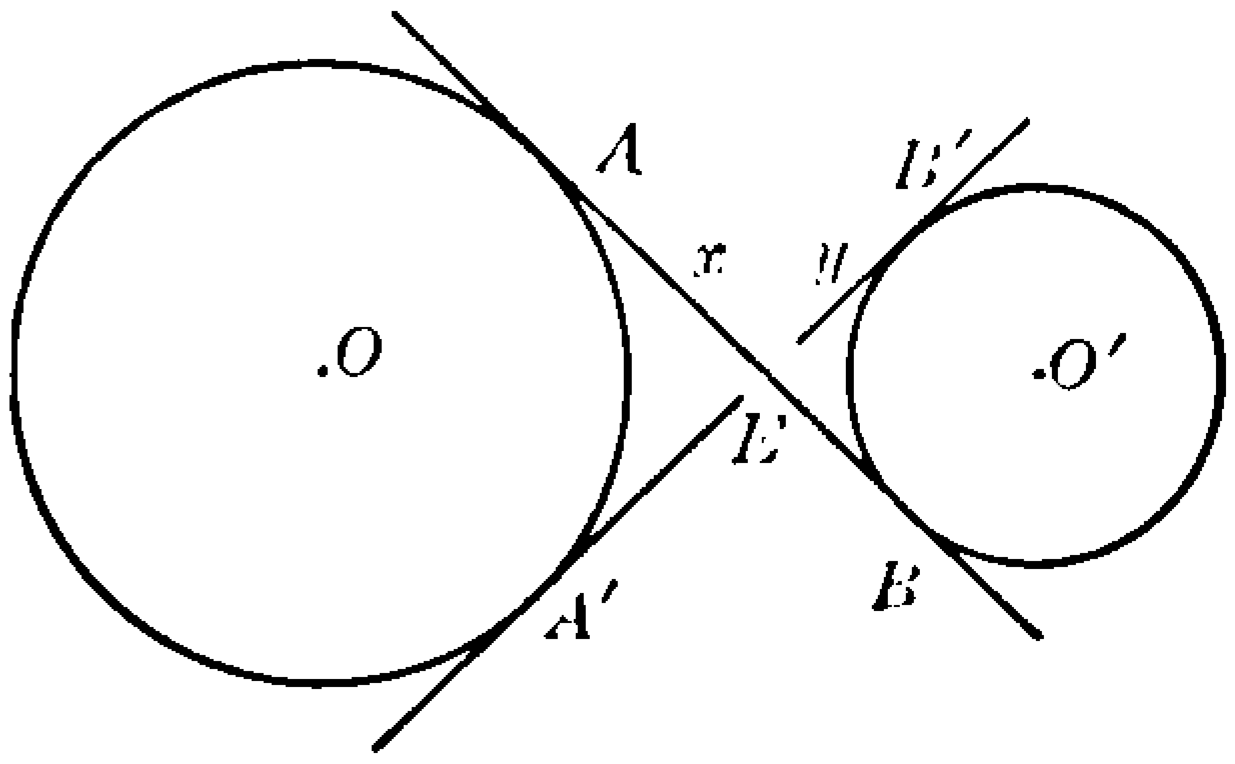
Two Circles

250. Definitions. A common external tangent to two circles is a line that is tangent to two circles on the same side of the line.

A common internal tangent to two circles is a line that is tangent to two circles on opposite sides of the line.



251. THEOREM. The common internal tangents to two circles are equal.



Given: AB and $A'B'$ common int. tans. to $\circledS O$ and O'

Prove: $AB = A'B'$

PROOF

Statements	*Reasons*
1. $AE = A'E$ $EB = EB'$	1. Tans. to $\odot$ from ext. pt. are =.
2. $AB = A'B'$	2. Add. Ax.

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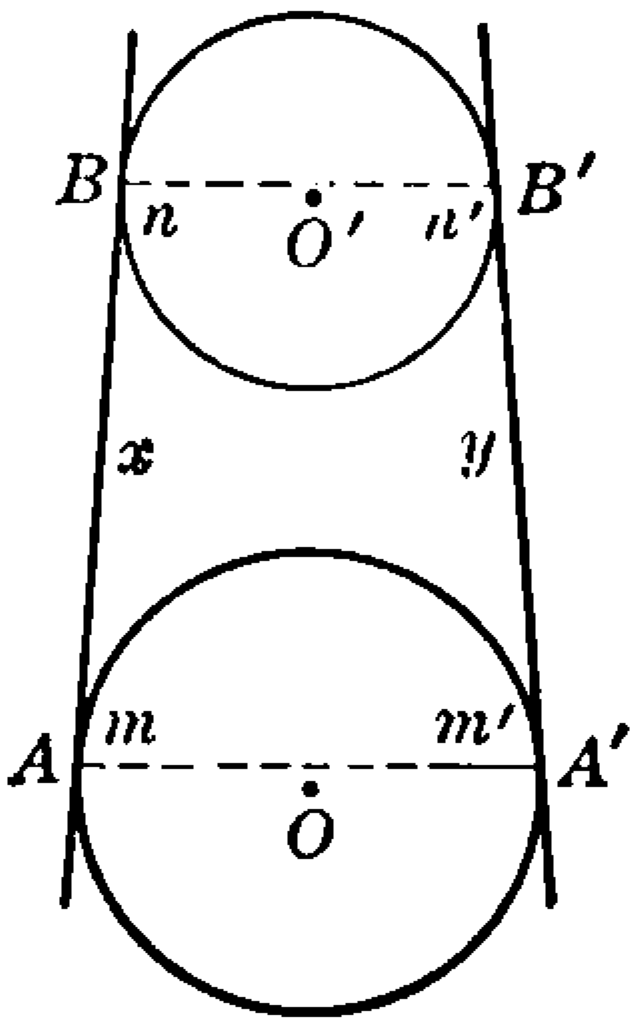
252. THEOREM. The common external tangents to two circles are equal.



Given: AB and $A'B'$ common ext. tans. to $\circledS O$ and O'

Prove: $AB = A'B'$

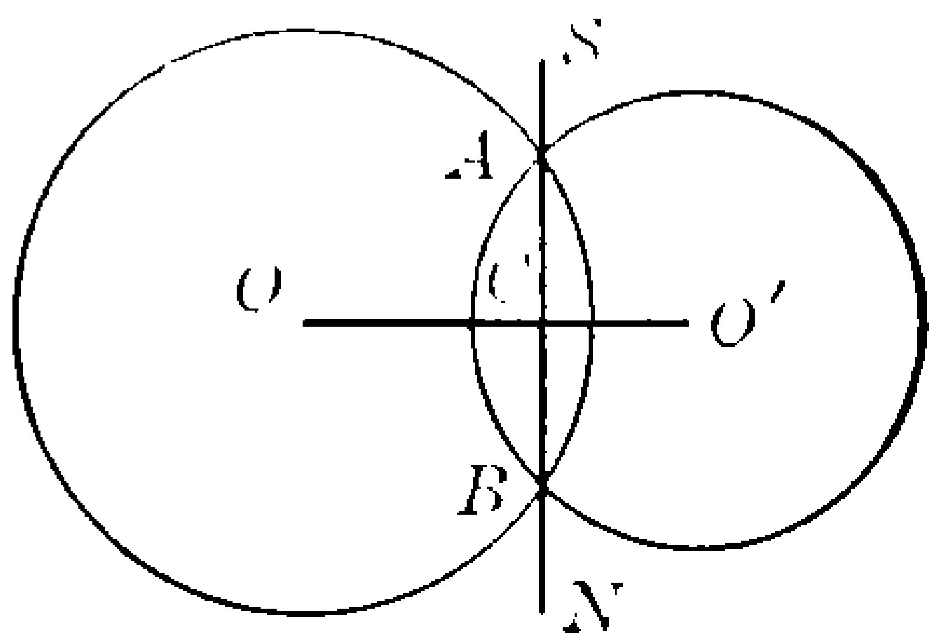
Cons.: Draw AA' and BB'.

PROOF

Statements	*Reasons*
1. $\angle m = \angle m'$, $\angle n = \angle n'$	1. An $\angle$ between tan. and chord to pt. of contact $\doteq \frac{1}{2}$ the intercepted arc (§ 238).
2. $\angle m + \angle n + \angle m' + \angle n' = 360°$	2. Sum of all the $\angle$s of a polygon = $(n - 2)$ st. $\angle$s.
3. $\angle m + \angle n = 180°$	3. Div. Ax.
4. $BB' \parallel AA'$	4. Int. $\angle$s sup., $\parallel$s.
5. If $AB \nparallel A'B'$, then $AB = A'B'$	5. Base $\angle$s of trap. are =, trap. isos.
6. If $AB \parallel A'B'$, then $AB = A'B'$	6. Opp. sides of $\square$ are =.

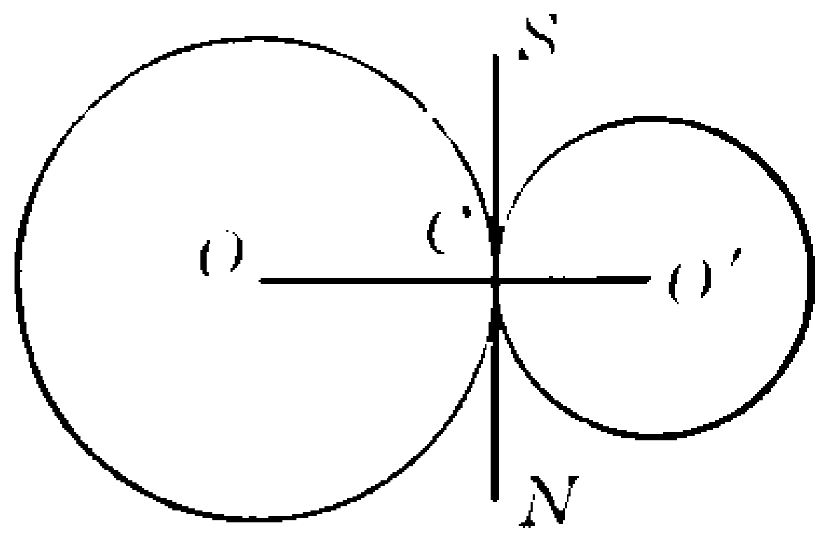
Note that in Steps 5 and 6 we are drawing the same conclusion from two opposite assumptions. This is known as the *dilemma* argument. In everyday life the conclusion is usually unpleasant and we speak of being "caught between the horns of a dilemma."

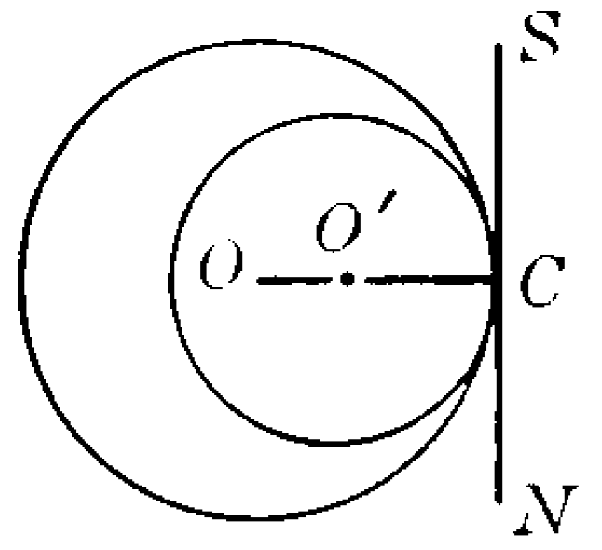
In the figure, circles O and O' intersect each other in A and B. AB is the common chord and OO' is called the *line of centers*. From the Two-Point Theorem (§ 151), we see that OO' is the perpendicular bisector of AB.



253. Corollary. If two circles intersect, the line of centers bisects the common chord at right angles.

As the two circles move about, the points A and B approach each other so that one of the two positions shown in the figure will result. In the first position SN becomes a common internal tangent, and the circles are tangent externally. In the second position SN becomes a common external tangent, and the circles are tangent internally.

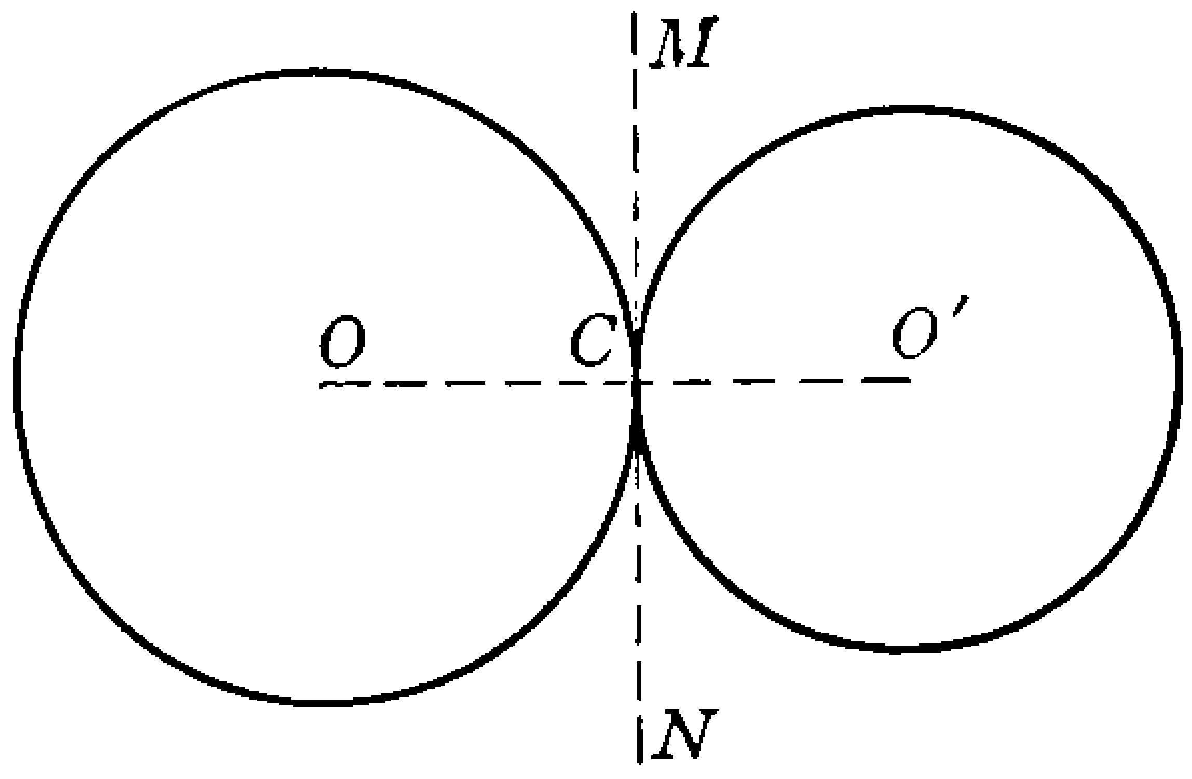




254. Definition. Two circles are tangent to each other if they are tangent to the same line at the same point.

We see that as the circles approach tangency, the common chord AB approaches zero and the midpoint C becomes the point of contact. Now, the line of centers always passes through C and continues to do so as C becomes the point of tangency.

255. THEOREM. If two circles are tangent to each other, the line of centers passes through the point of contact.



Given: ⊙O and O' tangent to each other at C

Prove: The line of centers OO' passes through C

Cons.: Draw OC and $O'C$.

PROOF

Statements	*Reasons*
1. There is a common tan. MN through C	1. Def. tan. ⊙ (§ 254).
2. $\angle OCM$ and $\angle MCO'$ rt. ∠s	2. Why?
3. OCO' a st. line	3. Why?

This proves indirectly that the line of centers OO' passes through C.

EXERCISES

1. (*a*) If two circles are equal, prove that the external tangents are parallel.
(*b*) If the external tangents to two circles are not parallel, prove that they are equal by prolonging them to meet.

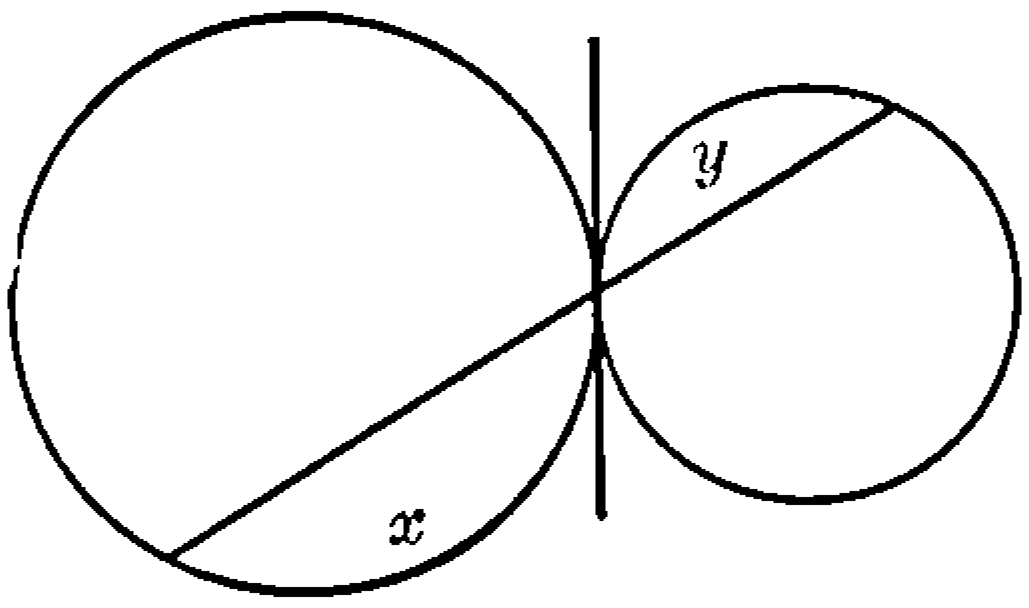
2. (*a*) If two circles are tangent to each other, and from any point in the common tangent through their point of contact two other tangents are drawn to the circles, these tangents are equal. (There are two cases.)
(*b*) If two circles are tangent to each other externally, the common internal tangent bisects the common external tangent.

3. (*a*) In two concentric circles, two equal chords of the smaller circle, when prolonged, will form equal chords of the larger circle.
(*b*) Prove: If the diameter of one circle is the radius of another circle, the circles are tangent to each other internally.

4. Prove: If two equal circles are tangent externally, and the ends of two parallel diameters are joined, then a rhombus is formed.

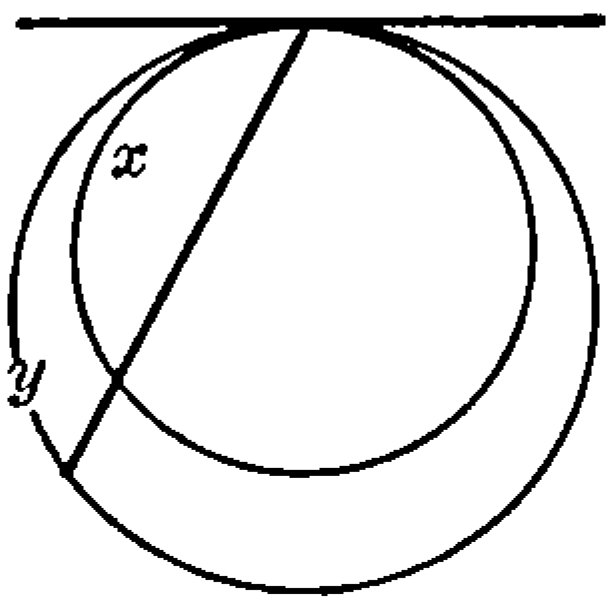
5. In each case, the circles are tangent to each other.

(*a*)



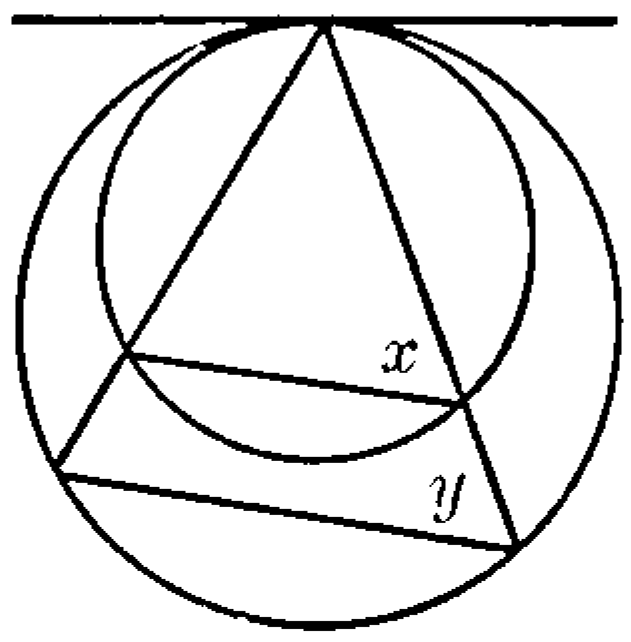
Prove:
$\widehat{x} \overset{\circ}{=} \widehat{y}$
(in angular measure)

(*b*)



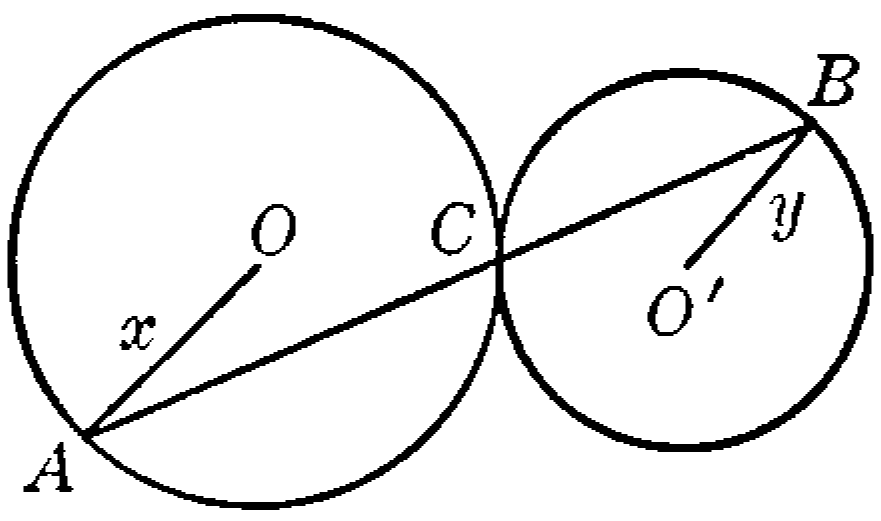
Prove:
$\widehat{x} \overset{\circ}{=} \widehat{y}$
(in angular measure)

(*c*)



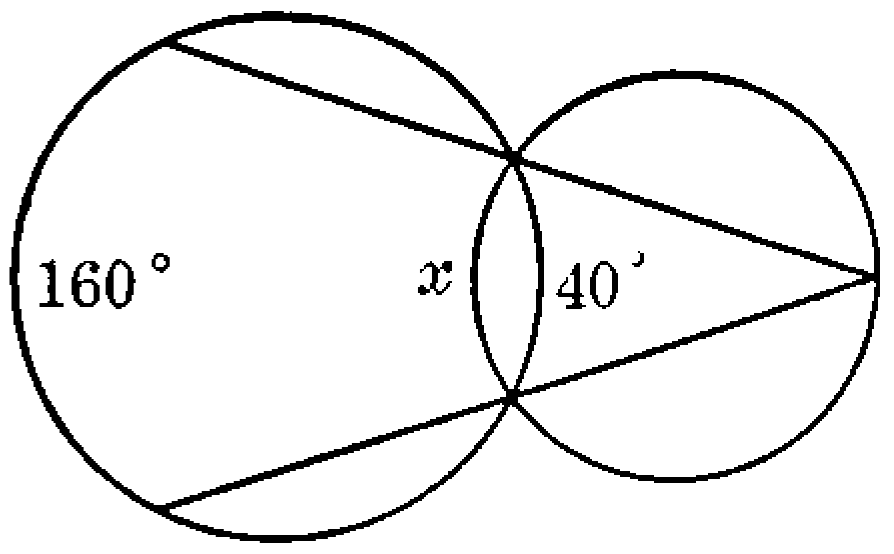
Prove:
$\angle x = \angle y$

6.



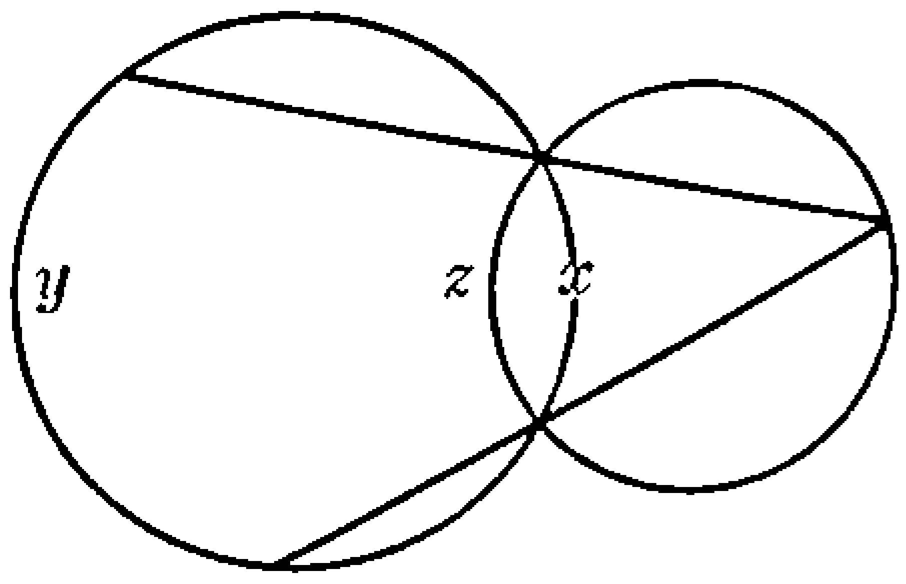
Given:
ⓢ O and O'
tan. at C
Prove:
$OA \parallel O'B$

7. (*a*)

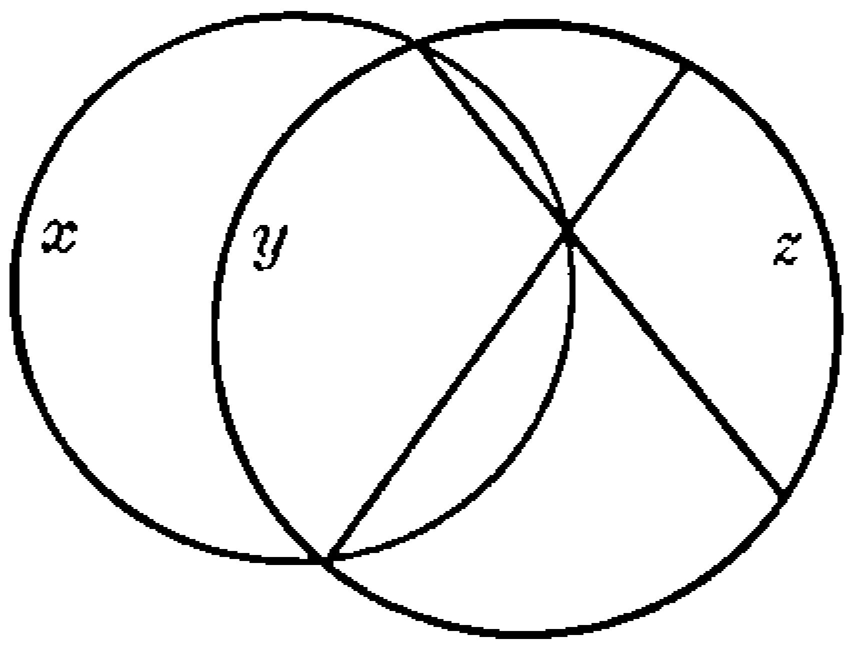


Find the number of degrees in arc x.

(*b*) x, y, and z the angular measures of the arcs. Write an equation giving the relationship among x, y, and z.

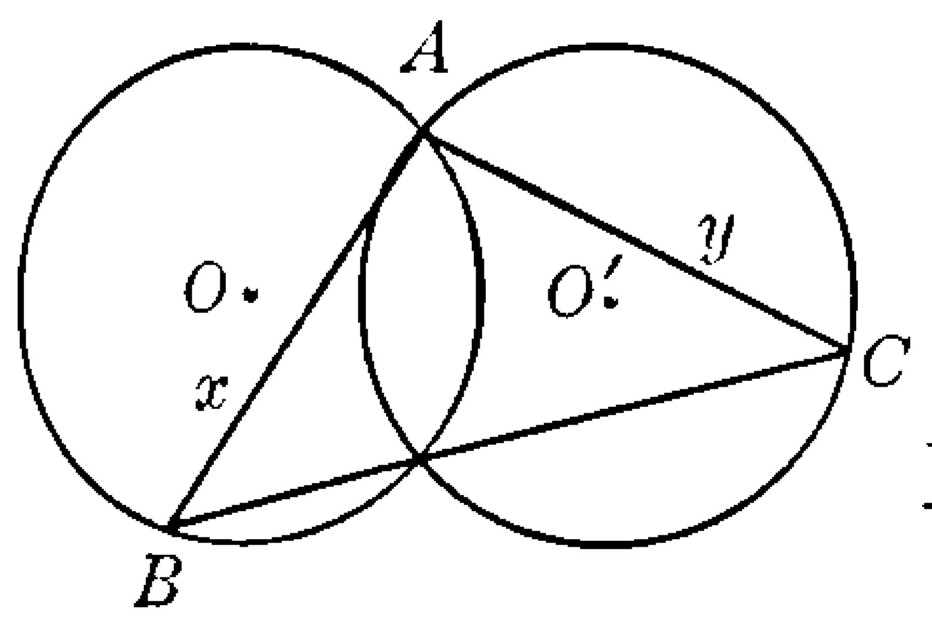


(*c*)



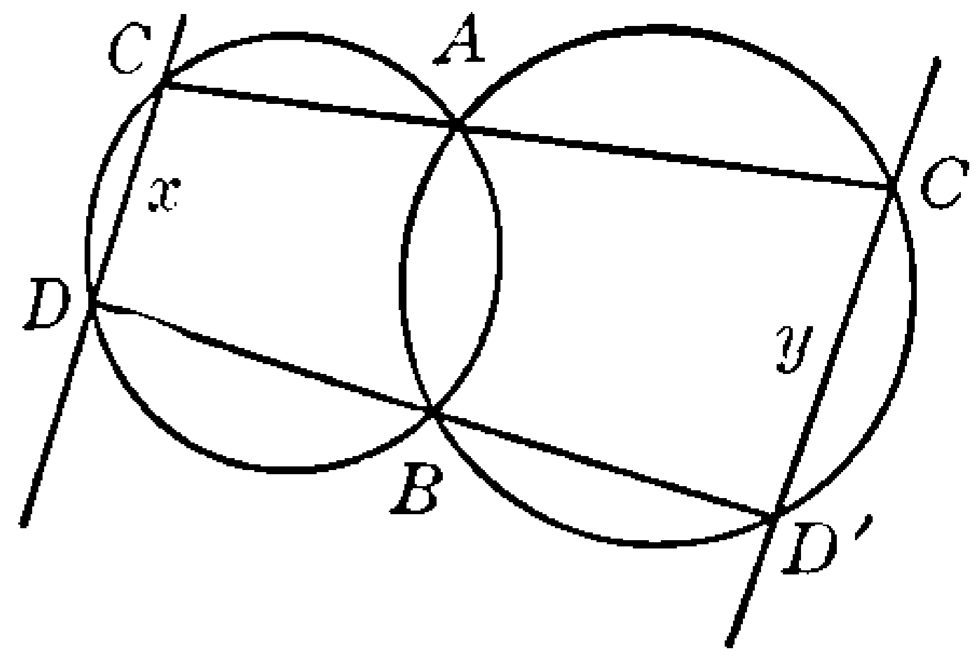
Write an equation connecting the angular measures x, y and z

8.



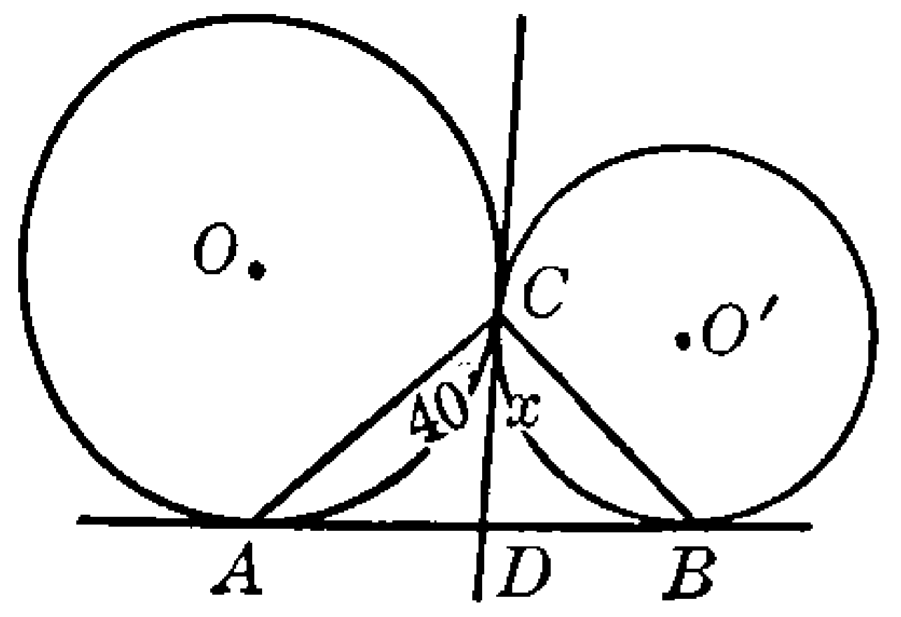
Given:
= ⓢ O and O' intersecting st. line BC
Prove:
$AB = AC$

9.



Given:
Intersecting ⓢ
CC' and DD' st. lines
Prove:
$CD \parallel C'D'$

10. (*a*)

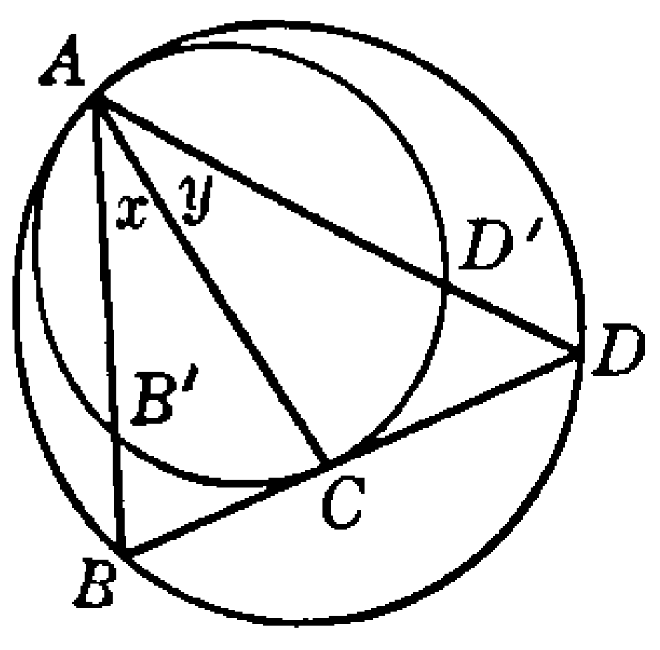


Given:
ⓢ O and O' tan. at C
Common int. tan. CD
Common ext. tan. AB
$\angle ACD = 40°$
Req.:
To find $\angle x$

(*b*) Prove: If two circles are tangent externally, and a common external tangent is drawn, then the three points of tangency are the vertices of a right triangle.
Hint: In the diagram above, replace 40° by y and prove $x + y = 90°$.

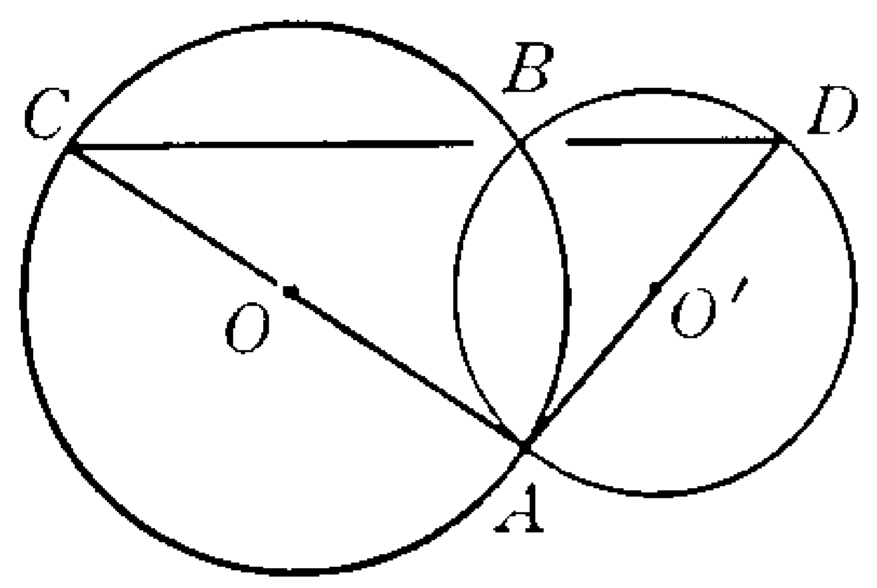
SUPERIOR WORK

S1.



Given:
ⓢ tan. internally
BD tan. to inner ⊙
Prove:
$\angle x = \angle y$
Help:
Draw $B'D'$ and prove it $\parallel BD$.

S2.



Given:
Diams. AC and AD
Prove:
CD passes through B

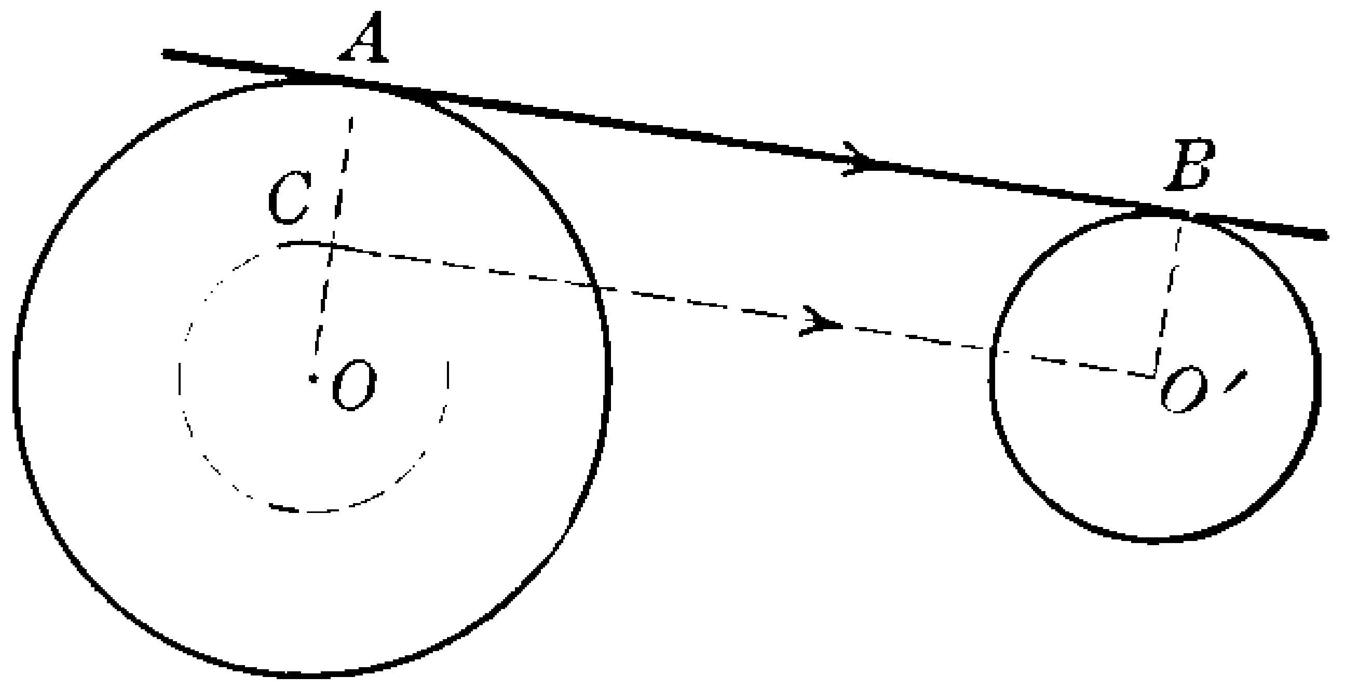
S3. As two circles move toward each other, they pass through five positions: (1) one circle is entirely outside the other circle; (2) they are tangent externally; (3) they cut in two points; (4) they are tangent internally; and (5) one is entirely inside the other.

State the necessary and sufficient condition for each position in terms of the number of common external and internal tangents; also, in terms of the relationship between the line of centers and the radii.

S4. (*a*) If two circles have two external tangents, then the four points of tangency are cyclic (they are on a circle).

(*b*) If two circles have two internal tangents, then the four points of tangency are cyclic.

S5. *Problem.* To construct a common external tangent to two circles.



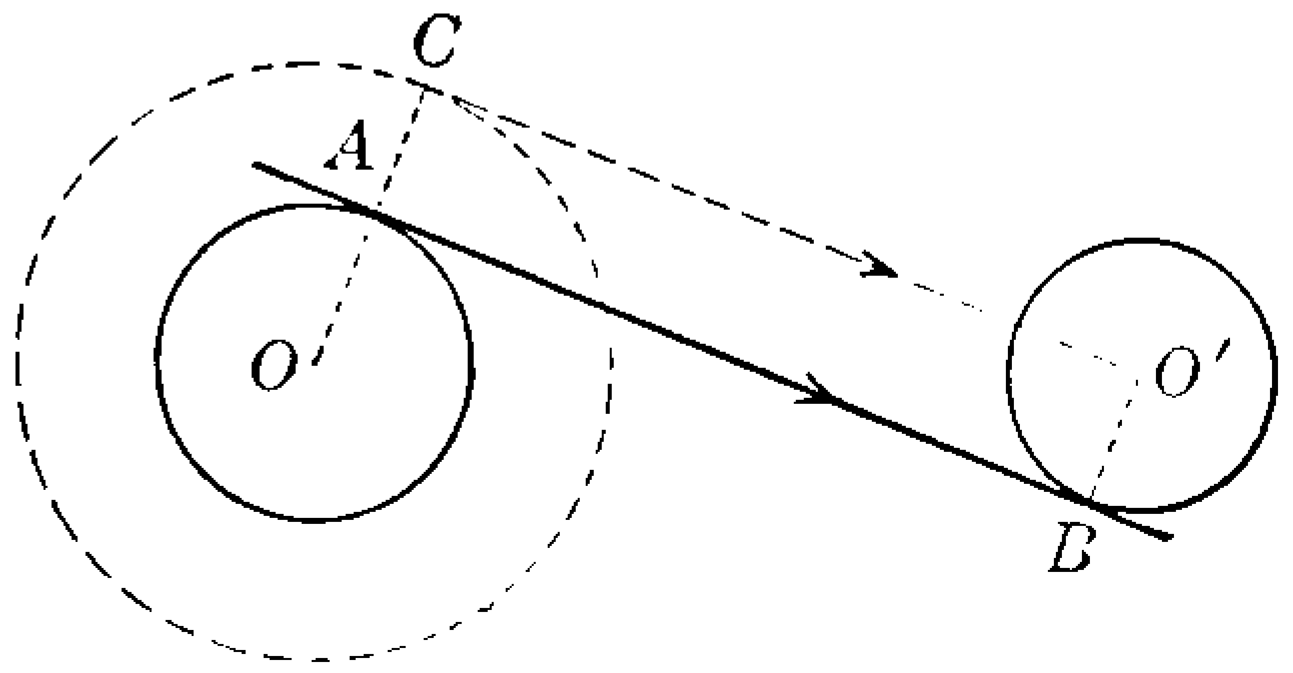
Analysis. Let AB be the required tangent. Draw radii OA and $O'B$; also draw $O'C$ parallel to AB.

$ABO'C$ is a rectangle and $AC = BO'$. If we draw an auxiliary circle with O as the center, and a radius equal to $OA - O'B$, $O'C$ will be tangent to it. This suggests that we begin by constructing an auxiliary circle, and then construct a tangent to it from O'.

Make the construction.

This figure and the next one are used in problems in which the length of a common tangent is computed. Finding the length of a belt connecting two pulleys illustrates such a problem.

86. *Problem.* To construct a common internal tangent to two circles.



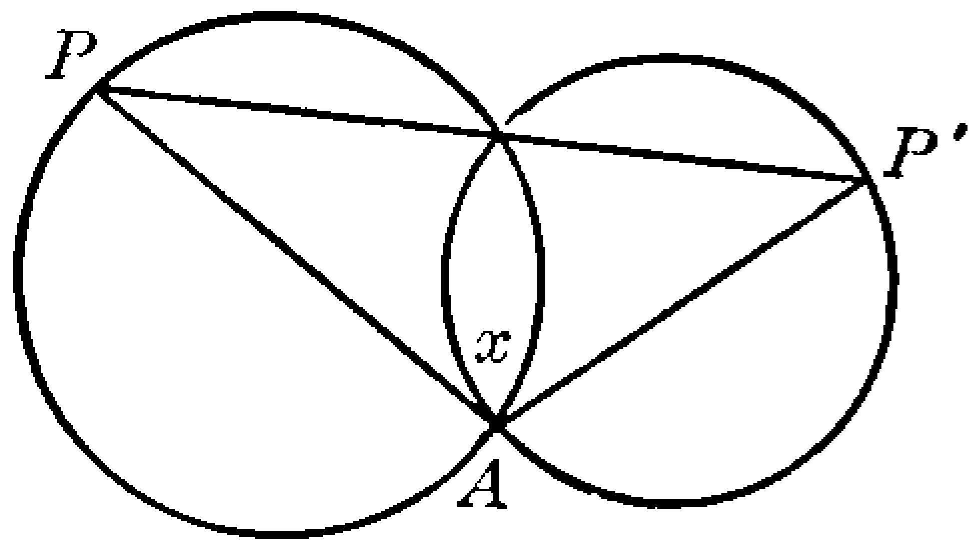
The analysis is similar to that in 85. This time the radius of the auxiliary circle is the *sum* of the radii of the given circles.

Make the construction.

HONOR WORK

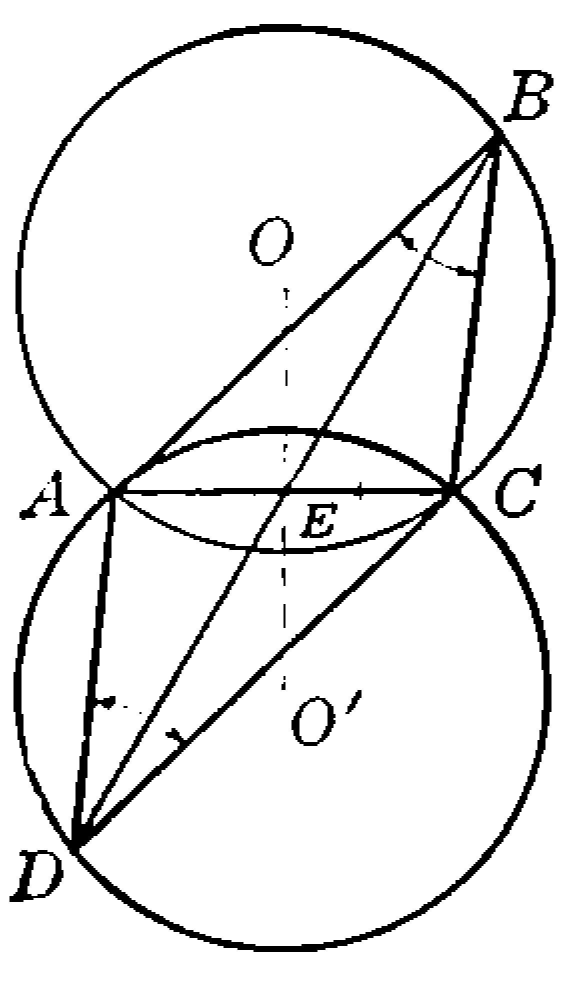
H1. Given: St. line PP' passes through one point of intersection of the two circles

Prove: $\angle A$ is constant as P (and P') moves on the circle



H2. (*a*) Prove: If a side and the angle opposite this side of one triangle are equal respectively to the corresponding parts of another triangle, then the circles circumscribed about these triangles are equal.

(*b*) Prove that Combination 12, § 174, is sufficient for a quadrilateral to be a parallelogram:



If one pair of opposite angles of a quadrilateral are equal, and the diagonal through the vertices of these angles bisects the other diagonal, then the figure is a parallelogram.

See diagram at the bottom of column 1.

Help: Circumscribe circles about the triangles ABC and ADC.

SUMMARY

256. Many theorems about circles were discovered and some were proved by moving lines and circles across one another. The most important of these theorems states that we may measure a pair of vertical angles by the algebraic sum of the arc degrees that they intercept. These angles may be thought of as having come from the center of the circle where (by the definition of the angular measure of an arc) the sum of the angles equals the sum of the arcs in degrees. This remains true everywhere because the translated sides cause one arc to lose whatever the other arc has gained. When the vertex of the angles is outside the circle, one arc has become negative.

From our study of the circle we now have the following main tools for proving propositions:

1. To prove line segments equal, show that:

(*a*) They are chords of a circle with equal arcs.

(*b*) They are the parts of a chord formed by a diameter perpendicular to the chord.

(*c*) They are chords equally distant from the center of the same circle; also the converse.

(*d*) They are common external or common internal tangents to two circles; or tangents to a circle from an external point.

2. To prove that arcs in the same circle are equal, show that:

(*a*) They have equal chords.

(*b*) They are included between the ends of a chord and the diameter perpendicular to the chord.

(*c*) They measure equal angles.

(*d*) They are included between parallel lines.

3. To prove angles equal, show that:
   (*a*) They are measured by equal arcs.
   (*b*) They are included between two tangents to a circle and a line to the center of the circle, all from the same point.

4. To prove that an angle is a right angle, show that:
   (*a*) It is inscribed in a semicircle.
   (*b*) It is formed by a tangent and a diameter to the point of contact.

5. To prove that two angles are supplementary, show that they are the opposite angles of an inscribed quadrilateral.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

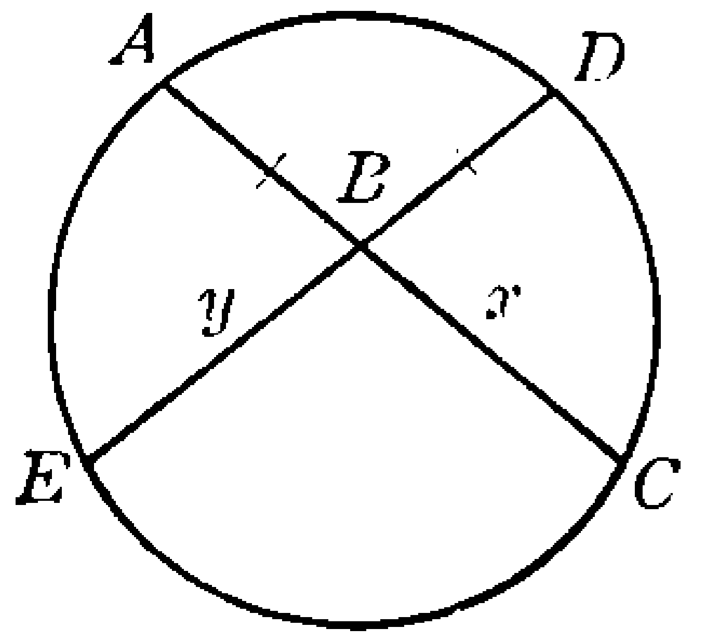
tangent (tan.)	§ 204
point of contact	§ 203
point of tangency	§ 203
secant	§ 205
chord	§ 206
diameter (diam.)	§ 207
central angle	§ 210
angular measure of an arc	§ 211
equal arcs	§ 213
semicircle	§ 215
major and minor arcs	§ 215
translation	§ 229
circumscribed circle	§ 234
inscribed circle	§ 245
circumscribed polygon	§ 246
inscribed polygon	§ 233
inscribed angle	§ 235
common internal and external tangents	§ 250
line of centers	§ 252
tangent circles	§ 254

REVIEW EXERCISES

1. (*a*) Prove: If the diagonals of an inscribed quadrilateral are equal, the figure is an isosceles trapezoid or a rectangle.
   (*b*) If two opposite sides of an inscribed quadrilateral are equal, and one angle is a right angle, the quadrilateral is a rectangle.

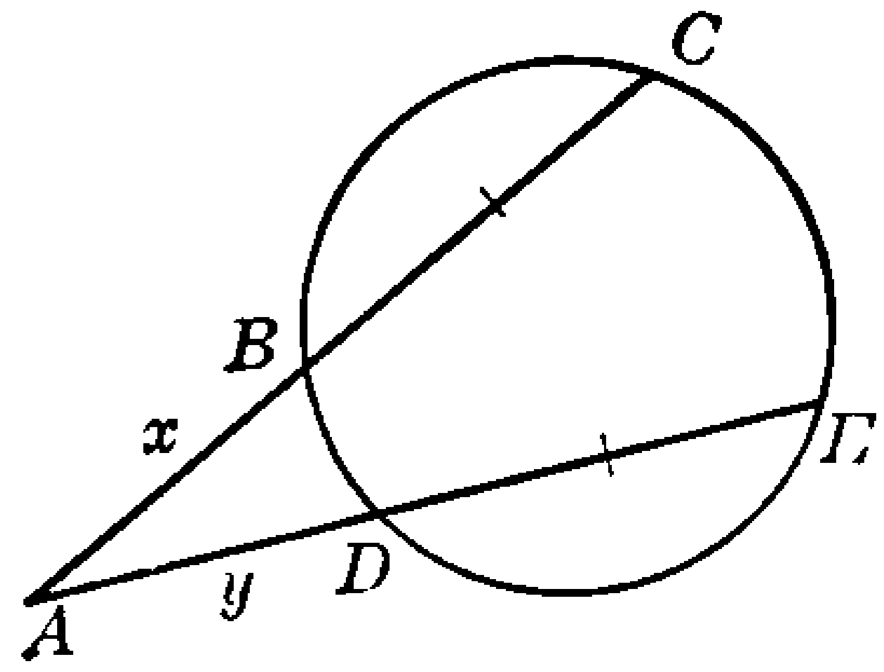
2. Prove: If a circle is inscribed in an isosceles triangle, the point of contact on the base is the midpoint of the base.

3.



Given:
$AB = BD$
Prove:
$CB = BE$

4. (*a*)

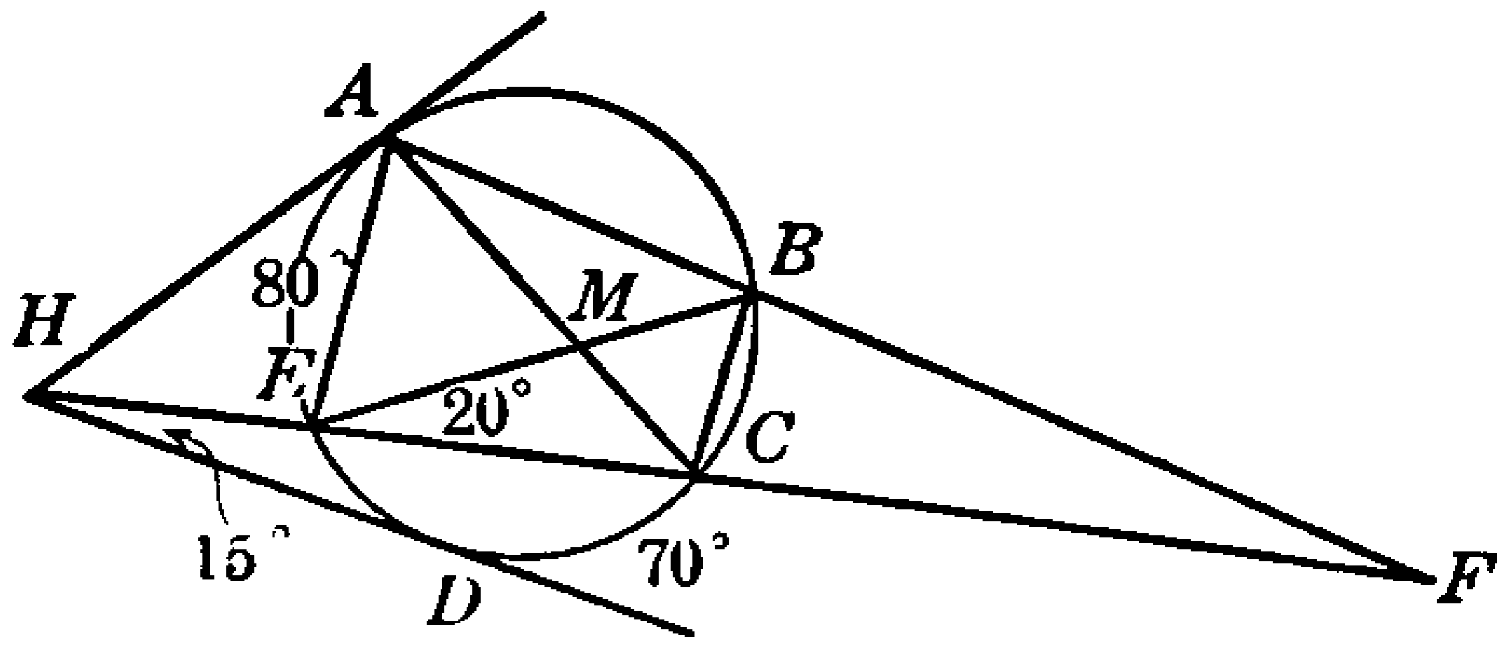


Given:
$BC = DE$
Prove:
$AB = AD$

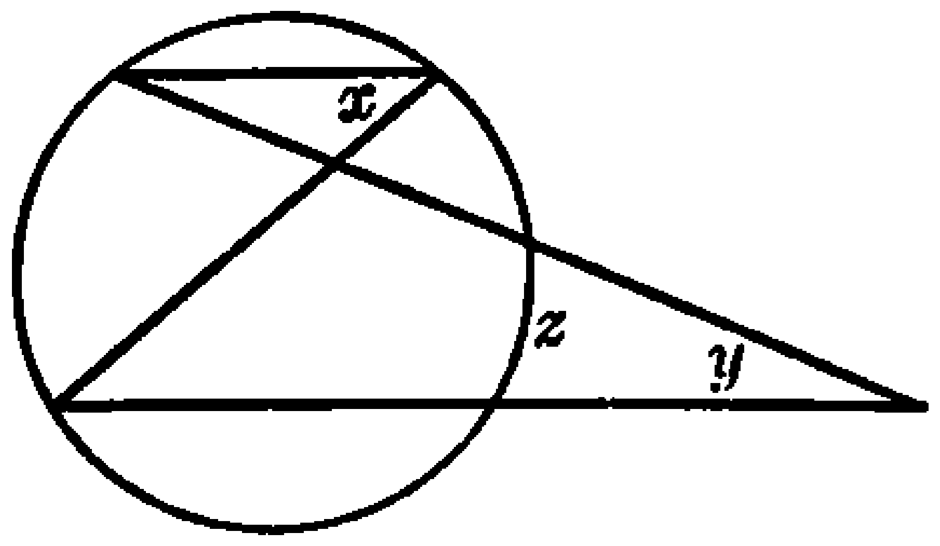
(*b*) State the proposition that results as the chords in (*a*) become zero.

5. Given: $\angle BEC = 20°$, $\angle EHD = 15°$
$\overset{\frown}{AE} = 80°$, $\overset{\frown}{CD} = 70°$
HA and HD are tans.
All the lines are straight.

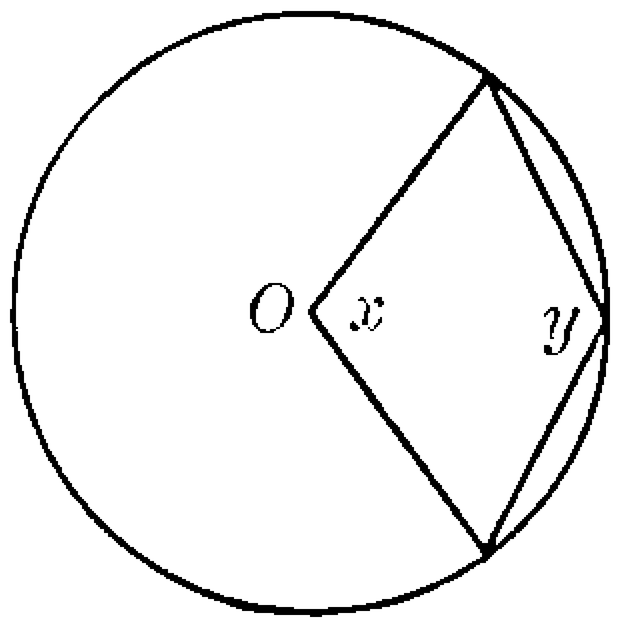
Find the number of degrees in the following: $\overset{\frown}{BC}$, $\angle BAC$, $\angle BMC$, $\angle F$, $\overset{\frown}{ED}$, $\angle EBC$, $\angle EAC$, $\angle AEB$, $\overset{\frown}{AB}$, $\angle AHC$.



6. (*a*) Write the relationship among the number of degrees in x, y, and z.



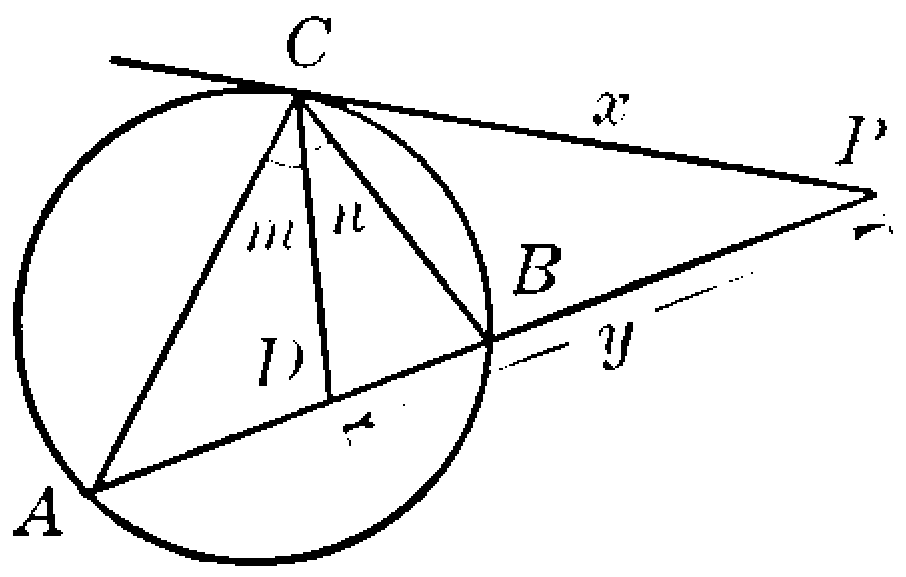
(*b*)



Write the relationship between x and y.

7. Prove: The midpoint of a chord is equidistant from the ends of any other chord parallel to the bisected chord.

8.



Given:
PC tan. to $\odot$ at C
$\angle m = \angle n$
Prove:
$\triangle PCD$ isos.
Help:
$\angle CDP = \angle A + \angle m$

9. A right triangle is inscribed in a circle. At the vertex of one acute angle, a tangent is drawn and the bisector of the other acute angle is prolonged to meet the tangent. Prove that the figure contains an isosceles triangle.

10. Draw a radius perpendicular to a diameter. From the end of the radius, draw a chord intersecting the diameter. At the other end of the chord, draw a tangent intersecting the diameter prolonged. Prove that the figure contains an isosceles triangle.

11. Prove the following:
   (*a*) If an inscribed polygon is equilateral, then it is equiangular.
   (*b*) If an inscribed n-gon is equiangular and n is odd, it is equilateral.
   (*c*) If a circle is divided into any number of equal parts, and the points of division are joined in succession, then a regular polygon (equilateral and equiangular) is formed.

12. Prove that if two chords are not diameters, they cannot bisect each other. (Use the method of *reductio ad absurdum.*)

SUPERIOR WORK

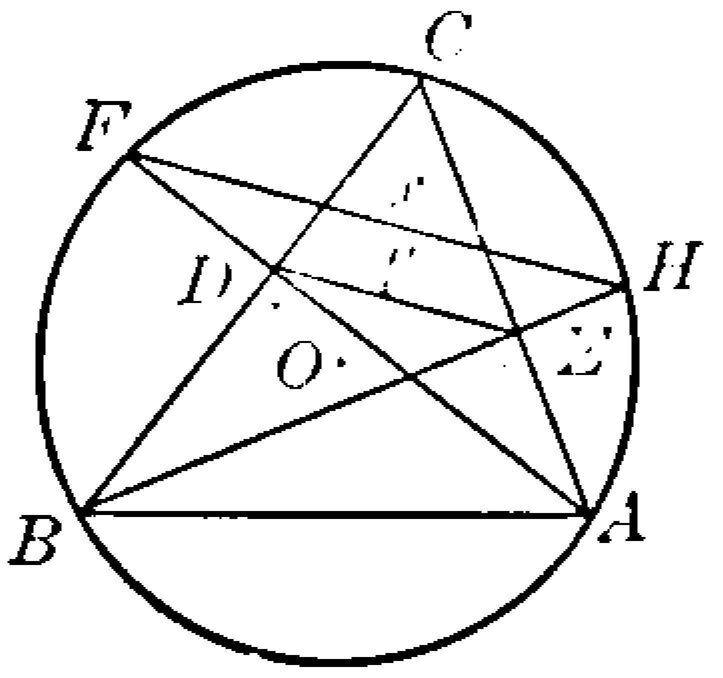
S1. (*a*) Two angles of a quadrilateral are 40° and 140°. What can you say about the quadrilateral?

(*b*) The sides of a quadrilateral taken in order are 4, 6, 7, and 5. What can you say about the quadrilateral?

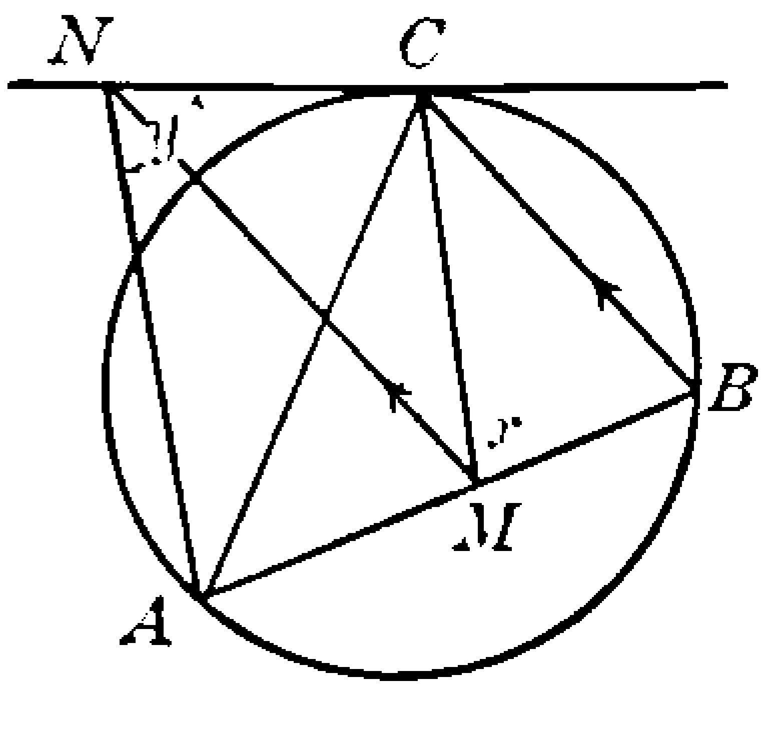
(*c*) State the following as two separate propositions and justify each:

If the sum of two opposite (sides, angles) of a quadrilateral equals the sum of the other two (sides, angles), then a circle can be (inscribed in, circumscribed about) the quadrilateral.

S2. Given: AD and BE, alts. of inscribed $\triangle ABC$, meet the $\odot O$ in F and H
Prove: $FH \parallel DE$
Help: Draw a circle through B, D, E, and A.

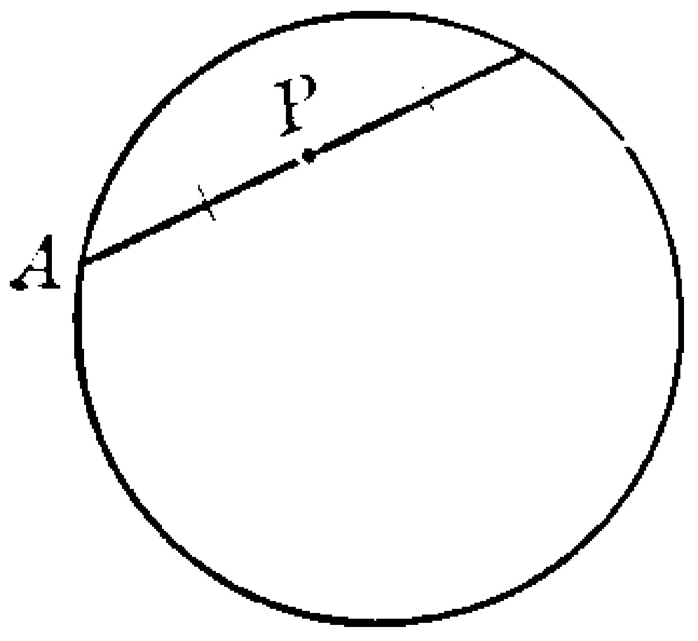


S3. Given: Inscribed $\triangle ABC$
NC tan. to the circle at C
MN BC
Prove: $\angle BMC = \angle ANC$
Help: Prove $\angle MAC = \angle MNC$, and draw a circle through M, A, N, and C.



S4. Prove: If the diagonals of an inscribed quadrilateral are perpendicular to each other, then the line through their intersection that is perpendicular to one side of the quadrilateral bisects the opposite side.

S5. (*a*) Prove: If the diameter of one circle is the radius of another circle, then a chord of the larger circle from the point of contact is bisected by the smaller circle.



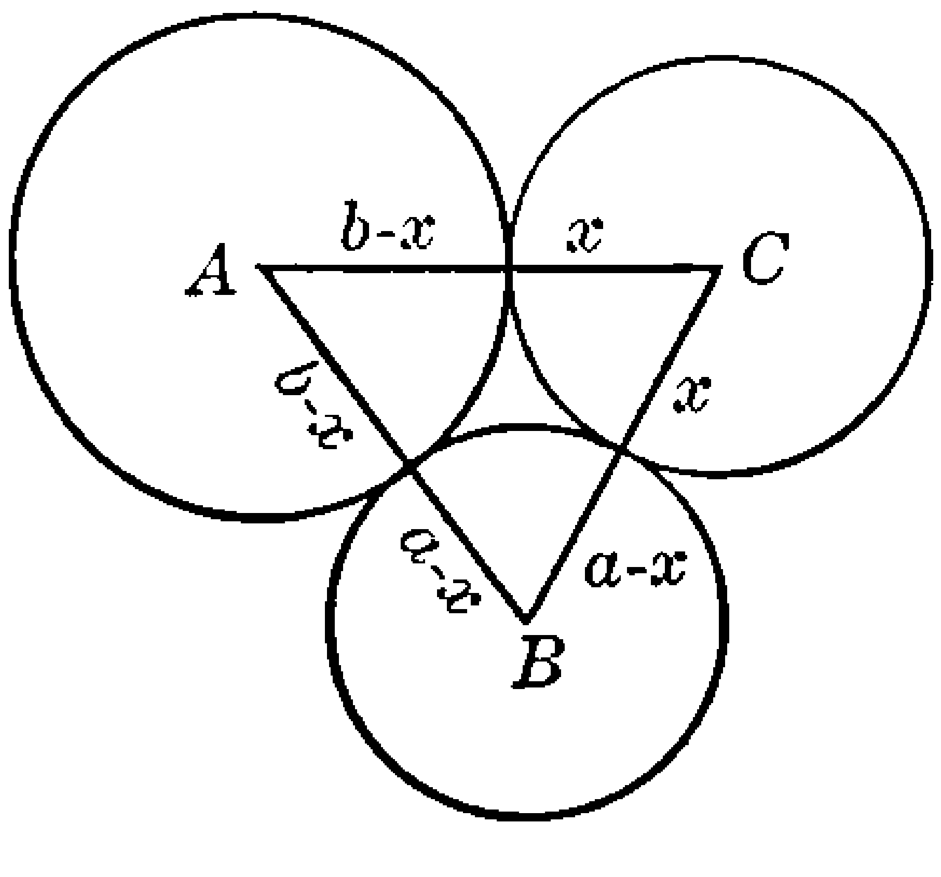
(*b*) In the figure, the chord rotates about the stationary point A on the circle. Find the locus of the midpoint P.

S6. In our study of arcs, it was not the actual lengths of these arcs that we considered, but the angular measures. This statement, of course, means that arcs having the same angular measure may differ greatly in length. We shall see later that this can happen only in circles of unequal radii.

The apparent motion of the sun is through an arc in the sky. The end of the hour hand moves through an arc on the dial of your watch. For the same interval of time, how do the angular measures of these arcs compare? When local standard time is 12 o'clock noon, in what direction is the sun? Now complete the following rule that will enable you to use a watch set to local standard time as a compass.

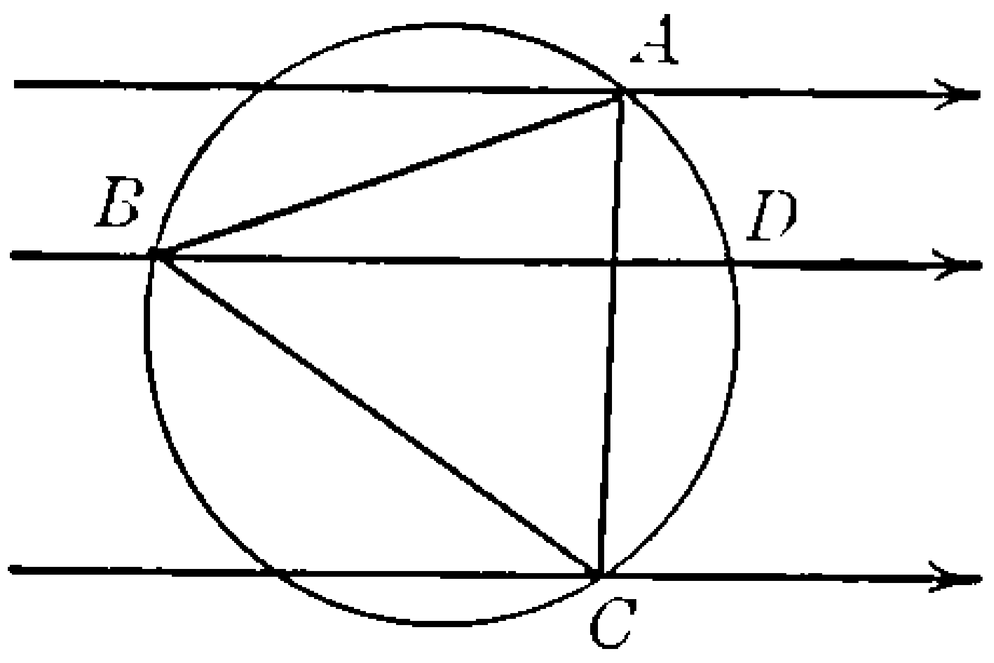
Point the hour hand to the sun. South is the point on your dial . . . .

S7. *Problem.* Given their centers, construct three circles tangent to one another.



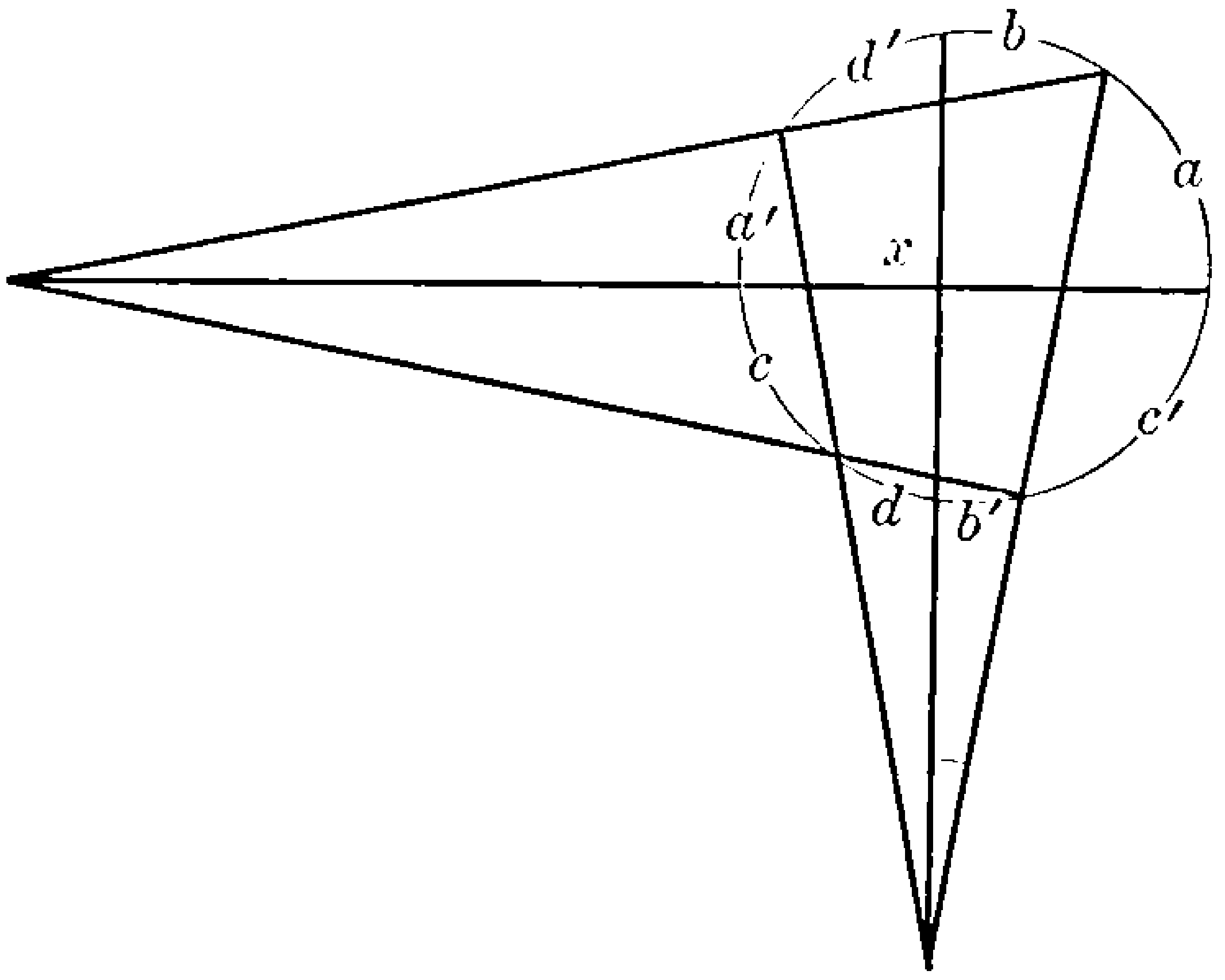
Analysis. Let A, B, and C be the given centers and a, b, and c the sides of $\triangle ABC$. $AB = c = (a-x)+(b-x)$; $x = \frac{1}{2}(a+b-c)$ Make the construction.

S8. *Problem.* Construct an equilateral triangle whose vertices are on three given parallel lines.



Analysis: Let ABC be the required equilateral triangle. Circumscribe a circle about it. How large is $\angle ADB$?

S9. Prove: The bisectors of the angles formed by prolonging the opposite sides of an inscribed quadrilateral are perpendicular to each other.



Help: Prove the following:

$$\overset{\frown}{a} + \overset{\frown}{b} + \overset{\frown}{c} + \overset{\frown}{d} = \overset{\frown}{a'} + \overset{\frown}{b'} + \overset{\frown}{c'} + \overset{\frown}{d'}$$

S10. Construct a triangle, given one side, the size of the opposite angle, and the length of the altitude to the given side. (A, a, h_a. See § S249.)

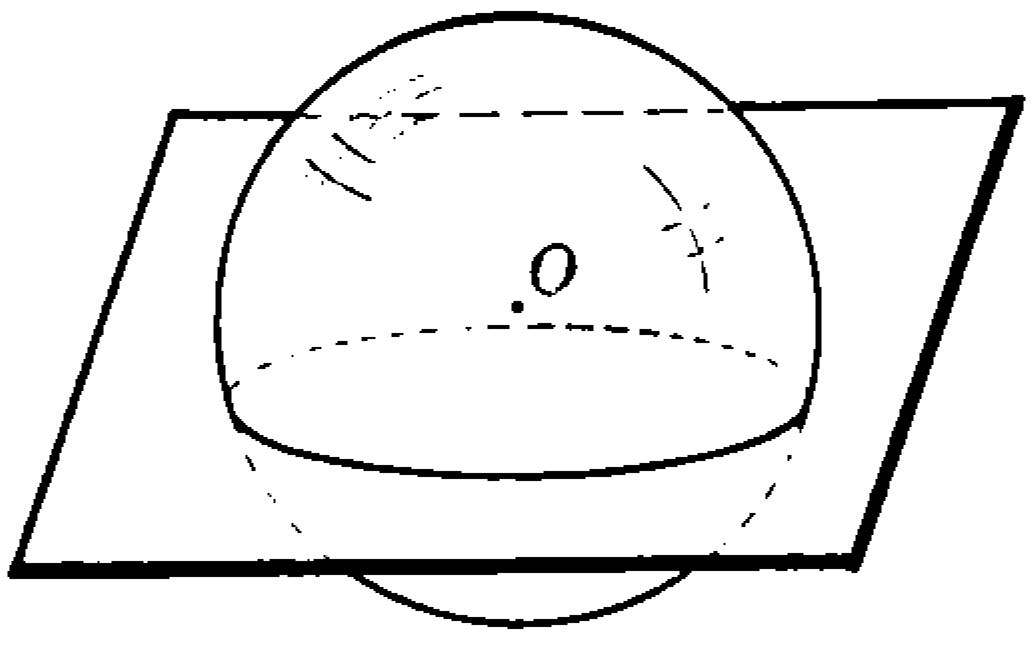
S11. Construct a triangle, given one side, the size of the opposite angle, and the length of the altitude to another side. (A, a, h_b.)

The Sphere

S12. The intersection of a plane with a sphere is called a circle of the sphere.

A *great circle* of a sphere is one whose plane passes through the center of the sphere.

A *small circle* of a sphere is one whose plane does not pass through the center of the sphere.



Parallel circles are circles whose planes are parallel.

(*a*) How can you pass a great circle through two points on the surface of a sphere?

(*b*) How many great circles can pass through two given points on a sphere if these points are the ends of a diameter of the sphere? if they are not?

(*c*) How many small circles can pass through two points on a sphere, if the two points are the ends of a diameter? if they are not?

(*d*) Through a given point, how many great circles can be drawn parallel to a great circle? to a small circle?

(*e*) In how many points do two great circles intersect? How are these points related? Why?

S13. The axis of the earth is the diameter about which the earth rotates. The ends of the axis are called the *poles.*

(*a*) How many great circles can pass through the north and south poles of the earth? These circles are called *meridians.*

(*b*) How many meridians can pass through a point, not a pole, on the earth's surface? Why?

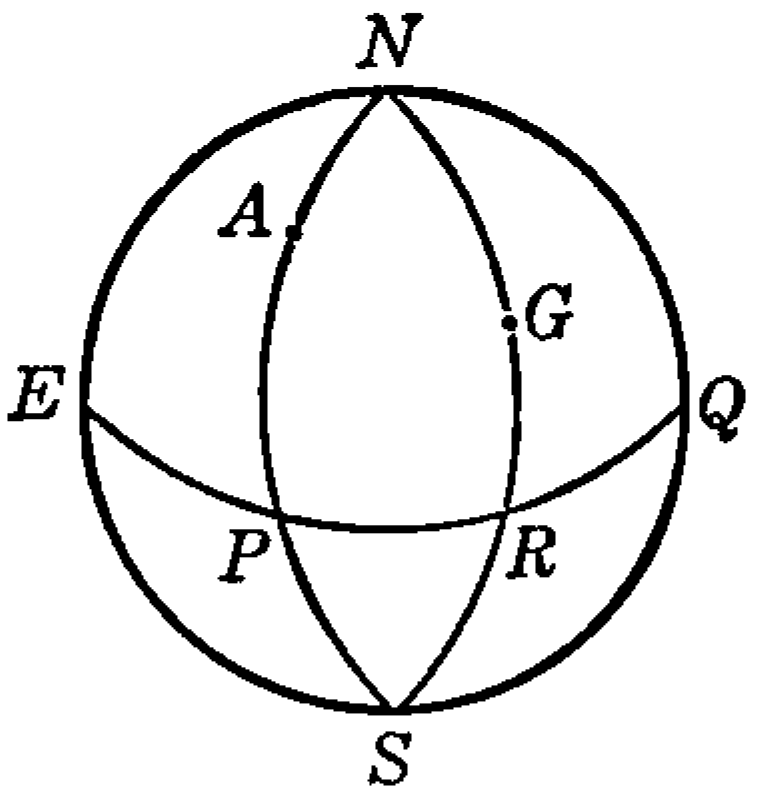
(*c*) Is the equator a great or a small circle? All the circles on the earth parallel to the equator are called circles of *latitude.* Are the circles of latitude great or small circles?

(*d*) How many circles of latitude can pass through a given point on the earth's surface? Why?

(*e*) If you travel directly east, what kind of circle are you following?

(*f*) What is the name of the circle that you follow when you go directly south?

S14. In the figure, N and S are the North and South poles of the earth, EQ is the equator, G is Greenwich, England, A is any other point on the earth, and circles $NGRS$ and $NAPS$ are meridians.



(*a*) The latitude of a point is the part of the meridian between that point and the equator. What is the latitude of G? of A? Are they north or south of the equator?

(*b*) The *longitude* of a point is the part of the equator between the meridian through that point and the meridian through Greenwich. What is the longitude of point A? Is it east or west of Greenwich?

(*c*) The longitudes of two points are 40° E. and 15° E. What is the difference in longitude (abbreviated DL_0) between these two points?

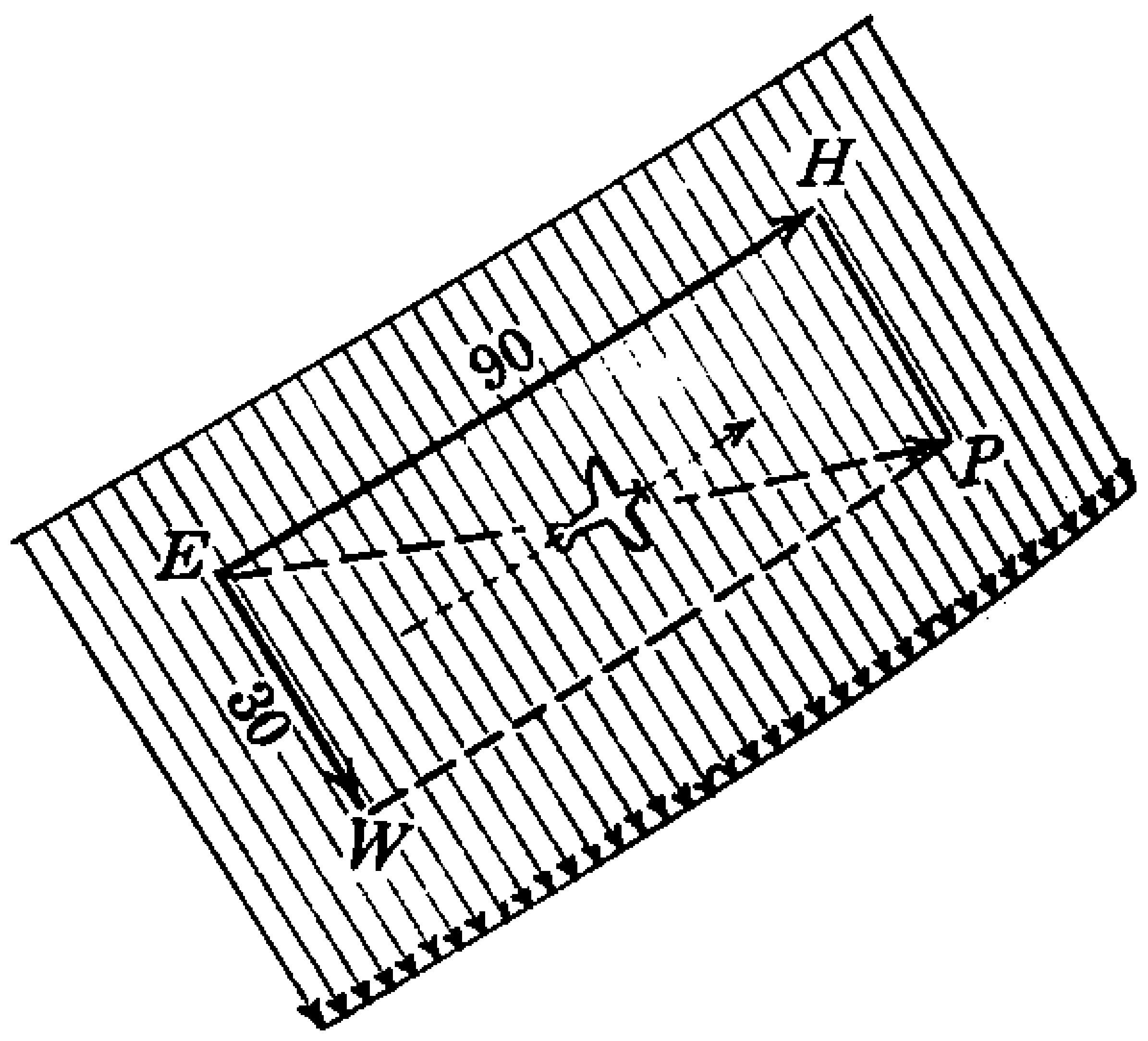
(*d*) Remembering that the earth rotates through 360° longitude in 24 hours, what is the difference in time between the two points in (*c*)?

(*e*) At the North Pole, which direction is east? north? south?

HONOR WORK

Air Navigation Problems

H257. *Vectors.* An aviator is headed in the direction EH at 90 miles per hour, and the wind is blowing in the direction EW at 30 miles per hour. The diagram



shows the flight in one hour. The line segments EH and EW represent the velocities. A velocity has both speed and direction.

It is clear from the diagram that the direction EH in which the plane is headed is not the course over the ground. By the time the plane has arrived at the point H in the air, the whole mass of air in which the plane is flying will have moved a distance of 30 miles in the direction of EW. Therefore, the plane will actually be at P with respect to the earth. In other words, the result of this flight is the same as if the plane had flown with a velocity of EP in still air.

We could arrive at the same conclusion by supposing that the plane were suspended in the air like a balloon for one hour, during which time the air would carry it to W. The plane would then fly for one hour in still air a distance of 90 miles, thus arriving at P. EP is called the *track* of the plane.

The track EP is the resultant of the velocity of the plane in still air and the wind velocity. It is seen from the diagram that the resultant is the diagonal of a parallelogram that is determined by EH and EW as sides. EH and EW are called *vectors* because they have both *magnitude* and *direction*. Force is another example of a vector, because the direction in which it acts is as important as its size. The following is a general theorem about adding vectors.

H258. THEOREM. The resultant of two vectors is the diagonal of the parallelogram determined by them.

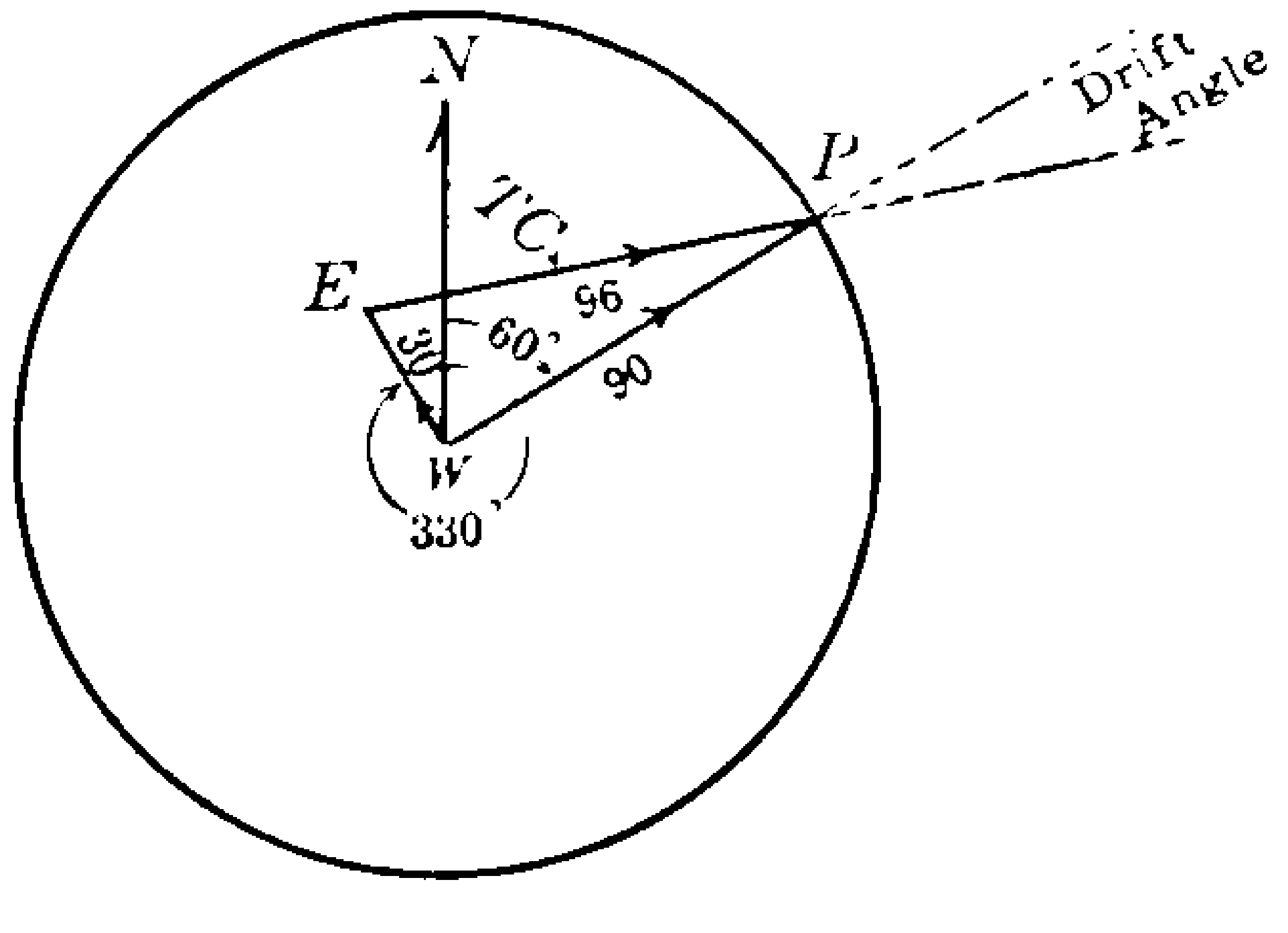
In practice, it is not necessary to work with the whole parallelogram. Since the parts of $\triangle WEP$ are duplicated in $\triangle HEP$, we need use only one of these triangles. $\triangle WEP$ is called the *wind triangle*. The letters W, E, and P are standard notation. W denotes *wind*; E, *earth*; and P, *plane*. The vector EP is read "plane with respect to earth." This vector contains the ground speed and the *true course*. The vector EW is read "wind with respect to earth." This means the wind force and the wind direction. The vector WP is read "plane with respect to wind (air)." This means the airspeed and the heading.

The angle between the heading WP and the true course EP is called the *drift angle*. It shows the number of degrees to the right or left that the plane is drifting away from the course in which it is headed.

In navigation problems we are given enough data to determine the $\angle WEP$.

Illustrative Example. A plane with an airspeed of 90 miles per hour is headed in the direction of 60°. The wind is 30 miles per hour *from* 330°. Find the true course (TC), the ground speed (GS), and the drift angle.

Since the navigator can always read the airspeed on his instrument panel, he begins by drawing the circle with W as the center and WP, the airspeed (to a suitable scale), as the radius. He then draws the heading



WP and the wind velocity *EW*. This enables him to complete the triangle *WEP*.

The required answers are found by measuring the diagram. The course is found by measuring the angle from the north line to *EP*. It is 76°. The length of *EP*, 96 miles per hour, is the ground speed. The drift angle is 16° to the *right*.

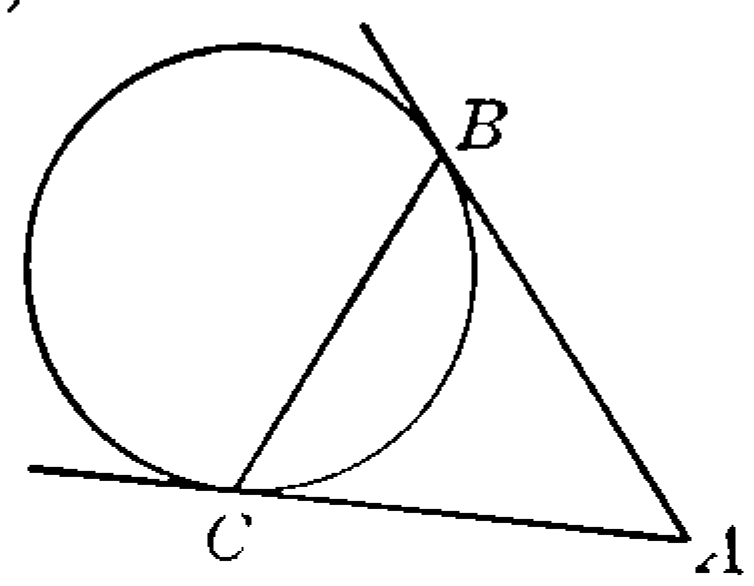
H1. The wind is 35 miles per hour from 260°. A flier is headed in the direction of 340° with an airspeed of 110 miles per hour. Find the true course, the ground speed, and the drift angle.

H2. The wind is 30 miles per hour from 200°. If the airspeed is 100 miles per hour, what heading must a plane have to make good a course of 70°? What is the ground speed?

TEST

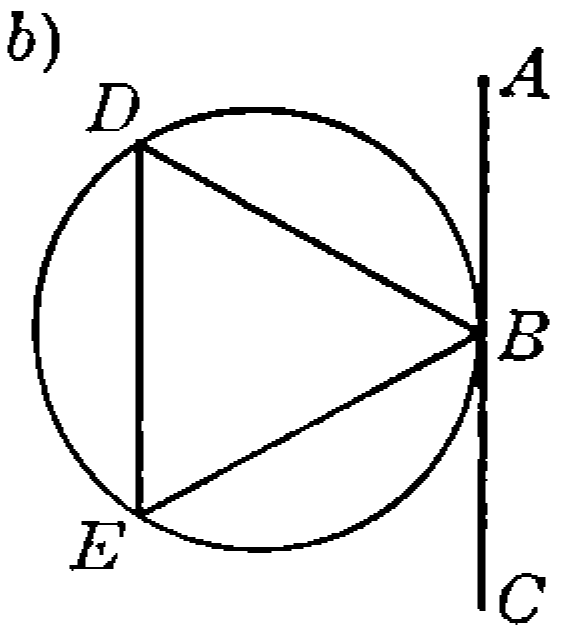
1. In the following diagrams, many angles *look* equal. In each diagram, name a pair of angles that you can *prove* equal from the given facts. Do not copy the diagrams.

(*a*)



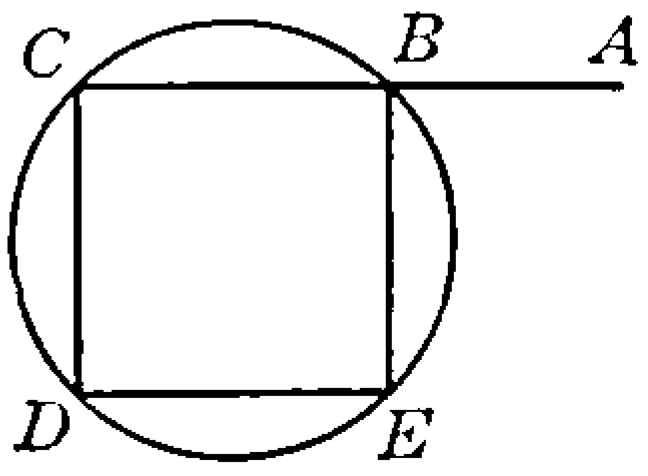
AB and *AC* are tangents.

(*b*)

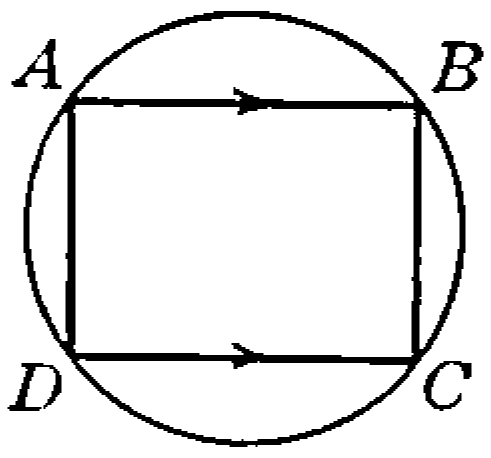


AC is a tangent.

(*c*)

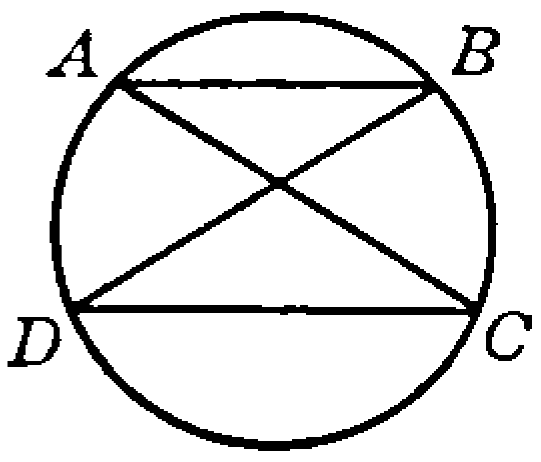


(*d*)



AB || *DC*

(*e*)

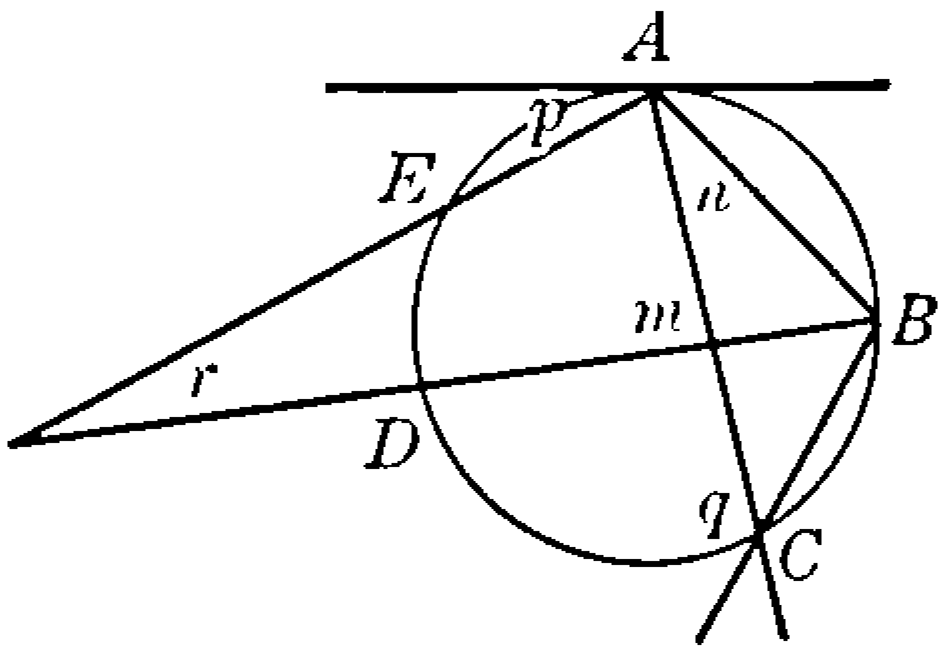


2. Prove: If a line is tangent to an arc at its midpoint, then it is parallel to the chord of the arc.

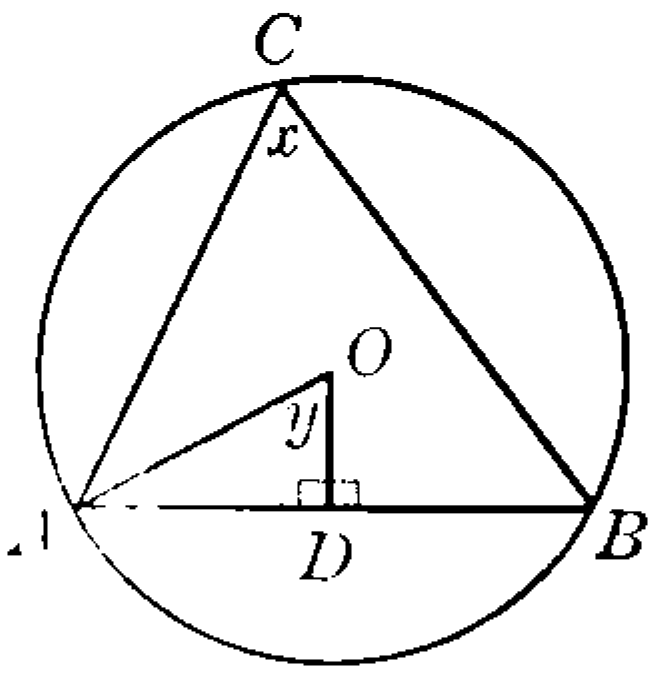
3. Prove: If two tangents to a circle from an external point form an angle of 60°, then the chord joining the points of contact equals either tangent.

4. Construct a rhombus and inscribe a circle in it.

5. Give the measures of the angles *m*, *n*, *p*, *q*, and *r* in terms of the arcs.

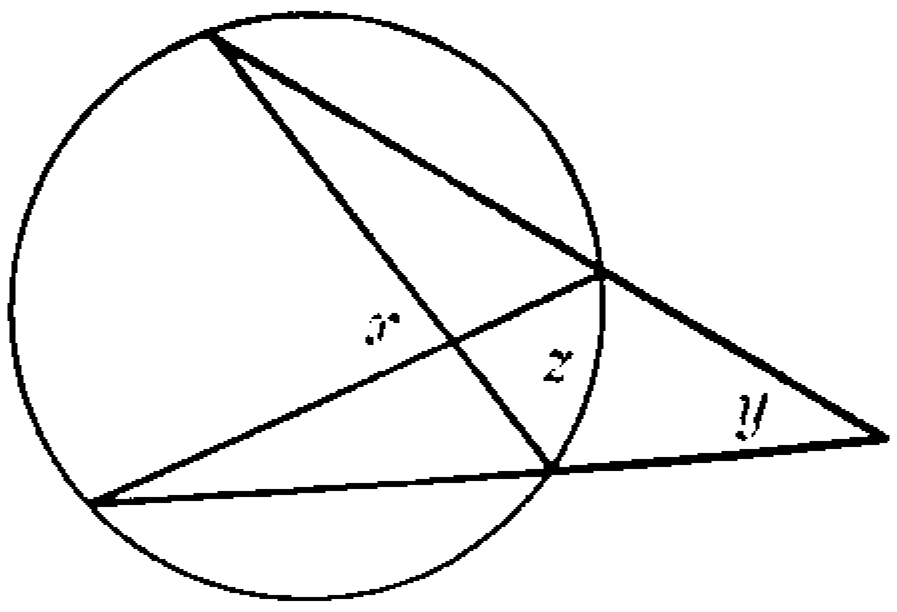


6.

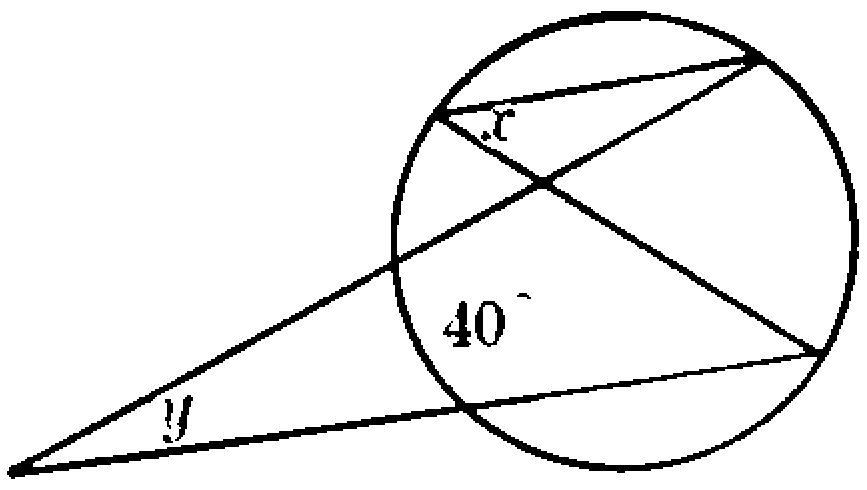


Given:
$OD \perp AB$
Prove:
$\angle x = \angle y$

7. (*a*) Find a relationship among the number of degrees in *x*, *y*, and *z*

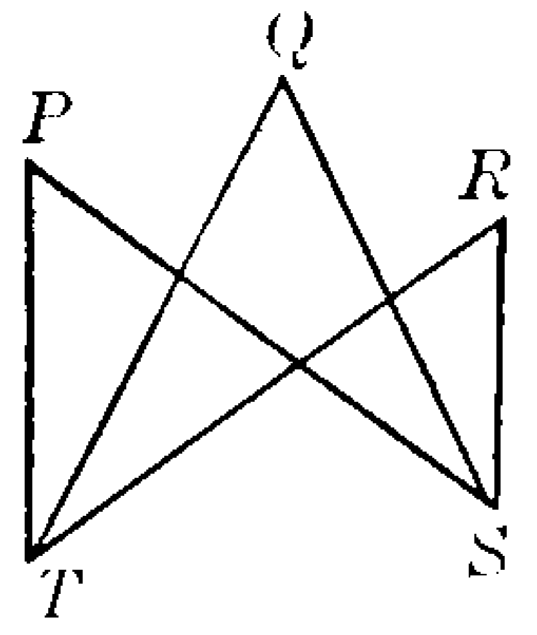


(*b*)



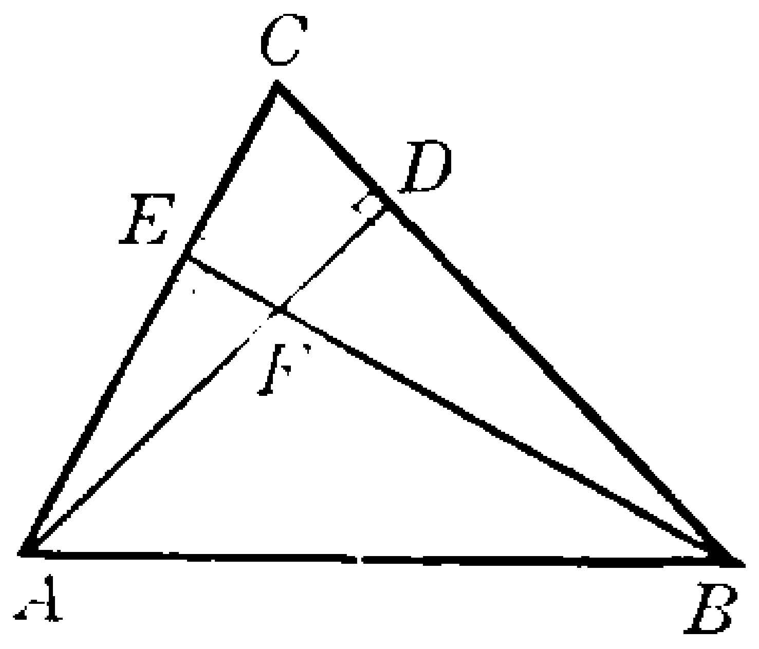
Find a relationship between ∠s *x* and *y*.

8. (*a*) What can you say about the points P, Q, R, S, and T?



$\angle P = \angle Q = \angle R$

(*b*) If a circle is drawn with AB as a diameter, what points lie on it?

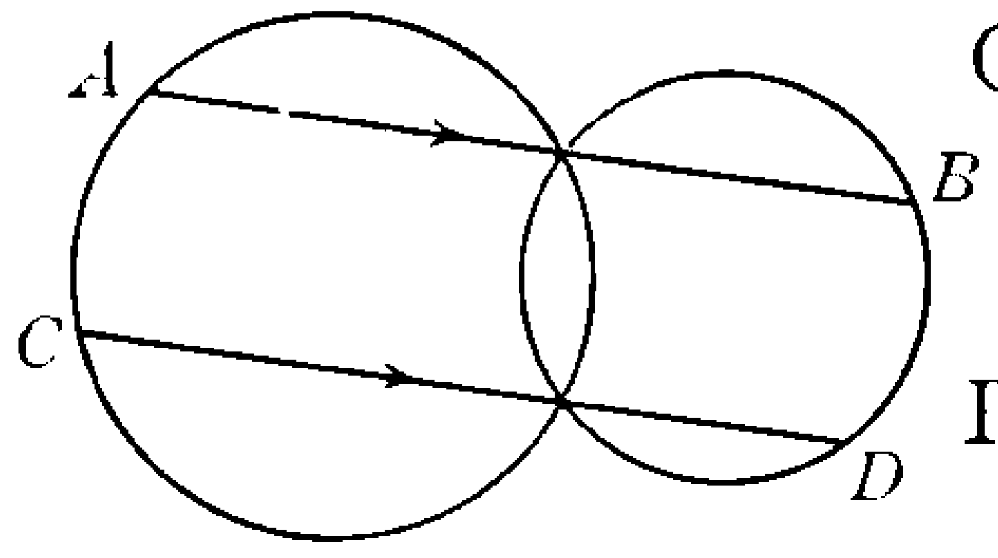


AD and BE altitudes

(*c*) If a circle is drawn with CF as a diameter, what points lie on it?

9. A diameter and a tangent at one end of the diameter are two sides of a triangle. At the point where the third side cuts the circle, another tangent is drawn. Prove that the first tangent is bisected. (*Hint:* Connect the two points of contact.)

10.



Given:
AB and CD
two ∥
double chords
Prove:
$AB = CD$

Help: Show that AB and CD are opposite sides of a ▱.

7 Dimensional Thinking

Area and Proportion

AREAS

Measuring Magnitudes

259. The measures needed to describe a magnitude are called its *dimensions*. How many dimensions are needed to measure a line? time? money? It is apparent that only one dimension is needed to measure each of these things. To measure most things, however, several dimensions are required. Thus, the extent of employment is often expressed in man-hours. This means the product of a number of men and the number of hours they worked.

Area of a Rectangle

When we measure the size of a surface, we say we are measuring its *area*. To express an area in numbers, we first choose a *unit of area*.

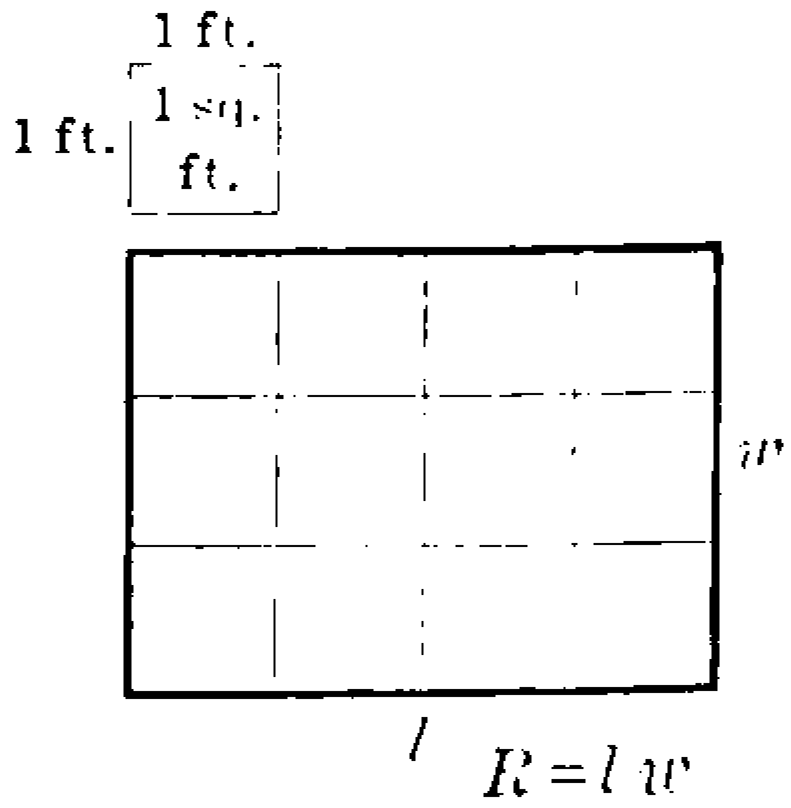
260. Definition. A unit of area is a square whose side is the unit of length.

Thus, if the unit of length is one foot, then the unit of area is a square whose side is one foot. This unit of area is called one square foot. If the unit of length is one inch, then the unit of area is a square whose side is one inch. This unit of area is called one square inch. Describe the terms "square yard" and "square mile."

261. Definition. The area of a surface is the number of unit squares that the surface contains.

262. Definition. Equal figures are figures that have equal areas.

If a rectangle is l units long and w units wide, then we can divide it into unit squares. The number of these squares will be lw. With any rectangle given, it may not be possible to measure each side in units of length. We shall see later that there are line segments that cannot be compared directly with the standard unit of length. We shall nevertheless assume that the number of unit squares that will fit into any rectangle is the number of linear units in its length times the number of linear units in its width.

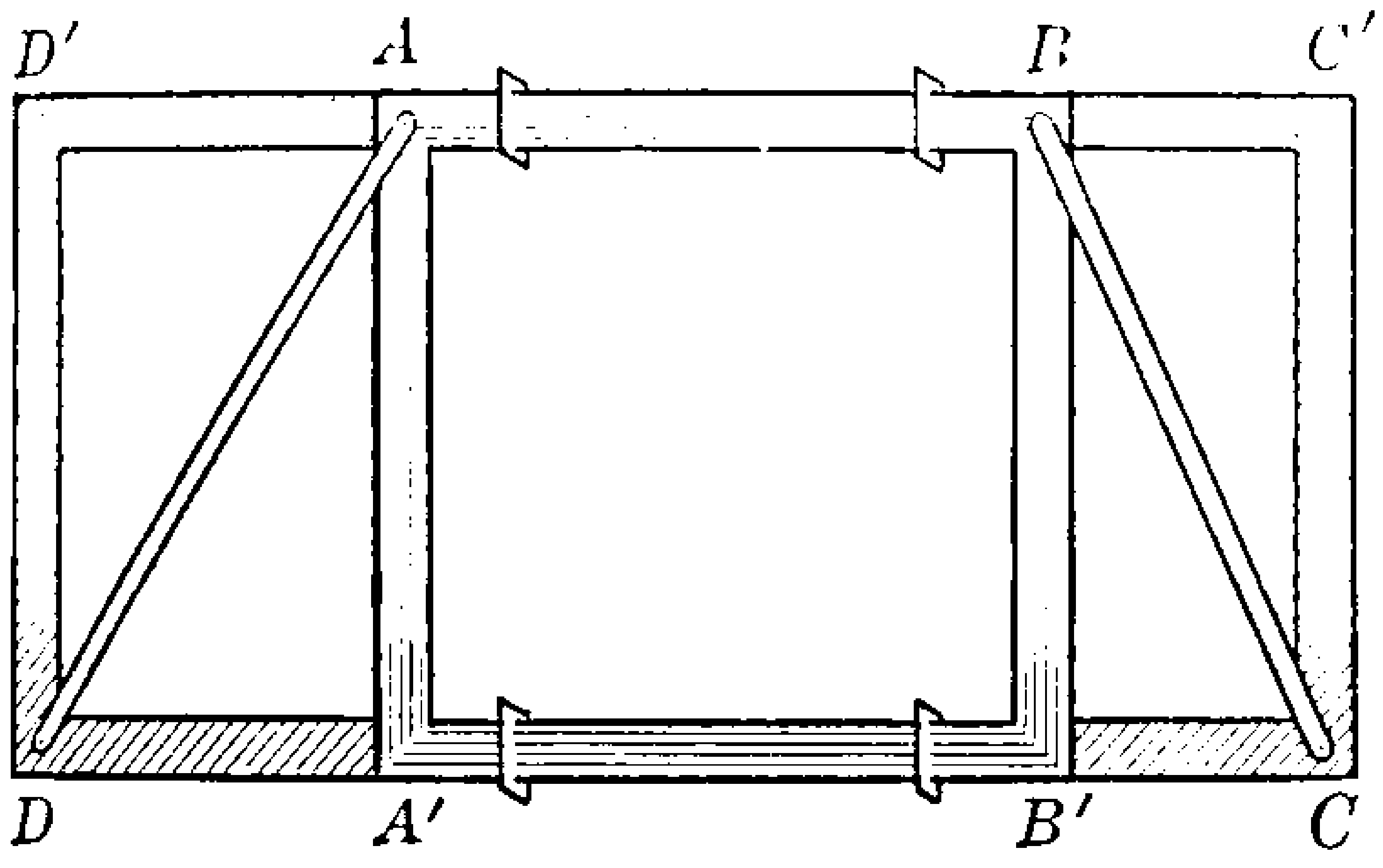


$R = lw$

263. Postulate. The area of a rectangle equals the product of its base and its altitude.

The Trapezoid

264. The model shown in the figure can easily be constructed of strips of wood. It consists of two rectangles $ABB'A'$ and $D'C'CD$, having equal altitudes. One rectangle is placed on the other, and the points A and D, as well as the points B and C, are connected by rubber bands. With the bases lined up, one rectangle slides over the other and the figure $ABCD$ remains a trapezoid.



In the figure shown, the trapezoid includes the smaller rectangle and one-half of what is left of the larger rectangle. In other words, the area of the trapezoid appears to be midway between the two rectangles, and therefore equal to their average. In symbols, this would mean that the area of a trapezoid whose height is h, and whose bases are b and b', is $\frac{1}{2}(bh + b'h)$. Factoring out h, we have $h\left(\frac{b + b'}{2}\right)$. This can be expressed as the height times the average of the bases. It can also be expressed as half the product of the altitude and the sum of the bases.

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265. THEOREM. The area of a trapezoid equals the average of its bases times its altitude.

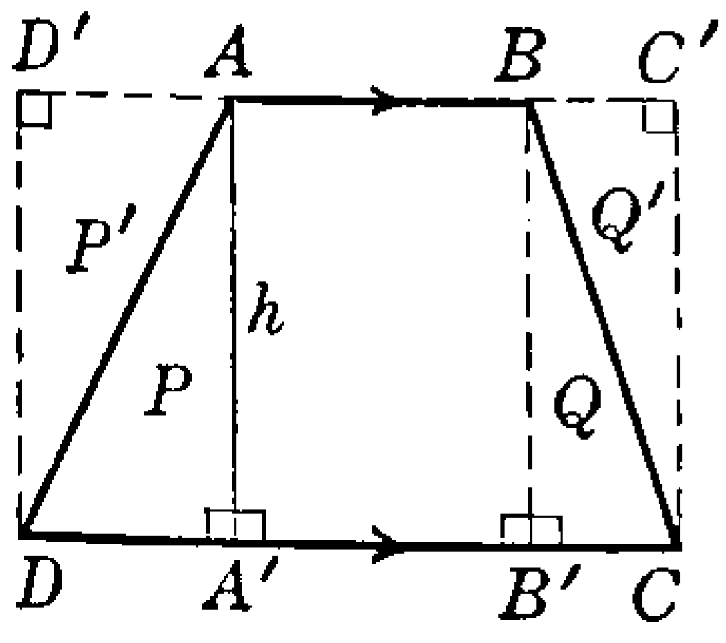


Fig. 1

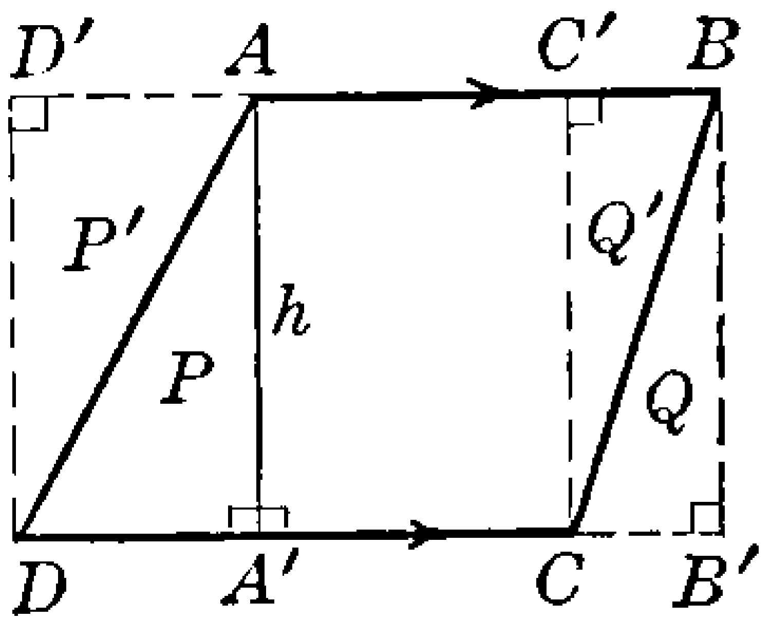


Fig. 2

Given: T = Area of trap. $ABCD$
h the altitude

Prove: $T = h\left(\frac{AB + DC}{2}\right)$

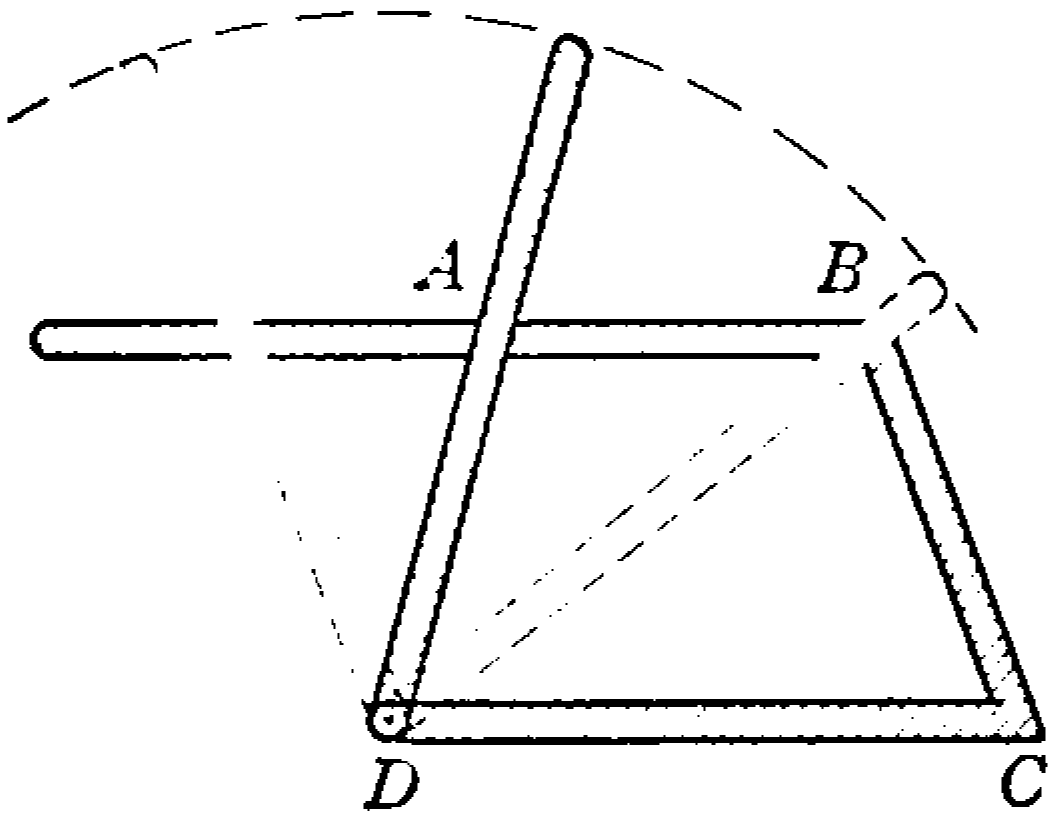
Cons.: Draw alts. from D, B, and C.

PROOF

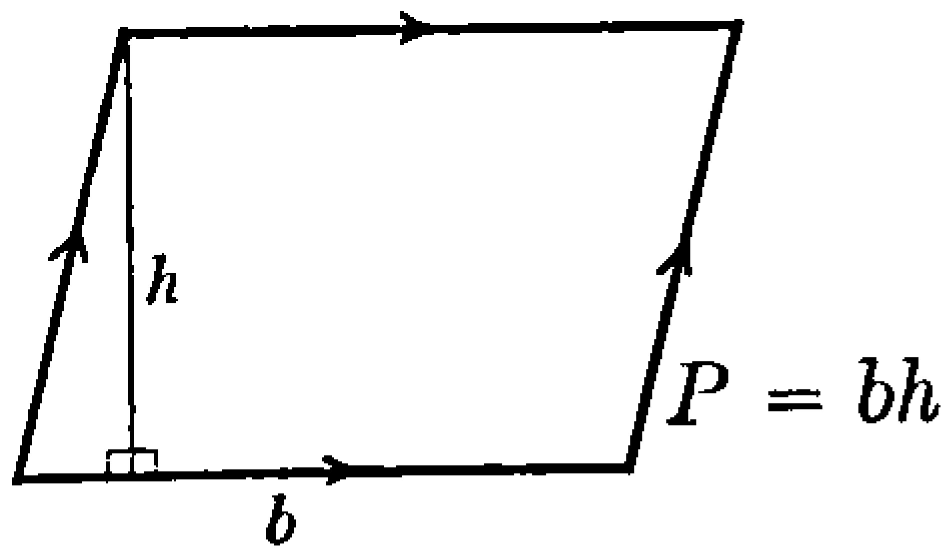
Statements	*Reasons*
1. $P \cong P'$ and $Q \cong Q'$	1. Diag. divides $\square$ in 2 $\cong$ $\triangle$s (§ 188).
2. $T = \square AB' + P + Q$ $\left(\text{In Fig. 2, } \square AB' + P - Q\right)$ $T = \square D'C - P' - Q'$ $\left(\text{In Fig. 2, } \square D'C - P' + Q'\right)$	2. Identity.
3. $2T = \square AB' + \square D'C$	3. Add. Ax.
4. But $\square AB' = AB \cdot h$ and $\square D'C = DC \cdot h$	4. Area $\square$ = base × alt. (§ 263).
5. $2T = AB \cdot h + DC \cdot h$ $= h(AB + DC)$	5. Subs. Ax.
6. $T = h\left(\frac{AB + DC}{2}\right)$	6. Div. Ax.

The Parallelogram and the Triangle

266. The model shown in the figure is a trapezoid constructed of wooden or cardboard strips. The leg AD is made free to pivot about the point D. If the leg AD is rotated so that it becomes parallel to BC, then the figure becomes a parallelogram. The area of the parallelogram would still be the average of the bases times the height. Since the bases in this case would be equal, their average would be the same as either base alone. To get the area of a parallelogram, we therefore consider it a special case of a trapezoid in which the two bases *have become* equal, so that their average is the same as either base alone.

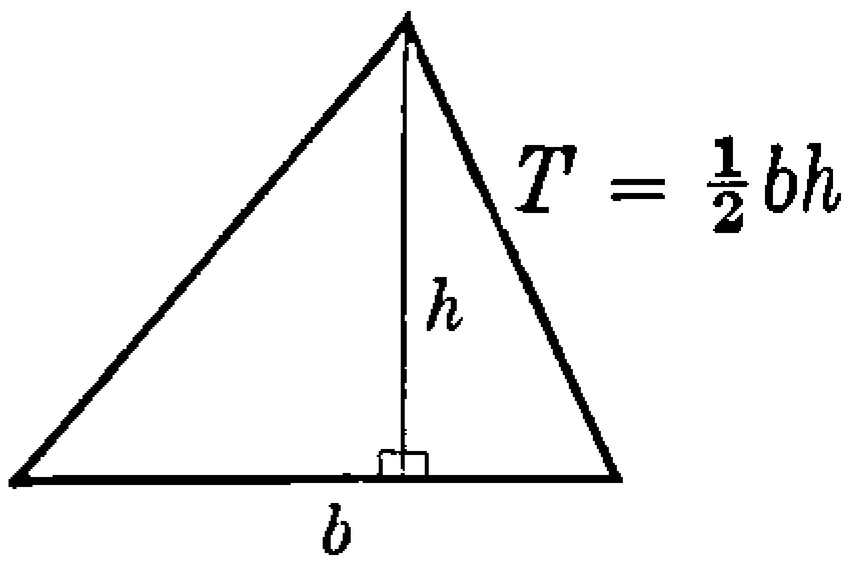


267. Corollary. The area of a parallelogram is the base times the height (altitude).



In our dynamic model, we may rotate the leg AD so that the base AB vanishes altogether. The figure is now a triangle. In other words, we may think of a triangle as a special case of a trapezoid in which one base *has become* zero. Its area would then be simply one-half the remaining base times the altitude to that base.

268. Corollary. The area of a triangle equals one-half a base times the altitude to that base.



EXERCISES

1. (*a*) Find the area of a rectangle whose dimensions are 27 feet by 24 feet.
   (*b*) The side of a square is 1 foot. Find its area in square inches.
   (*c*) Find the area of a rectangle 3 feet by 8 inches.
   (*d*) Find the area of a rectangle 3 feet 6 inches by 4 feet 8 inches.
   (*e*) The area of a rectangle is 48 square feet and the base is 12 feet. Find the altitude.
   (*f*) The area of a rectangle is 6 square feet and the altitude is 8 inches. Find the base.
2. (*a*) On graph paper, draw a trapezoid making the bases 8 units and 12 units, and the altitude 6 units. By counting the number of squares in the trapezoid, and estimating the parts of the squares partially enclosed to the nearest half-square, find the approximate area of the trapezoid. Check this by finding the area according to the theorem.
   (*b*) The bases of a trapezoid are 24 feet and 18 feet. The altitude is 7 feet 4 inches. Find the area of the trapezoid.
   (*c*) The area of a trapezoid is 148 square inches. Its bases are 40 inches and 34 inches. Find the altitude.
   (*d*) The area of a trapezoid is 92 square feet. The altitude is 8 feet and one base is 14 feet. Find the other base.
   (*e*) A trapezoid has $\frac{2}{3}$ the area of a rectangle. The altitude of the trapezoid equals the altitude of the rectangle; and one base of the trape-

zoid, which is 12, is also the base of the rectangle. Find the other base of the trapezoid.

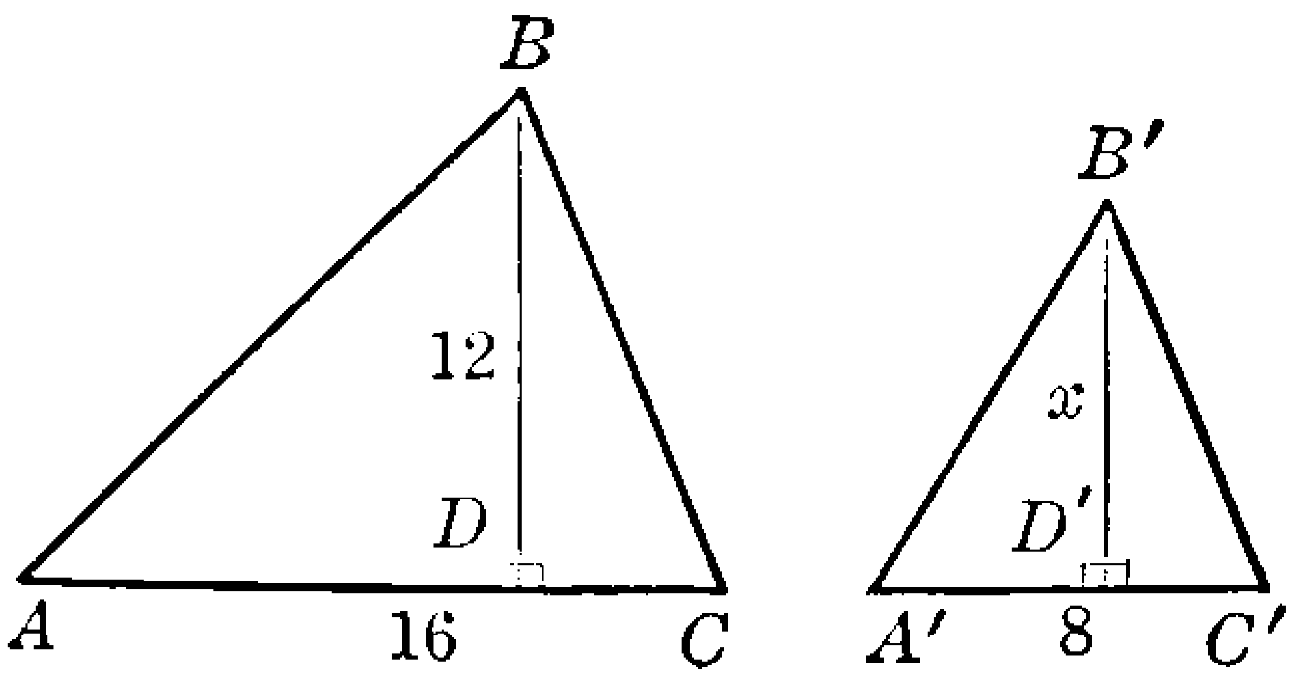
3. (*a*) Find the area of a triangle in which one side is 12 feet and the altitude to that side is 7 feet.

(*b*) In a triangle, an altitude 5 inches long is drawn to a side that is 3 feet long. Find the area of the triangle.

(*c*) The area of a triangle is 22 square inches and an altitude is 8 inches. Find the side to which the altitude is drawn.

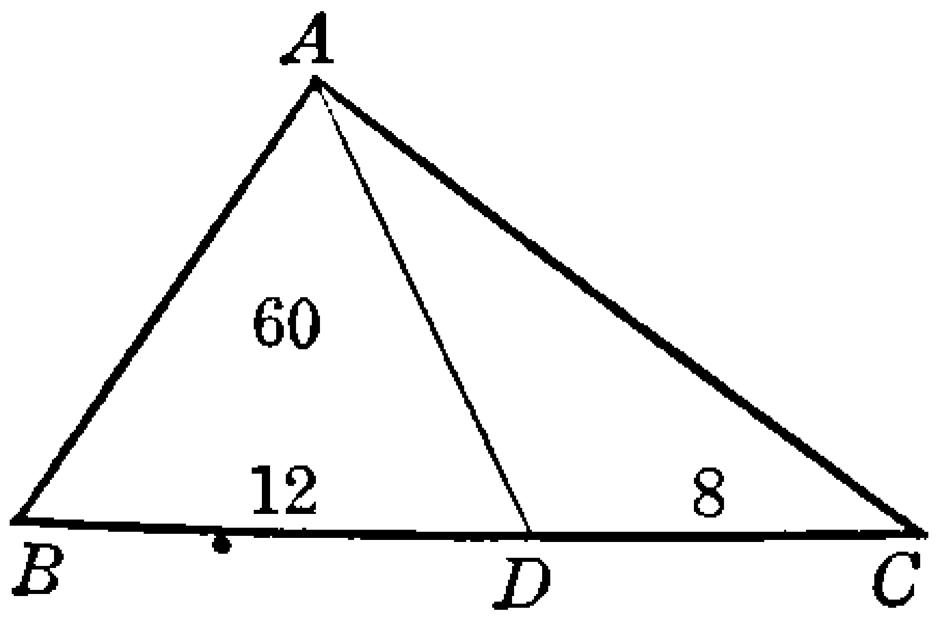
(*d*) The area of a triangle is 3 square feet and one side is 1 foot 4 inches. Find the altitude to this side.

4. (*a*) In the figure, $\triangle ABC = 3 \cdot \triangle A'B'C'$, $AC = 16$, $BD = 12$, and $A'C' = 8$. Find $B'D'$.



(*b*) Two sides of a triangle are 16 units and 24 units. If the altitude to the side 16 is 6 units, find the altitude to the side 24.

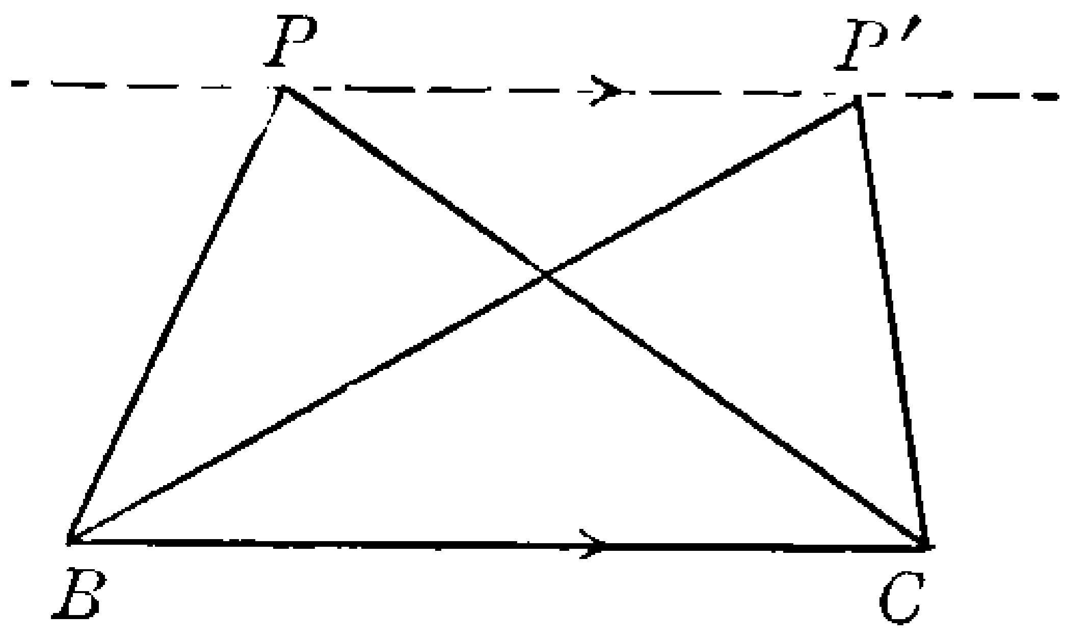
5. (*a*) In the figure, draw the altitudes of the triangles ABC, ABD, and ADC from the vertex A. How do they compare?



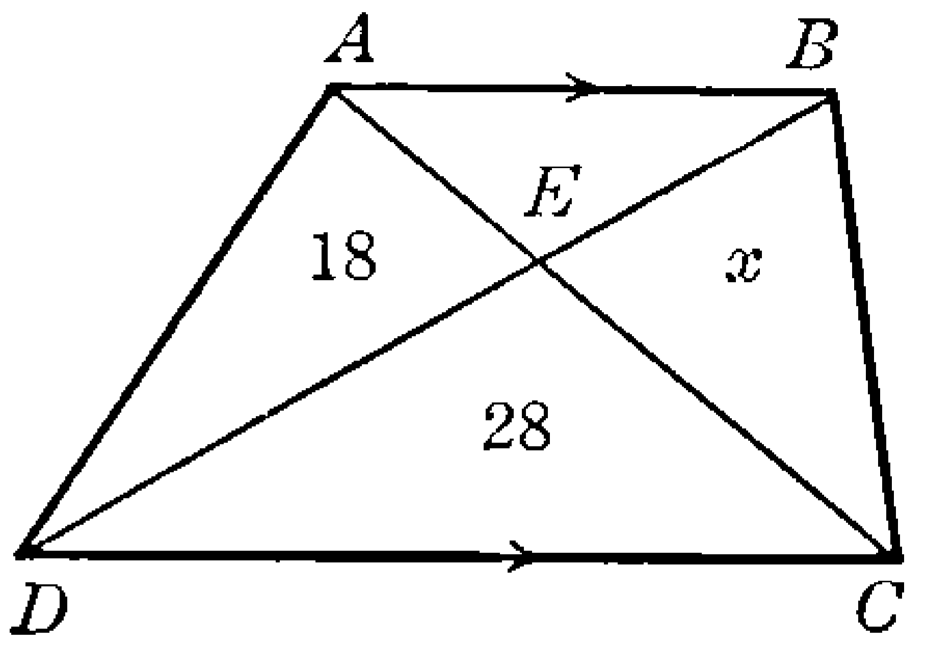
(*b*) In the figure above, the area of $\triangle ABD = 60$, $BD = 12$, and $DC = 8$. Find the area of $\triangle ADC$.

(*c*) In the same figure, let AD become the median of $\triangle ABC$. Find the areas of $\triangle ABD$ and $\triangle ADC$.

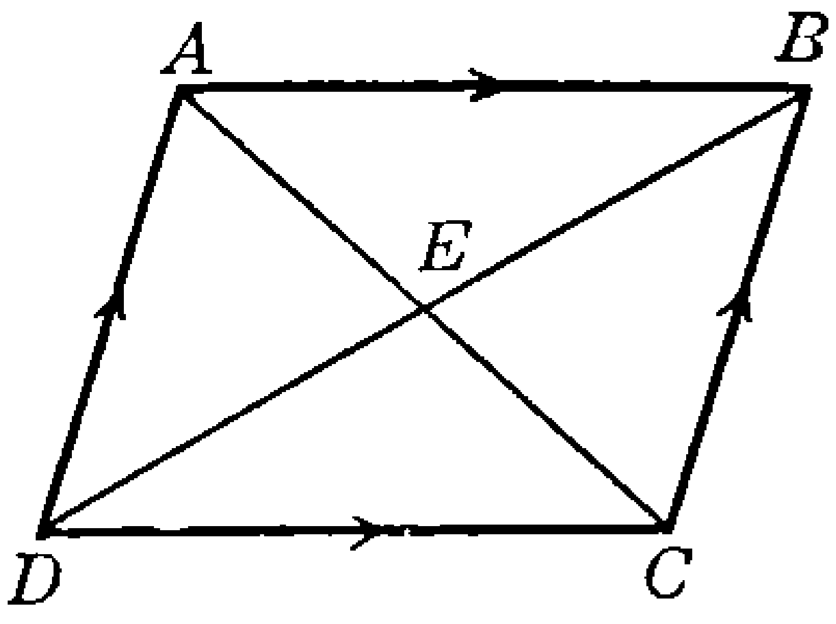
6. (*a*) In the figure below, PP' is parallel to BC. Draw the altitudes to BC of the triangles BPC and $BP'C$. How do they compare? Why? How are the triangles BPC and $BP'C$ related in area?



(*b*) In the figure, $ABCD$ is a trapezoid. $\triangle DEC = 28$ and $\triangle AED = 18$. Find the area of $\triangle BEC$.



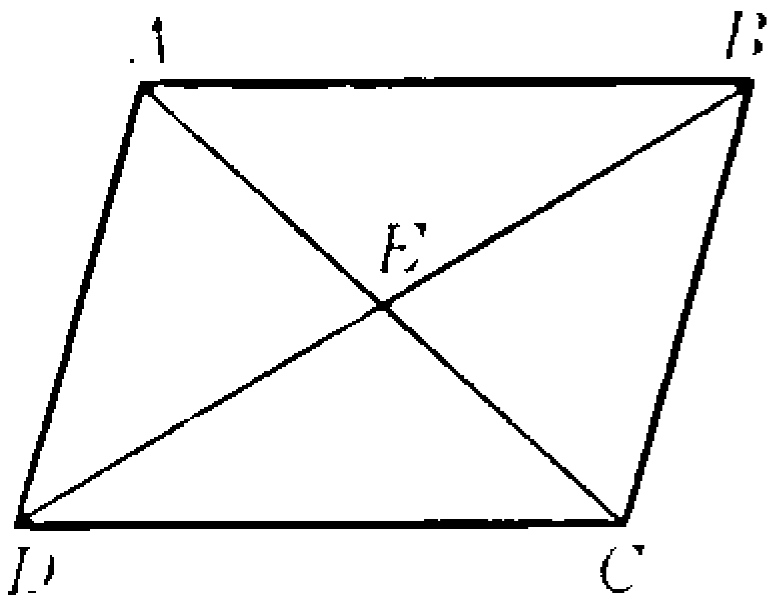
7. (*a*) In the figure below, $ABCD$ is a parallelogram. What triangles having DC as a common side are equal (have equal areas)? What triangles having BE in common are equal? What triangles having AE in common are equal?



(*b*) In the same figure, the area of $\square ABCD$ is 100. What is the area of $\triangle ABC$? of $\triangle ABE$?

(*c*) Is the parallelogram above divided into four equal triangles? Under what condition would the parallelogram be divided into four congruent triangles by its diagonals?

8. (a) In the figure below, $ABCD$ is a quadrilateral. The triangles AEB and AED are equal. How does BE compare with DE? How does $\triangle BEC$ compare with $\triangle DEC$?



(b) $\triangle AEB = \triangle BEC$. How does AE compare with EC?

(c) $\triangle AED = \triangle AEB = \triangle BEC$. What kind of figure is quadrilateral $ABCD$? Why?

SUPERIOR WORK

S1. A survey covers a hexagon $ABCDEF$

AB = 200 feet Bearing: Directly east

BC = 230 feet Bearing: N 30° E. (page 37)

CD = 180 feet Bearing: N 40° W.

DE = 270 feet Bearing: S 80° W.

EF = 150 feet Bearing: S 10° W.

(a) Using your protractor and a scale of 100 feet to the inch, draw the hexagon.

(b) Draw the diagonal CF and drop perpendiculars from the vertices to CF. This will divide the figure into right triangles and trapezoids. By measuring their bases and altitudes, find the area of the hexagon.

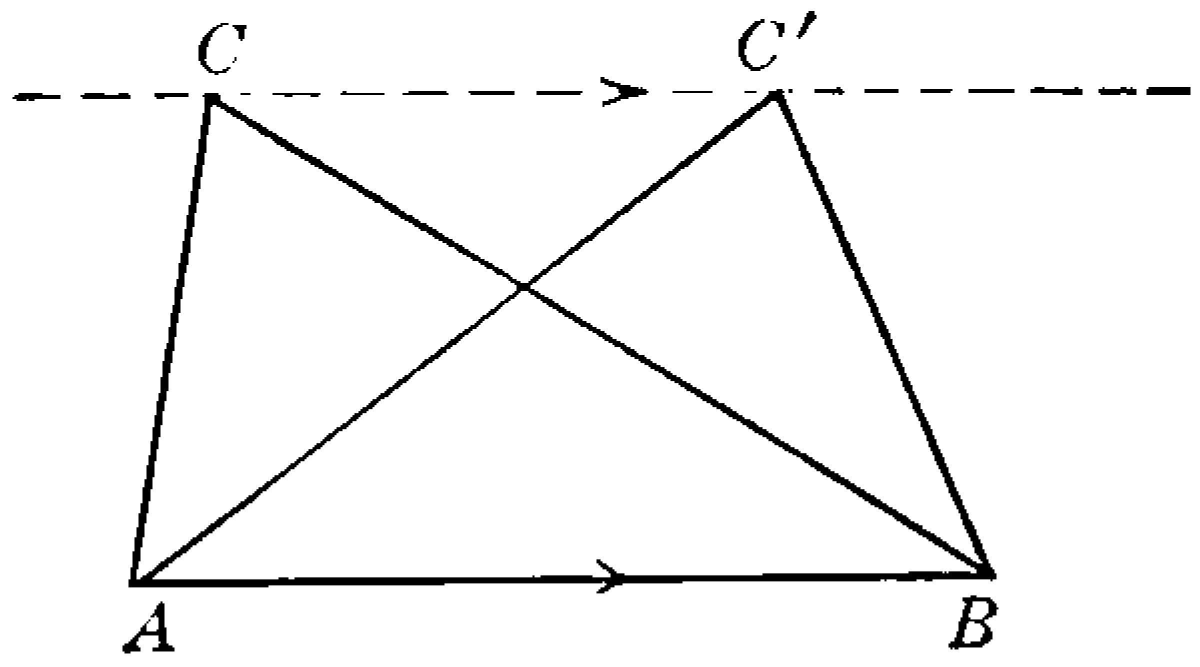
Useful Corollaries

269. Corollary 1. If two triangles have equal bases (or the same base) and equal altitudes (or the same altitude), they are equal.

270. Corollary 2. If two equal triangles have equal bases (or the same base), they have equal altitudes.

271. Corollary 3. If two equal triangles have equal altitudes (or the same altitude), they have equal bases.

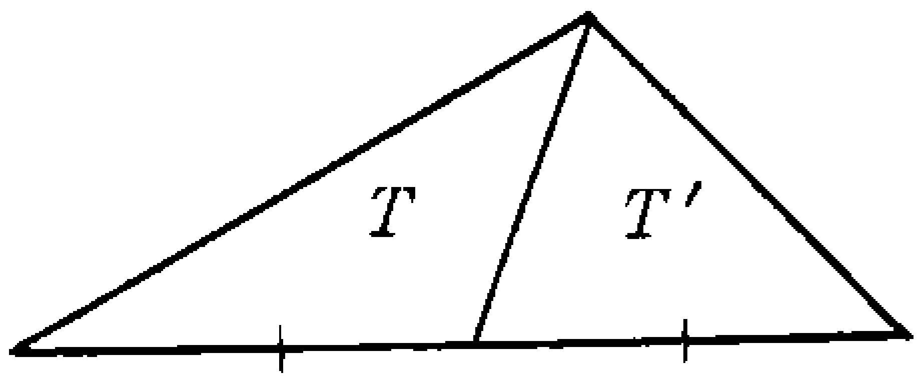
272. Corollary 4. If two triangles have the same base, and the vertices opposite the common base are on a line parallel to the base, the triangles are equal.



$\triangle ABC = \triangle ABC'$

Since the altitudes are perpendiculars included between parallels, the triangles are equal by Corollary 1.

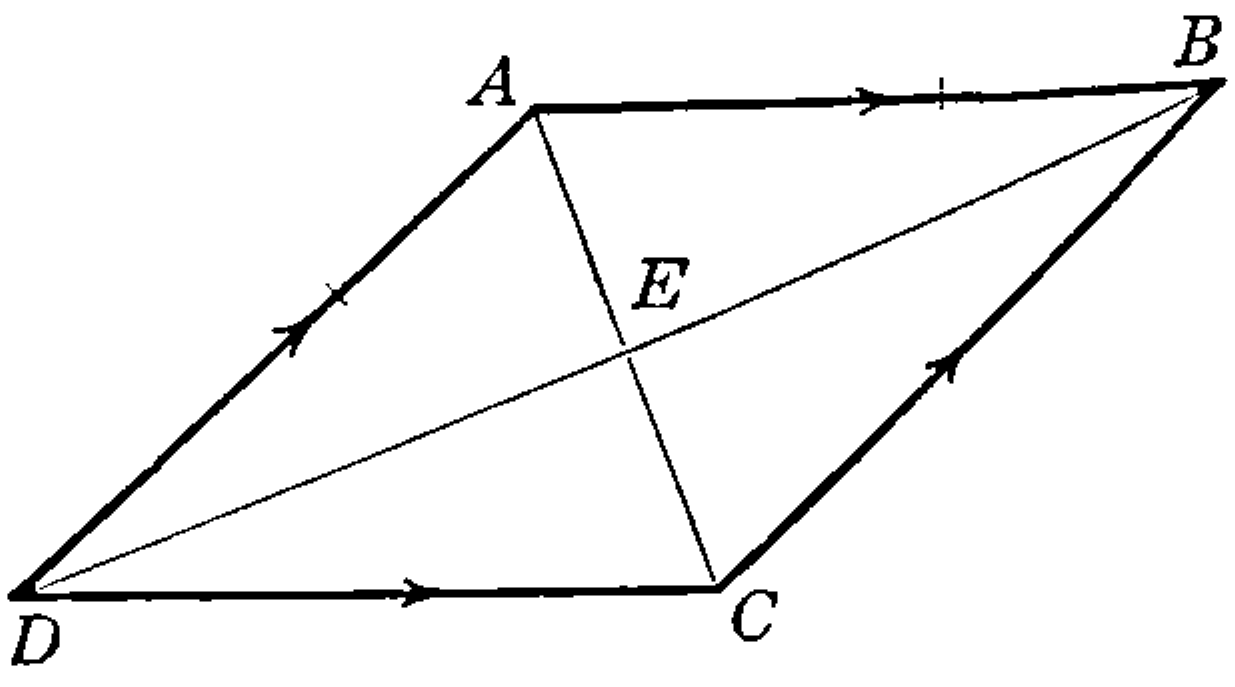
273. Corollary 5. The median of a triangle bisects the triangle.



$T = T'$

$\triangle T$ and T' have equal bases and the same altitude.

274. Corollary 6. The area of a rhombus equals half the product of its diagonals.

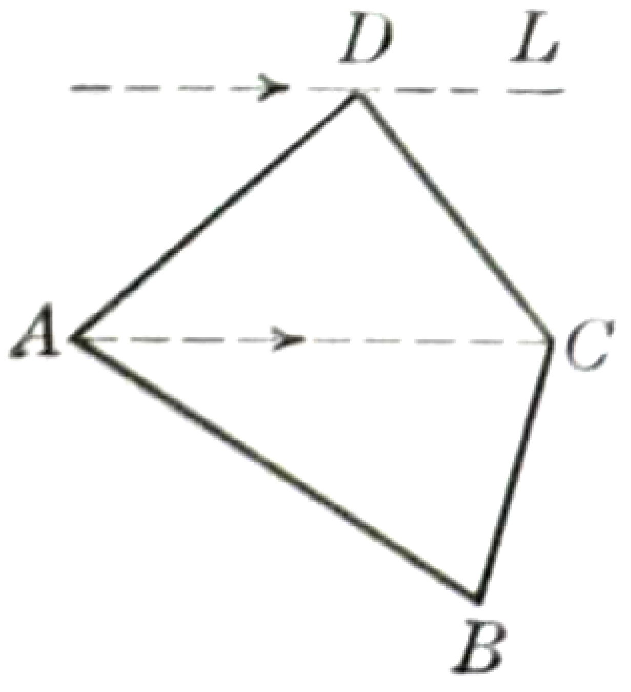


The diagonals of a rhombus are perpendicular to each other (§ 181).

$$\begin{aligned} \text{Rhombus } ABCD &= 2\,\triangle ABD \\ &= 2\left(\tfrac{1}{2}\cdot BD\cdot AE\right) \\ &= \tfrac{1}{2}\cdot BD\cdot(2\cdot AE) \\ &= \tfrac{1}{2}\cdot BD\cdot AC \end{aligned}$$

275. Definition. To transform a figure means to change it into another figure having the same area.

276. PROBLEM. Transform a quadrilateral into a triangle.



Given: Quad. $ABCD$

Req.: To transform quad. $ABCD$ into a △

Help: Cons. $DL \parallel AC$.

If we move D along the line DL, the area of $\triangle ADC$ will not change. We therefore move D so that it falls on the prolongation of BC (or BA), thus causing the two sides DC and CB to become one side.

Perform the actual construction and state the conclusion.

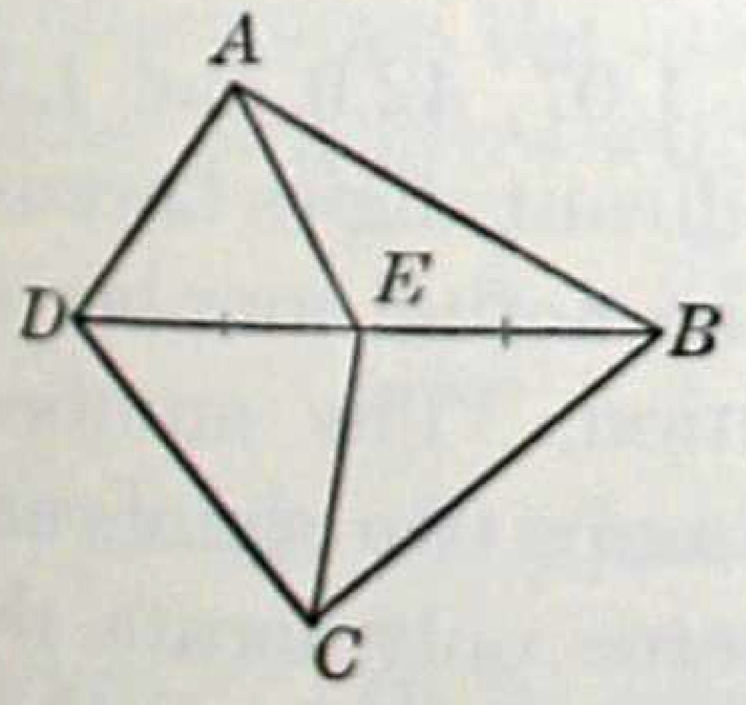
EXERCISES

1. Prove that the diagonals of a parallelogram divide the figure into four equal triangles.

2. Prove that if the diagonals of a quadrilateral divide it into four equal triangles, the figure is a parallelogram.

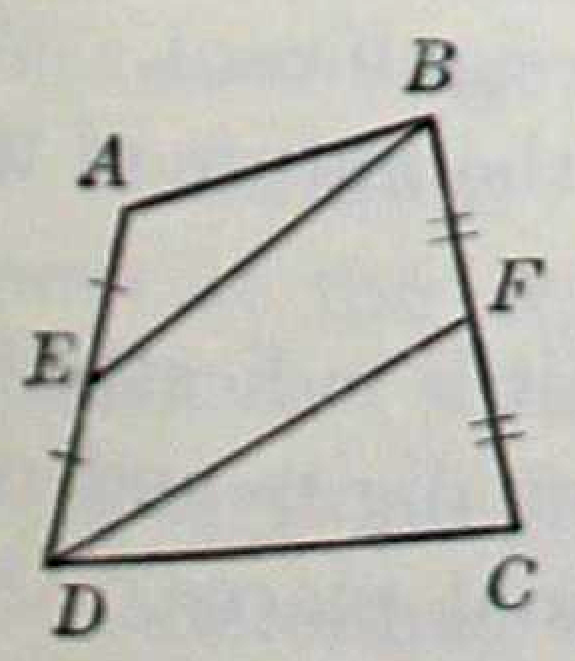
3. If any point in the base of a triangle is joined to the midpoints of the other two sides, the smallest quadrilateral thus formed is one-half the original triangle. *Hint:* Join the point in the base to the opposite vertex.

4. If the midpoint of a diagonal of a quadrilateral is joined to the other two vertices, two equal quadrilaterals are formed.



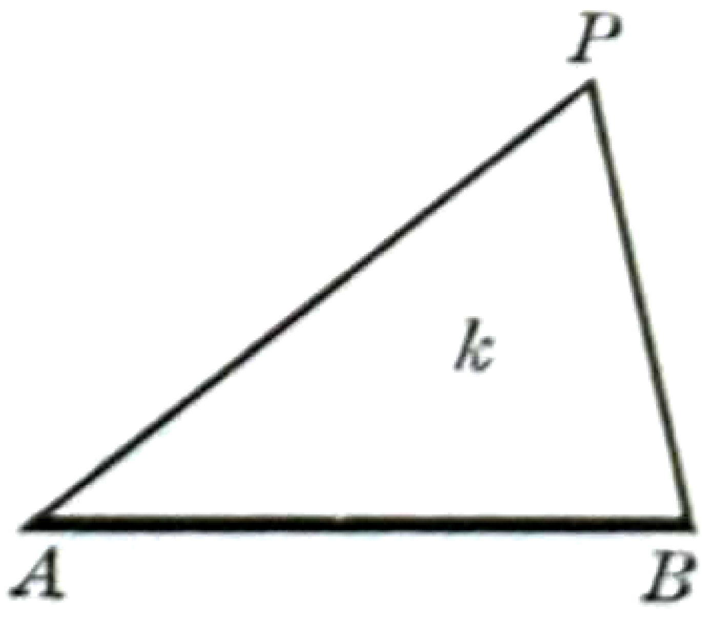
Prove:
Quad. $ABCE$
= quad.
$ADCE$

5.



Given:
In quad. $ABCD$
$AE = ED$
$BF = FC$
Prove:
Quad. $BFDE = \triangle ABE + \triangle CDF$

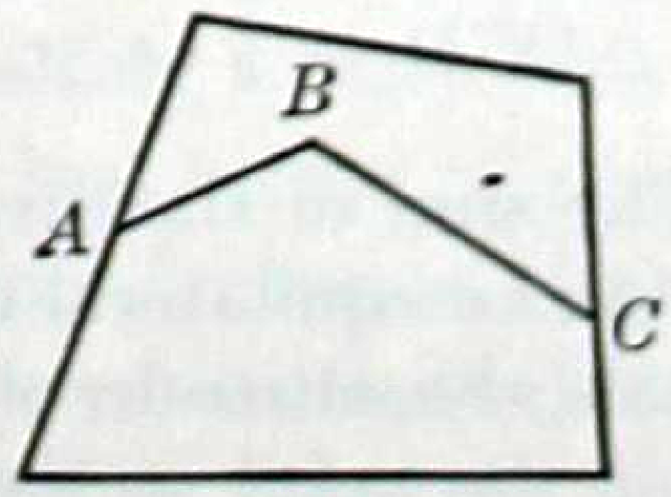
6. (*a*) If one side AB of a triangle is fixed and its area is constant, what is the locus of the third vertex P?



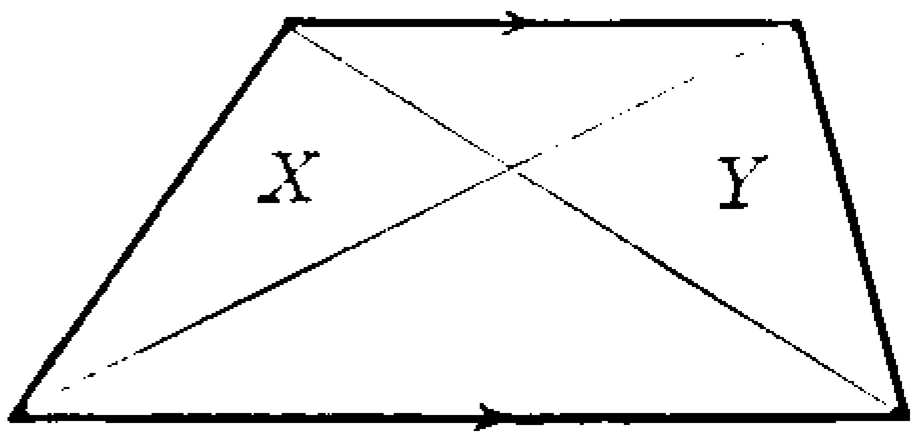
(*b*) Transform a triangle into a right triangle.

(*c*) Transform a triangle into an isosceles triangle.

7. Two farmers have adjoining land separated by a fence ABC. Make a construction showing how the fence could be made into a straight line without changing the amount of land that each farmer owns. *Help:* Draw AC and transform $\triangle ABC$.

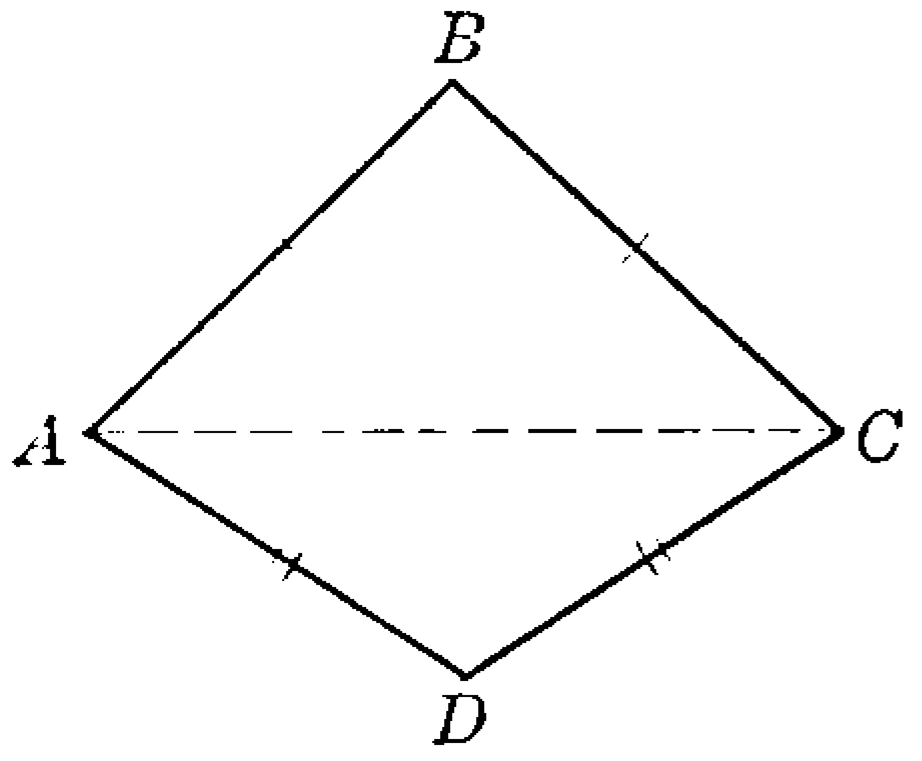


8. The diagonals of a trapezoid divide the figure on the top of the next page into four triangles. Prove that the two tri-



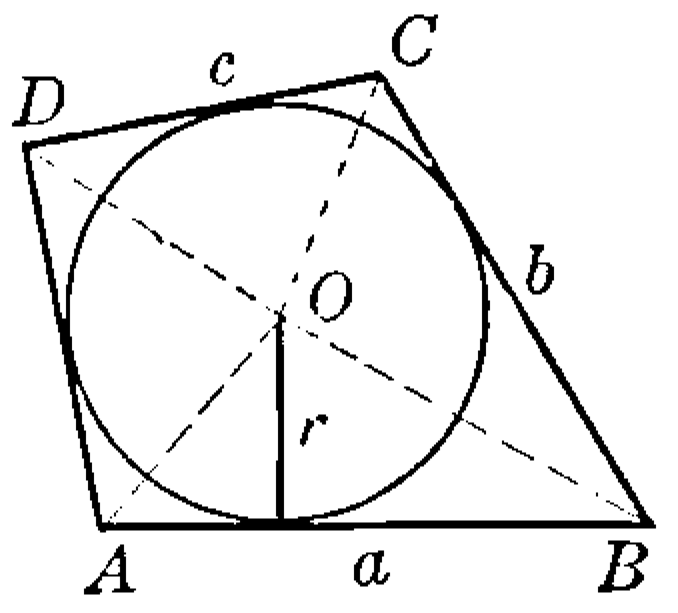
angles that have the legs of the trapezoid as sides are equal. Prove: $X = Y$.

9. A kite is a quadrilateral formed by the legs of two isosceles triangles on each side of a common base.



Prove that the area of a kite equals half the product of its diagonals. Quad. $ABCD$ is a kite.

10. Prove: The area of a polygon having an inscribed circle equals half its perimeter times the radius of the inscribed circle.



Help: $\triangle OAB = \frac{1}{2} \cdot a \cdot r$
$\triangle OBC = \frac{1}{2} \cdot b \cdot r$
$\triangle OCD = \frac{1}{2} \cdot c \cdot r$, etc.

11. Prove: The sum of the distances from a point within an equilateral triangle to the sides equals the altitude of the triangle. *Help:* Join the point to the vertices and find the areas of the triangles thus formed.

12. Prove: The sum of the distances from any point in the base of an isosceles triangle to the legs equals the altitude to one of the legs.

Using a Table of Square Roots

277. On page 280 there is a table that gives the square roots of the numbers from 1 to 100. To find the square root of a larger number, like 175, first divide it by 100 and find the square root of 1.75, which is 1.323.

Now

$$\sqrt{175} = \sqrt{1.75} \times \sqrt{100} = 1.323 \times 10 = 13.23$$

Similarly,

$$\sqrt{.175} = \sqrt{17.5} \div \sqrt{100} = 4.183 \div 10 = .4183$$

Note that the decimal point is shifted an *even* number of places to bring the number into the range of the table. It is then shifted back *half* the number of places in the answer.

Significant Figures

In the decimal system, a digit may serve two purposes:

(1) It may signify the number of counters in its class, that is, the number of units, tens, hundreds, etc. A digit that does this is called *significant.*

(2) It may locate the decimal point. A digit that *only* locates the decimal point is not significant.

The numbers 754, 1.07, 42.0 and 1.30 each have three significant digits because each digit signifies the actual number of counters in its position. The numbers .042 and .0063 have only two significant digits because the zeros only locate the decimal point. If a number is written merely as 400, we do not know whether or not the zeros are significant. We usually judge this from the nature of the measurement.

The accuracy of a result is indicated by the number of significant digits and not by the number of decimal places. Our square root table is accurate to four digit places.

It is frequently desirable to round off a number to a smaller number of digits. In rounding off a digit that is 5 or more, we usually increase the preceding digit by 1. In rounding off a digit that is less than 5, we do not change the preceding digit. Thus, rounded off to three figures, 1.486 becomes 1.49, 6845 becomes 6850, 19,486 becomes 19,500, 12.036 becomes 12.0, and 14.96 becomes 15.0.

In computing with approximate numbers, round off all your numbers so that they have but one more place of accuracy than the least accurate of the given measurements, then round off the last digit in the final answer.

EXERCISES

In these exercises, use the table of square roots on page 280.

1. The area of a square is 432 square inches. Find the side of the square correct to three digit places.
2. Find the side of a square that is equal to a rectangle 24.3 ft. by 16.4 ft. (three significant digits).
3. The side of a square is 64.5 ft. Find the side of a square that is twice as large.
4. The area of a rectangle is 390 sq. ft. It is twice as long as it is wide. Find its dimensions.
5. The area of an isosceles right triangle is 306 sq. in. Find the legs.
6. The side of a square is 10. By using Corollary 6 (§ 274), find the diagonal of the square.

PROPORTION

Forming Proportions

278. In order to state that quantities involving more than one dimension are equal, we must write an equation between two products or two quotients. Since distance equals the product of rate and time, two distances will be equal if $rt = r't'$. Since rate equals the quotient of distance and time, two rates will be equal if

$$\frac{d}{t} = \frac{d'}{t'}$$

This kind of equation is called a *proportion*. The fraction $\frac{d}{t}$ is called the ratio of d to t. The proportion is read, "d is to t as d' is to t'."

279. Definition. A proportion is a statement of the equality of two ratios (fractions).

280. Every proportion contains four terms. Therefore, we cannot say that two numbers are proportional to each other, but only that they are proportional to two other numbers. Thus, it does not make sense to say that *3* is proportional to *4*. We must say that *3* and *4* are proportional to *6* and *8*.

Although we cannot say that two fixed quantities are proportional, we can say that one variable d is proportional to another variable t. This would mean that any two values of d are proportional to the corresponding values of t.

The first and last terms of a proportion are called the end terms or the *extremes;* the other two terms are called the middle terms or the *means*. In the proportion in § 278, d and t' are the extremes; d' and t are the means. If we clear fractions in that proportion (by multiplying both members of the equation by tt'), we get $dt' = d't$. This is expressed in the following theorem.

281. THEOREM. In any proportion, the product of the means equals the product of the extremes.

---

282. CONVERSE THEOREM. If the product of one pair of numbers equals the product of another pair of numbers, either pair may be made the means and the other pair the extremes of a proportion.

Given: $ad = bc$

Prove: $\frac{a}{b} = \frac{c}{d}$ (1)

$\frac{b}{a} = \frac{d}{c}$ (2)

$\frac{a}{c} = \frac{b}{d}$ (3) etc.

PROOF

Statements	*Reasons*
1. $ad = bc$	1. Given.
2. $bd = bd$	2. Identity.
3. $\frac{a}{b} = \frac{c}{d}$ (1)	3. Div. Ax.
4. $ac = ac$	4. Identity.
5. $\frac{d}{c} = \frac{b}{a}$ or $\frac{b}{a} = \frac{d}{c}$ (2)	5. Div. Ax. (Eq. 1 ÷ Eq. 4).
6. $cd = cd$	6. Identity.
7. $\frac{a}{c} = \frac{b}{d}$ (3) etc.	7. Div. Ax. (Eq. 1 ÷ Eq. 6).

Each identity in this proof is the lowest common denominator of the proportion that we seek to prove.

---

283. Definition. If the means of a proportion are equal, each mean is said to be the mean proportional between the extremes.

If $\frac{a}{b} = \frac{b}{c}$, then b is the mean proportional between a and c.

Show that $b = \sqrt{ac}$; that is, the mean proportional between two numbers equals the square root of their product.

EXERCISES

1. Solve the following proportions for x:

   (*a*) $\frac{3}{4} = \frac{6}{x}$, (*b*) $\frac{2}{5} = \frac{x}{10}$, (*c*) $\frac{x}{8} = \frac{2}{x}$,

   (*d*) $\frac{5}{8} = \frac{8}{x}$, (*e*) $\frac{8}{x} = \frac{x}{18}$

2. A man has $24,000 invested in a business, and his income from it is $1200. He figures that improvements costing $4000 will bring a "proportionate return." What total income is he expecting?

3. (*a*) $7x = 5y$. Complete the proportion: $\frac{x}{y} = \text{—?—}$.

(*b*) $x^2 = mn$. Write a proportion with n as the last term.

(*c*) $x = \frac{a^2}{b}$. Write a proportion with x as the last term.

(*d*) $x = \sqrt{3y}$. Write a proportion with y as the last term.

(*e*) $\frac{x}{y} = \frac{3}{4}$. Complete the proportion: $\frac{x+y}{y} = \text{—?—}$.

4. Given: $pq = rs$. Write eight proportions using p, q, r, and s as the terms.

5. Given: $\frac{x}{3} = \frac{y}{4}$. First express the equation as the product of the means and the product of the extremes. Then derive seven proportions using x, y, 3, and 4 as the terms.

6. (*a*) Divide 42 into two parts proportional to 2 and 5.

(*b*) The area of a rectangle is 48 and the ratio of the length to the width is 4 : 3. Find the length and the width.

(*c*) Find the mean proportional between 12 and 3.

Transforming Proportions

284. GENERAL TRANSFORMATION THEOREM. A proportion may be transformed in any way that results in the same equation between the product of the means and the product of the extremes.

Given: $\frac{a}{b} = \frac{c}{d}$ (1)

Prove: $\frac{b}{a} = \frac{d}{c}$ (2)

$\frac{a}{c} = \frac{b}{d}$ (3) etc.

All these proportions lead to $ad = bc$ as the equation between the product of the means and the product of the extremes.

PROOF

Statements	*Reasons*
1. $\frac{a}{b} = \frac{c}{d}$ (1)	1. Given.
2. $bc = ad$	2. Product means = product extremes (§ 281).
3. $\frac{b}{a} = \frac{d}{c}$ (2) $\frac{a}{c} = \frac{b}{d}$ (3) etc.	3. If product of one pair nos. = product another pair, either pair can be the means of proportion with other pair as the extremes (§ 282).

Changing the proportion $\frac{a}{b} = \frac{c}{d}$ to the proportion $\frac{b}{a} = \frac{d}{c}$ is called *inversion*, because each ratio is inverted. Changing $\frac{a}{b} = \frac{c}{d}$ to $\frac{a}{c} = \frac{b}{d}$ is called *alternation*, because the first term is to the third as the second is to the fourth.

285. Corollary 1. If four quantities are in proportion, they are in proportion by inversion or alternation.

286. Corollary 2. In any proportion, the sum of the first two terms is to the first (or the second) as the sum of the last two terms is to the third (or the fourth).

This corollary is usually referred to in its abbreviated form as the *Addition Transformation.*

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{a} = \frac{c+d}{c},$$

$$\text{or } \frac{a+b}{b} = \frac{c+d}{d}$$

Show that in each of these three proportions, the equation between the product of the means and the product of the extremes reduces to $ad = bc$. By the General Transformation Theorem (§ 284), therefore, they are equally valid.

287. Corollary 3. In any proportion, the difference between the first and second terms is to the first (or the second) as the difference between the third and fourth terms is to the third (or the fourth).

This corollary is usually referred to in its abbreviated form as the *Subtraction Transformation.*

$$\text{If } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a-b}{a} = \frac{c-d}{c},$$

$$\text{or } \frac{a-b}{b} = \frac{c-d}{d}$$

---

288. THEOREM. In a series of equal ratios, the sum of any number of numerators is to the sum of their denominators as any numerator is to its denominator.

Given: $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$

Prove: $\frac{a+c+e}{b+d+f} = \frac{e}{f}$

The theorem is true in a simple proportion (two equal ratios) because the equation in the product of the means and the product of the extremes in the new proportion reduces to the same as in the old proportion.

That is, from $\frac{a}{b} = \frac{c}{d}$

we get $\frac{a+c}{b+d} = \frac{c}{d}$ (§ 284)

or $\frac{a+c}{b+d} = \frac{e}{f}$ (Subs. Ax.)

Applying our theorem to this simple proportion, we have $\frac{a+c+e}{b+d+f} = \frac{e}{f}$

To tell quickly whether two proportions like $\frac{m}{p} = \frac{2}{x}$ and $\frac{x}{p} = \frac{2}{m}$ are equivalent, we need only compare the equations in the product of the means and the product of the extremes. Here we get $mx = 2p$ in both cases. Therefore, the proportions are equivalent. You should practice the transformations until you can recognize them quickly. For example, the following proportions are interchangeable:

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{b}{d} = \frac{a}{c}, \quad \frac{a+b}{c+d} = \frac{b}{d},$$

$$\frac{a+c}{b+d} = \frac{a}{b}, \quad \frac{b-a}{d-c} = \frac{a}{c}, \quad \frac{b-d}{a-c} = \frac{d}{c}, \text{ etc.}$$

EXERCISES

1. Name the specific transformation applied in deriving each proportion from the preceding one:

$$\frac{15}{6}=\frac{5}{2}\quad(1),\qquad \frac{6}{15}=\frac{2}{5}\quad(2),\qquad \frac{6}{2}=\frac{15}{5}\quad(3),$$

$$\frac{8}{2}=\frac{20}{5}\quad(4),\qquad \frac{28}{7}=\frac{8}{2}\quad(5),\qquad \frac{21}{28}=\frac{6}{8}\quad(6)$$

2. Write this proportion: $\frac{3}{4}=\frac{6}{8}$. Then proceed as follows: (*a*) Alternate this proportion; (*b*) Invert the second proportion; (*c*) Transform the third proportion by subtraction; (*d*) Transform the fourth proportion by addition; and (*e*) Transform the fifth by adding the numerators.

3. Describe how each proportion (or equation) is derived from the preceding one.

$$\frac{m-3}{1-n}=\frac{3}{n}\quad(1),\qquad \frac{m}{1}=\frac{3}{n}\quad(2),$$

$$\frac{1}{m}=\frac{n}{3}\quad(3),\qquad \frac{1}{n}=\frac{m}{3}\quad(4),$$

$$\frac{n}{3}=\frac{1}{m}\quad(5),\qquad mn=3\quad(6)$$

4. How is each of the following proportions derived from the preceding equation:

$$s^2=tu\quad(1),\qquad \frac{t}{s}=\frac{s}{u}\quad(2),$$

$$\frac{s}{t}=\frac{u}{s}\quad(3),\qquad \frac{s+t}{s}=\frac{s+u}{u}\quad(4),$$

$$\frac{s+u}{s+t}=\frac{u}{s}\quad(5),\qquad \frac{u-t}{s+t}=\frac{u-s}{s}\quad(6)$$

5. Write a proportion from this equation: $x^2=7y$. Now transform your proportion by addition, the new proportion by alternation, then by subtraction, and finally by adding the numerators.

6. Two rectangles are equal. The dimensions of one rectangle are l and w; the dimensions of the other, l' and w'. Which of the following proportions are valid?

$$\frac{l}{w}=\frac{l'}{w'}\quad(1),\qquad \frac{l}{l'}=\frac{w'}{w}\quad(2),$$

$$\frac{l+l'}{w+w'}=\frac{l}{w}\quad(3),\qquad \frac{l-w}{l'-w'}=\frac{l}{l'}\quad(4)$$

$$\frac{l-w'}{l'-w}=\frac{w'}{w}\quad(5),\qquad \frac{w+w'}{l+l'}=\frac{w'}{l}\quad(6)$$

7. In two equal triangles, which of the following statements are true and which are false?

(*a*) The product of the base and the altitude to that base in one triangle equals the product of the base and the altitude to the base in the other triangle.

(*b*) The base of one triangle is to its altitude as the base of the other triangle is to its altitude.

(*c*) The bases of the triangles are proportional to the altitudes of the triangles.

(*d*) The base of the first triangle is to the base of the second triangle, as the altitude to the base in the second triangle is to the altitude to the base in the first triangle.

(*e*) The sum of the bases and the sum of the altitudes are proportional to the bases.

(*f*) The sum of the bases is to the sum of the altitudes as the base of one triangle is to the altitude of the other triangle.

8. Determine which of the following statements are true and which are false.

(*a*) If the means of a proportion are equal, the extremes are equal.

(*b*) In the proportion $\frac{x}{3}=\frac{27}{x}$, x has the same value as the mean proportional between 3 and 27.

(*c*) If a square equals a rectangle, the side of the square is the mean proportional between the length and the width of the rectangle.

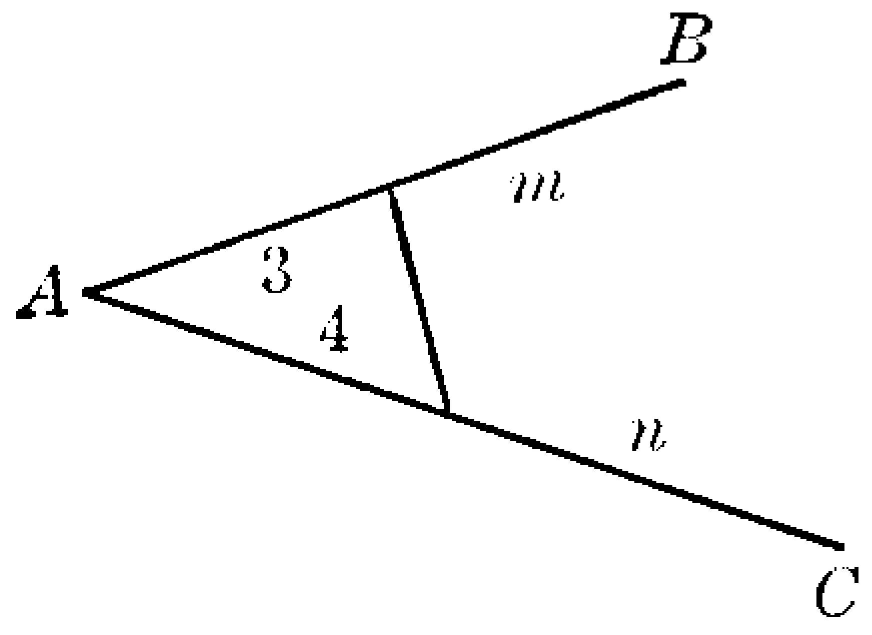
(*d*) If three terms of one proportion are equal to three terms of another proportion, the fourth terms are equal.

(*e*) If the first two terms of a proportion are equal, the last two terms are equal.

(*f*) In two isosceles triangles, the sides of one triangle are proportional to the sides of the other triangle.

(*g*) In any proportion, the sum of the first two terms is to their difference as the sum of the last two terms is to their difference.

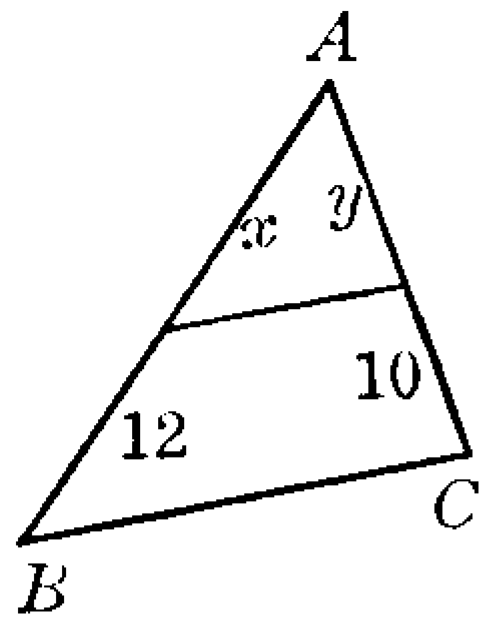
9. (*a*)



Given: $\dfrac{m}{n} = \dfrac{3}{4}$

Find: $\dfrac{AB}{AC}$

(*b*)

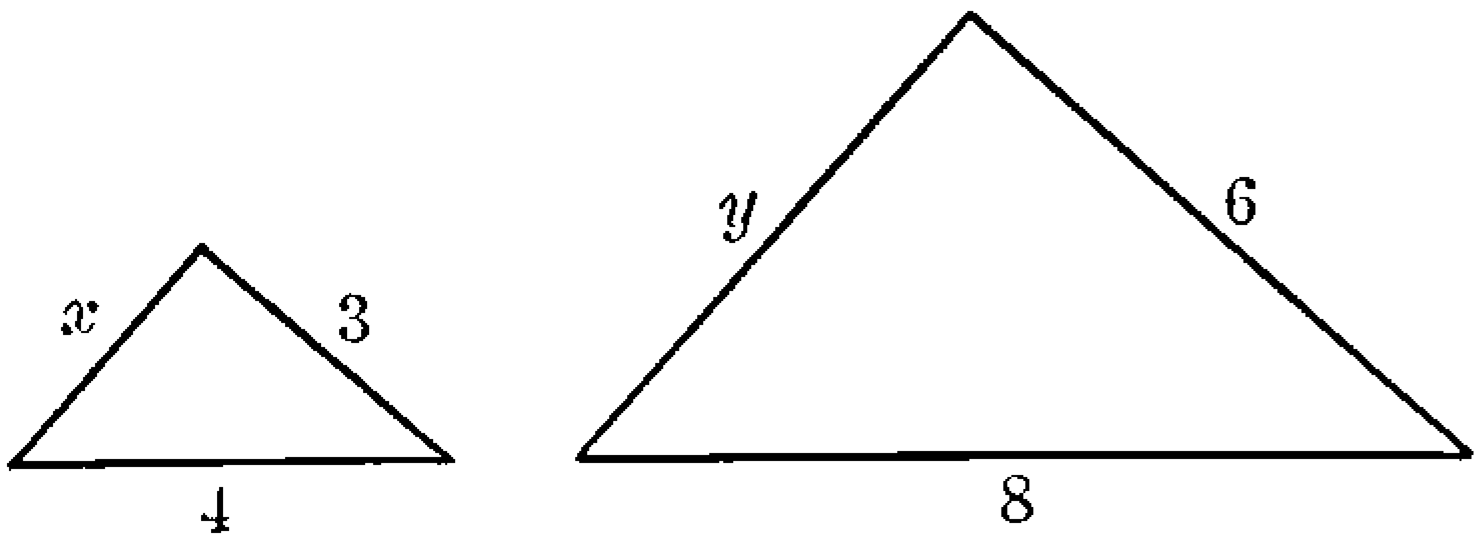


Given: $\dfrac{AB}{AC} = \dfrac{6}{5}$

Find: $\dfrac{x}{y}$

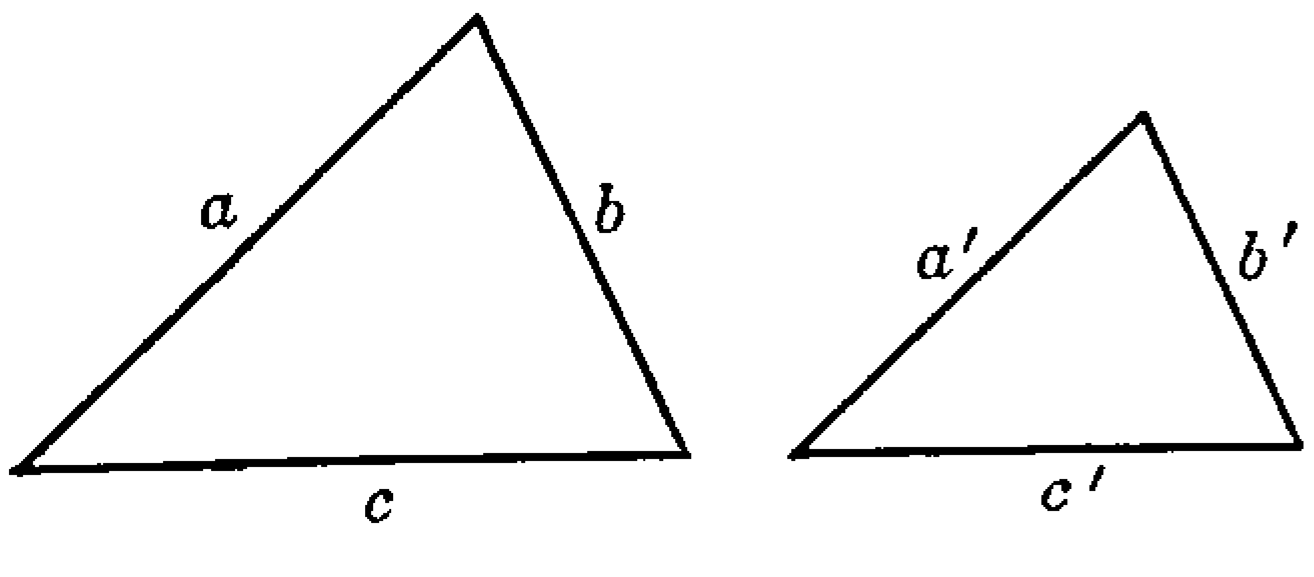
10. (*a*) Given: $\dfrac{x}{y} = \dfrac{1}{2}$

Find: $\dfrac{p}{p'}$ the ratio of the perimeters of the triangles

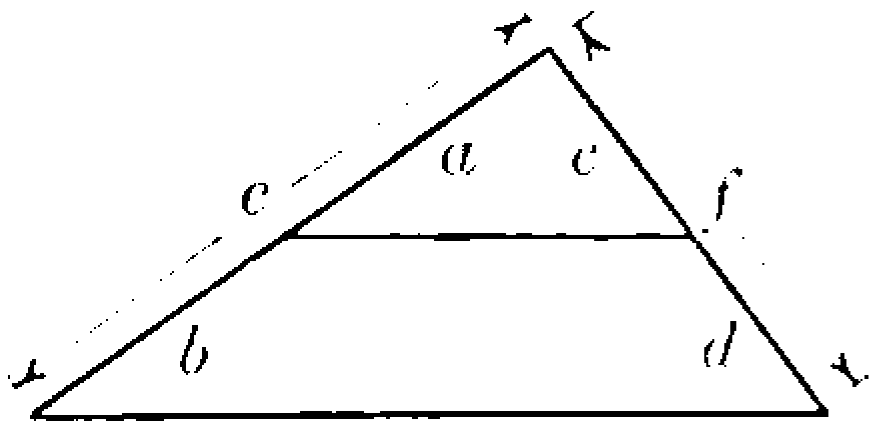


(*b*) Given: $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'} = \dfrac{4}{3}$

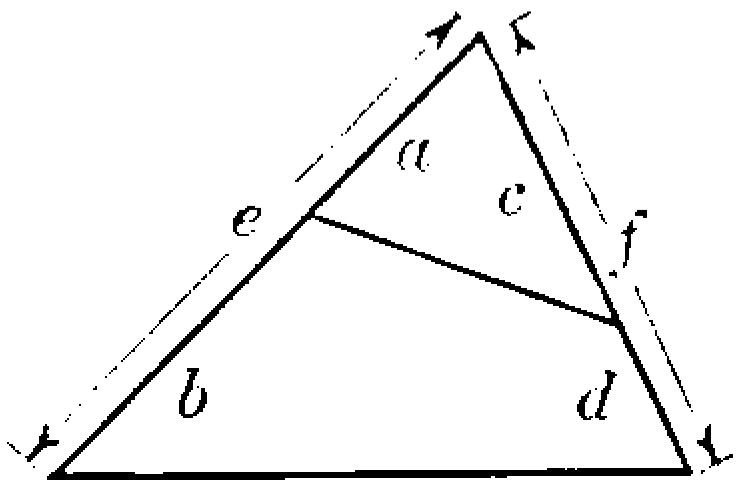
Find: $\dfrac{p}{p'}$ the ratio of the perimeters of the triangles



11. In the diagram, it is given that $ad =$ Write three proportions using: (1) a, b, and d; (2) a, c, e, and f; (3) b, d, e, and



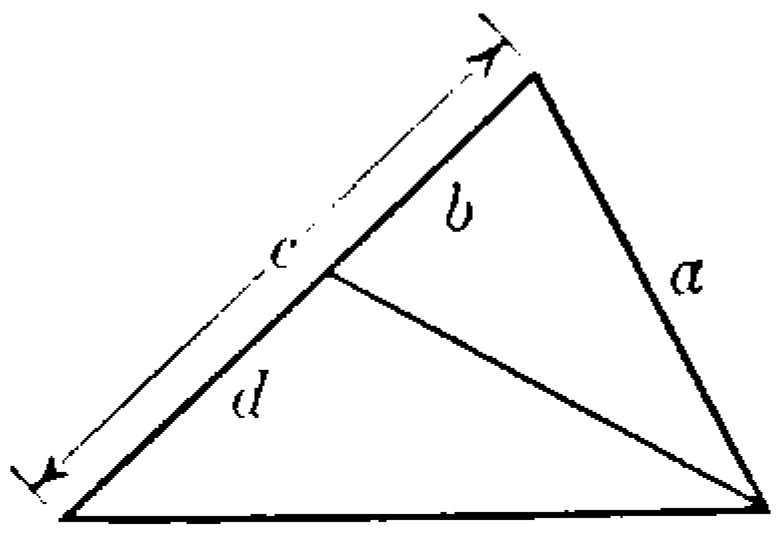
12. In the diagram, it is given that $ae = c$ Which of the following proportions are valid?



$\dfrac{a}{f} = \dfrac{c}{e}$ (1), $\quad \dfrac{f}{e} = \dfrac{a}{c}$ (2),

$\dfrac{a}{b} = \dfrac{c}{d}$ (3), $\quad \dfrac{c}{b} = \dfrac{f}{d}$ (4)

13. In the figure, it is given that $\dfrac{b}{a} = \dfrac{a}{c}$. Which of the following are valid?



$a^2 = bc$ (1), $\quad \dfrac{a}{b} = \dfrac{c}{a}$ (2),

$\dfrac{a^2}{b^2} = \dfrac{c}{b}$ (3), $\quad \dfrac{a}{b} = \dfrac{c}{d}$ (4)

14. (*a*) Given: $\dfrac{a}{b} = \dfrac{c}{d}$

Prove: $\dfrac{a-c}{b-d} = \dfrac{c}{d}$

(*b*) Given: $\dfrac{m+x}{n+y} = \dfrac{m}{n} = \dfrac{3}{5}$

Find: $\dfrac{x}{y}$

Variation

Let us suppose that a train is running at a constant speed k. We have seen that the variable distance d is proportional to the variable time t (§ 280). Another way of expressing this is that "d varies as t" or $\frac{d}{t} = k$. In terms of static proportion, we have $\frac{d}{t} = \frac{d'}{t'}$, where all the letters are fixed numbers.

S289. Definition. If the ratio of two variables is constant, one variable is said to vary as the other.

Illustrative Example. In 6 hours a motorist drives 240 miles. How far will he drive in 9 hours at the same speed?

Solution. Since the speed is constant, we have the variation $\frac{d}{t} = k$.

Therefore, $\frac{d}{t} = \frac{d'}{t'}$. Here $d = 240$, $t = 6$, $d' =$ —?— and $t' = 9$.

Substituting, we have $\frac{240}{6} = \frac{d'}{9}$, and $d' = 360$ miles.

k is called the constant of the variation. In this case, k (the speed) equals $\frac{240}{6} = 40$.

S1. If a dozen eggs cost 42 cents, how much will 22 eggs cost? Let $c =$ the total cost, and $n =$ the number of eggs bought. Since the price is presumed to be constant, $c = kn$, or $\frac{c}{n} = k$.

S2. If $60 interest is earned in 4 years, how much will be earned in 7 years under the same conditions?

The formula for simple interest is $i = prt$. Since the principal and the rate of interest do not change, we may substitute a constant k for pr and get $i = kt$, or $\frac{i}{t} = k$; that is, the interest varies as the time.

S3. (*a*) Assuming that the base of a rectangle is fixed, write a variation connecting the area and the altitude. Express the variation in words.

(*b*) When the altitude of a rectangle is 4 units, the area is 48 square units. Find the area of the rectangle when the altitude is 6 units.

S4. In the formula $L = k\frac{d}{2}SV^2$, L is the lift of an airplane, k is a constant (called the coefficient of lift), d is the density of the air, S is the area of the wings, and V is the velocity of the plane.

An airplane model with a wing surface of 2 square feet shows a lift of 2 pounds in a wind tunnel. What would be the lift of a full-size plane of similar design with a wing area of 1500 square feet? The air conditions and the speed are the same as in the wind tunnel.

S5. A plane flying at an altitude where the density of the air is .0023 slugs per cubic foot (a slug is approximately 32.2 pounds) has a lift of 1500 pounds. What will be the lift if it flies to an altitude where the density is only .0018 slugs per cubic foot? (Use the formula in S4.)

Inverse Variation

Sometimes the *product* of two variables is constant. In the distance formula, for example, if D is constant, then $RT = k$. If this were written $\frac{R}{\frac{1}{T}} = k$, we would see that the ratio of R to $\frac{1}{T}$ is constant. $\frac{1}{T}$ is the inverse of T. According to our definition of a variation (§ S289), R is said to vary as the inverse of T, or "R varies inversely as T." We therefore have the following corollary.

S290. Corollary. If the product of two variables is constant, then one variable varies inversely as the other.

Illustrative Example. At 6% it requires a principal of \$7200 to yield a certain amount of interest. What investment is required for the same yield at $4\frac{1}{2}\%$, other conditions remaining the same?

Solution. In the formula for simple interest $i = prt$, i and t are constant. We may write $pr = \frac{i}{t}$ or $pr = k$.

Therefore, $pr = p'r'$ $\left(\text{or } \frac{p}{p'} = \frac{r'}{r}\right)$

Here $p = 7200$, $r = 6\%$, $p' = ?$ and $r' = 4\frac{1}{2}\%$.

Substituting, $7200 \cdot 6\% = p' \cdot 4\frac{1}{2}\%$, $p' = \frac{7200 \cdot 6}{4\frac{1}{2}} = \$9600$.

S6. A boy uses his money to buy 6 apples at 4 cents apiece. How many apples could he have bought if the price had been 3 cents?

S7. If 12 men can do a job in 6 days, how long would it take 18 men to do the job?

The whole job J equals the number of men M times the number of days they work D times the amount of work each man does in a day W; that is, $J = MDW$. Since J and W are constant, we have $MD = k$.

S8. (*a*) The lengths of two sides of a triangle are 12 units and 15 units. The length of the altitude to the 12-unit side is 5 units. Find the altitude to the 15-unit side.

(*b*) Write a statement in words telling how two altitudes of a triangle are related to the sides to which they are drawn.

S9. In the flow of an electric current, *Ohm's Law* states that $E = I \cdot R$, where E is the electromotive force in *volts*, I is the intensity of the current in *amperes*, and R is the resistance of the circuit in *ohms*.

(*a*) Express in words the relationship between I and R, if E is constant. If R is constant, what is the relationship between E and I? If I is constant, how does E depend on R?

(*b*) If a lamp, having a resistance of 220 ohms, uses $\frac{1}{2}$ ampere, how many amperes does a heater with a resistance of 22 ohms on the same line use?

S10. The *Gas Law* states that $PV = kT$, where P is the pressure of a gas, V is its volume, and T is the absolute temperature (the centigrade temperature plus 273°).

(*a*) Write three statements telling in words how two of the variables are related while the third is held constant.

(*b*) The volume of the air in a tire under a pressure of 30 pounds per square inch is 3 cubic feet. How many cubic feet will the air from this tire occupy after a blowout, considering that the pressure of the atmosphere is 15 pounds per square inch?

(*c*) We could consider the product PV as one quantity and write the Gas Law $T = k(PV)$. This relationship can be expressed in words as follows: The absolute temperature varies as the *product* of the pressure and the volume.

In the formula for the area of a triangle $T = \frac{1}{2}bh$, consider the product of the base and the altitude to that base as one two-dimensional quantity. Write a statement telling how the area of a triangle changes as compared with the *product* of the base and the altitude to that base.

S11. The kinetic energy *K.E.* that a body possesses because of its motion is given by the formula $K.E. = \frac{1}{2}mv^2$, where m is the mass of the body and v is its speed (velocity).

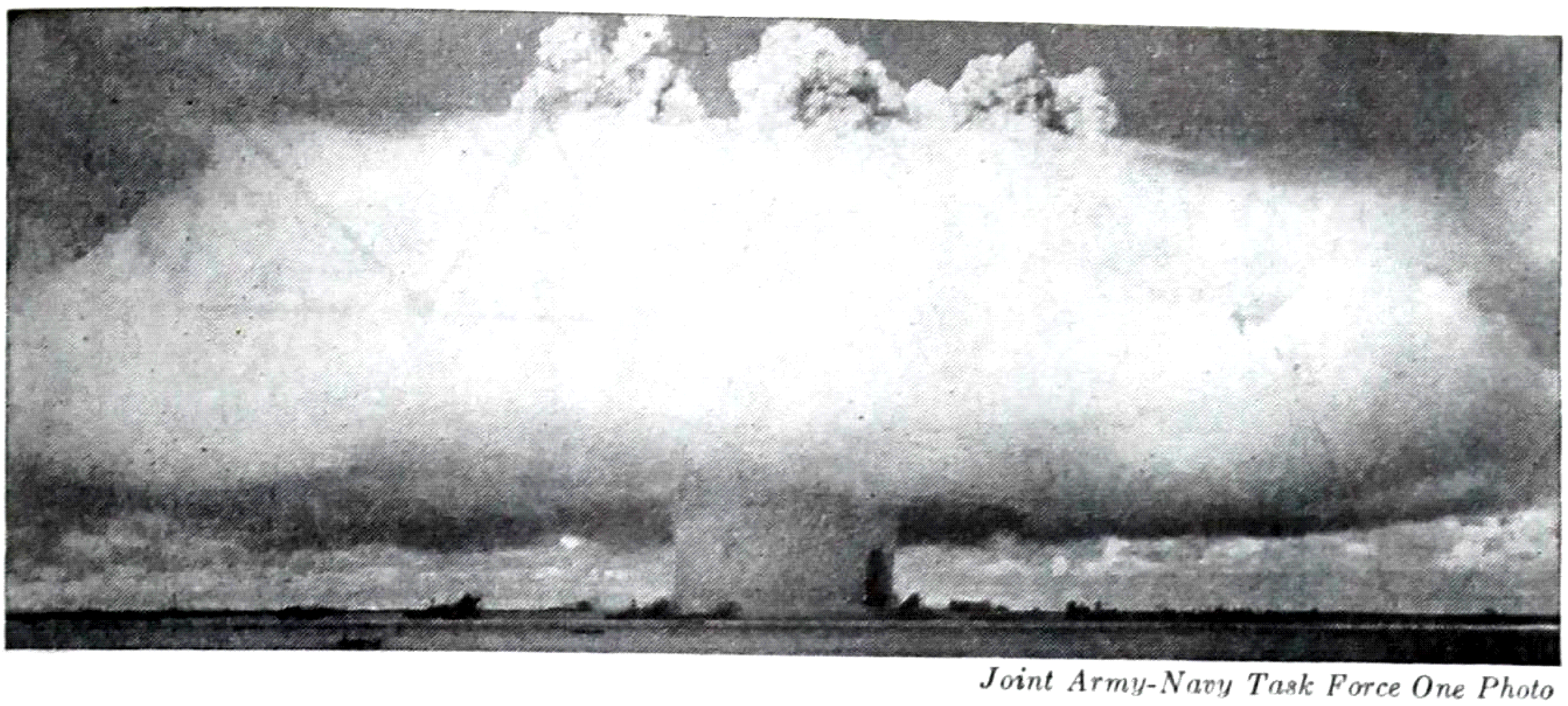
(*a*) A car when driven at 20 miles per hour has 80,000 foot-pounds of kinetic energy. What will be its kinetic energy when driven at 50 miles per hour?

Since m is constant, we have $K.E. = kv^2$. Here the kinetic energy varies directly as the *square* of the speed.

(*b*) Compare the distance required to stop a car going 50 miles per hour with the distance needed to stop it when it is going 20 miles per hour.

S12. Although in the preceding problem the kinetic energy was expressed in foot-pounds, a foot-pound is too small a unit for most practical purposes. The unit more frequently used is the kilowatt-hour (*kw-h*). A kilowatt-hour is 2.66 million foot-pounds (2.66×10^6). Electricity is sold by the kilowatt-hour.

(*a*) A 1200-pound naval shell (a 14-inch shell) has a muzzle velocity of 3000 feet per second. At the moment of firing the shell has 63.3 kw-h of kinetic energy. What would be the atomic energy of the shell, if the shell could be completely converted into energy?



Joint Army-Navy Task Force One Photo

The formula for atomic energy $A.E.$ is similar to the formula for kinetic energy. The speed c in atomic energy is the speed of light, 186,000 miles per second.

$$A.E. = mc^2$$

$$K.E. = \tfrac{1}{2}mv^2$$

$$\frac{A.E.}{K.E.} = \frac{2c^2}{v^2}$$

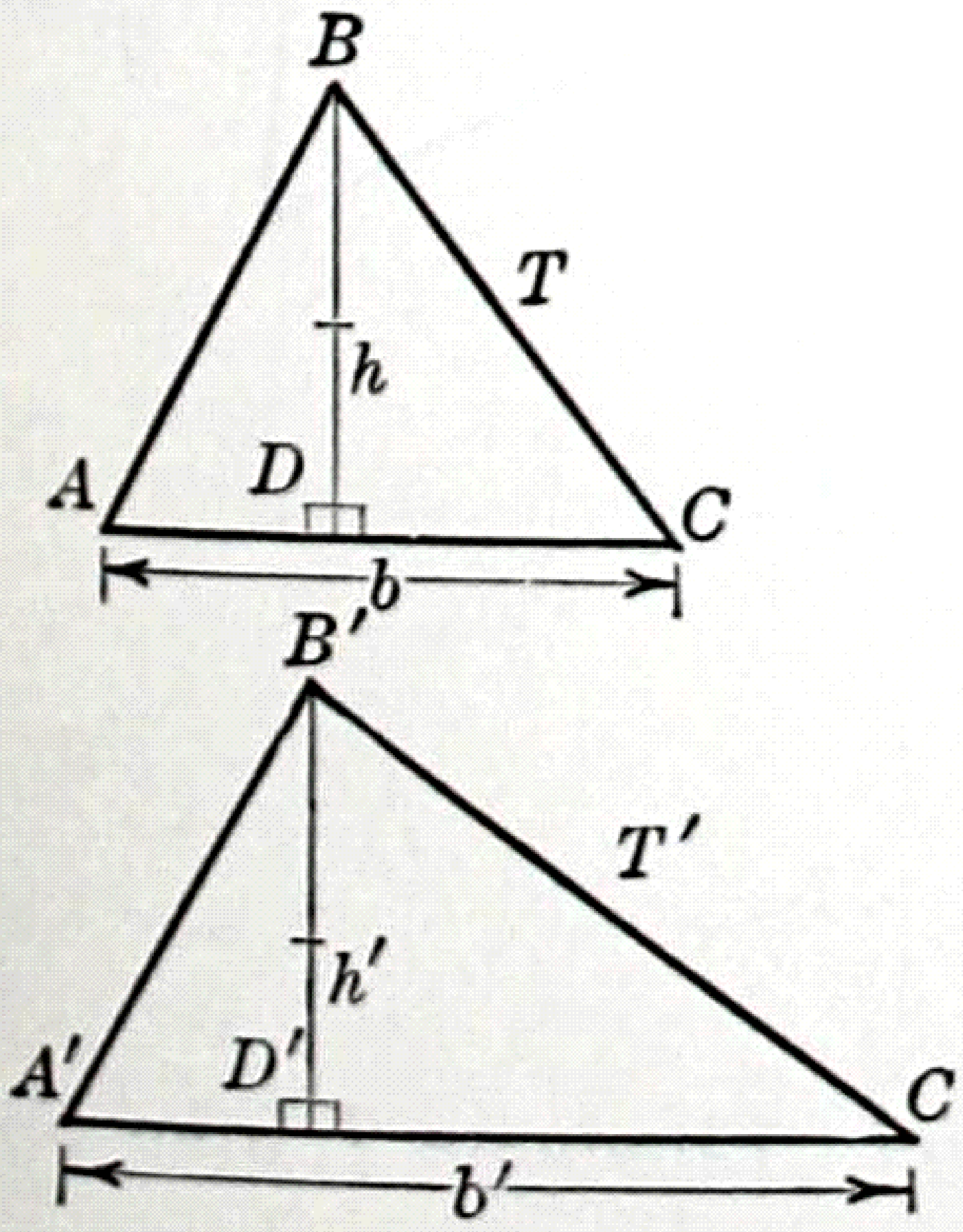
At the present time the atomic bomb gives about .1% of the possible atomic energy.

(*b*) A pound of coal, as a result of combustion, gives 3.86 kw-h of energy. What is the ratio of the possible atomic energy to the combustion energy of coal?

Triangles that Agree in an Altitude

We shall now prove that if two triangles agree in an altitude, then their areas are proportional to the bases.

291. THEOREM. If two triangles have equal altitudes, their areas are proportional to their bases.



Given: $T = \text{Area } \triangle ABC$

$T' = \text{Area } \triangle A'B'C'$

$\text{Alt. } h = \text{alt. } h'$

Prove: $\dfrac{T}{T'} = \dfrac{b}{b'}$

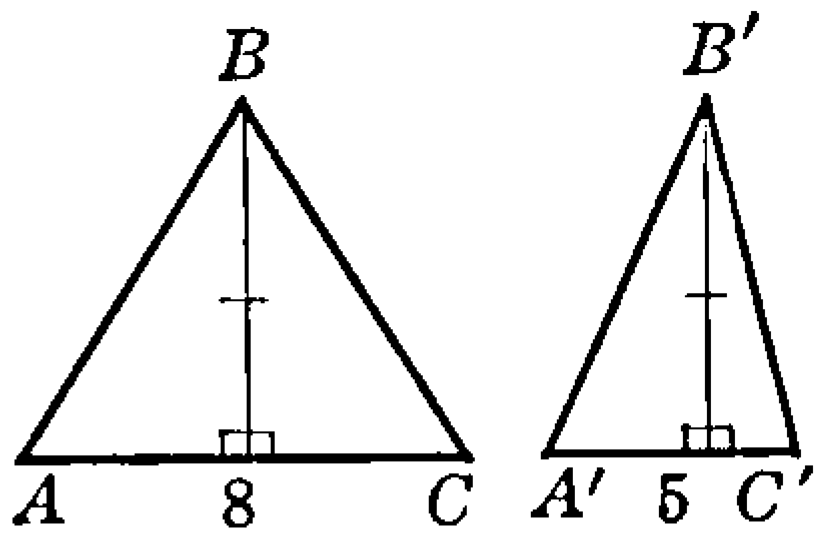
PROOF

Statements	*Reasons*
1. $T = \frac{1}{2}bh$ $T' = \frac{1}{2}b'h'$	1. Area $\triangle = \frac{1}{2} \cdot \text{base} \times \text{alt.}$
2. But $h = h'$	2. Given.
3. $\dfrac{T}{T'} = \dfrac{b}{b'}$	3. Div. Ax.

EXERCISES

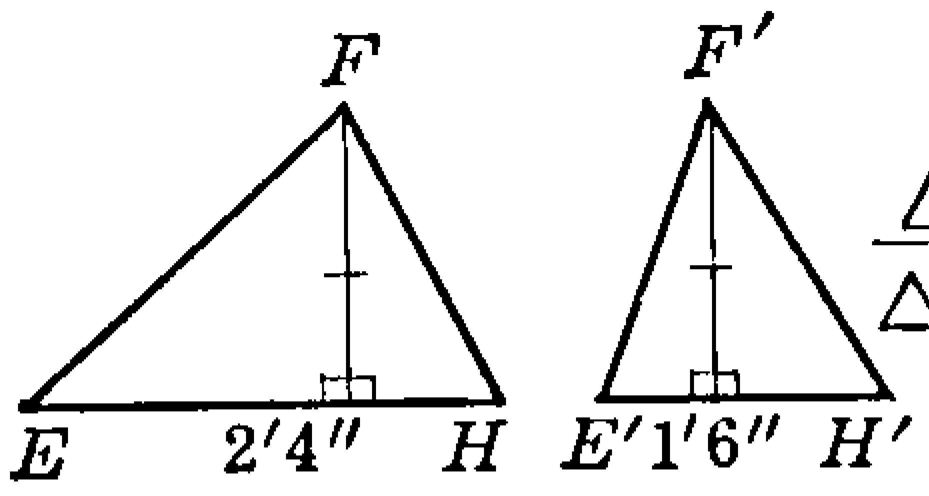
From the figures shown complete the following proportions:

1.



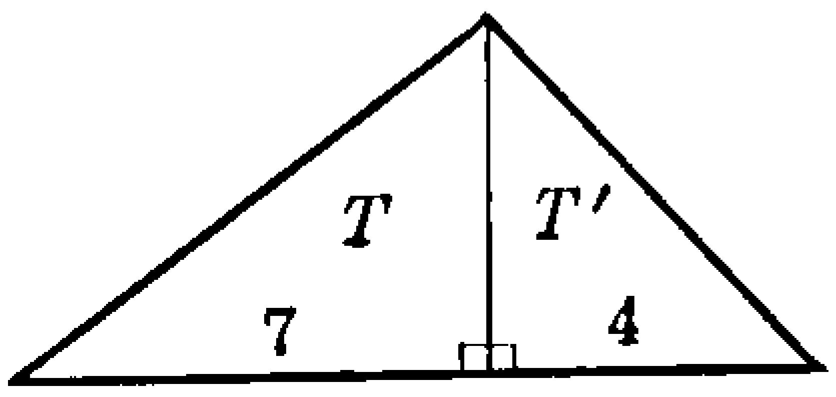
$\dfrac{\triangle ABC}{\triangle A'B'C'} =$ —?—

2.



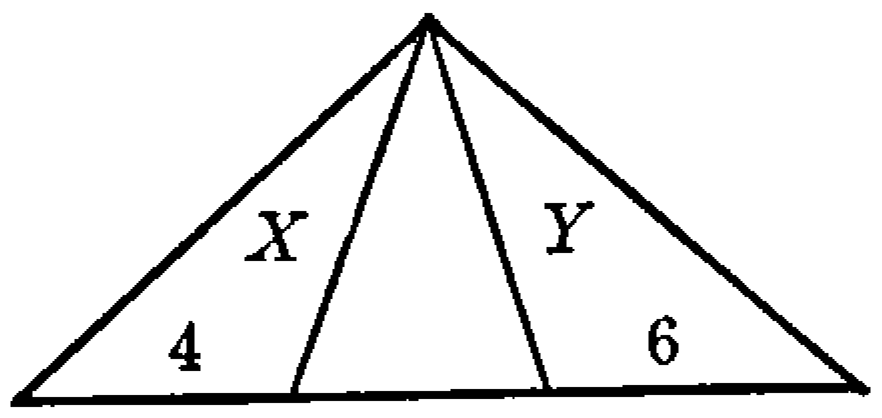
$\dfrac{\triangle EFH}{\triangle E'F'H'} =$ —?—

3.



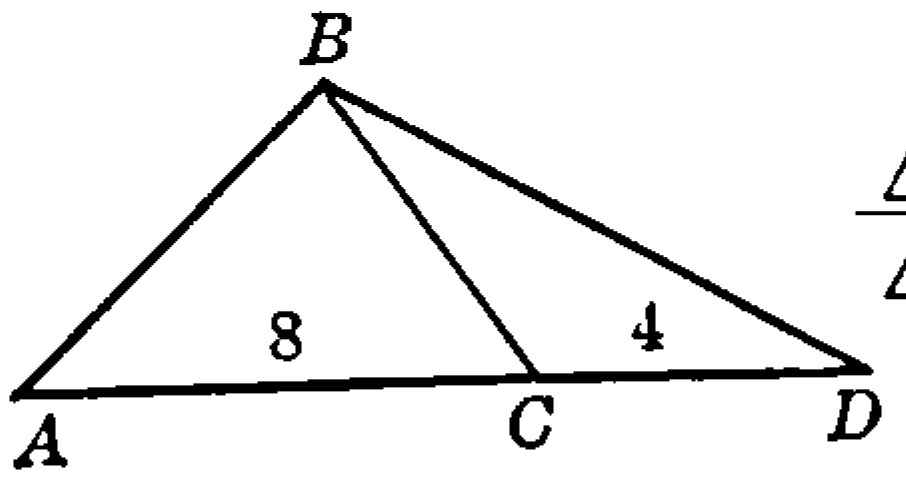
$\dfrac{T}{T'} =$ —?—

4.



$\dfrac{X}{Y} =$ —?—

5.

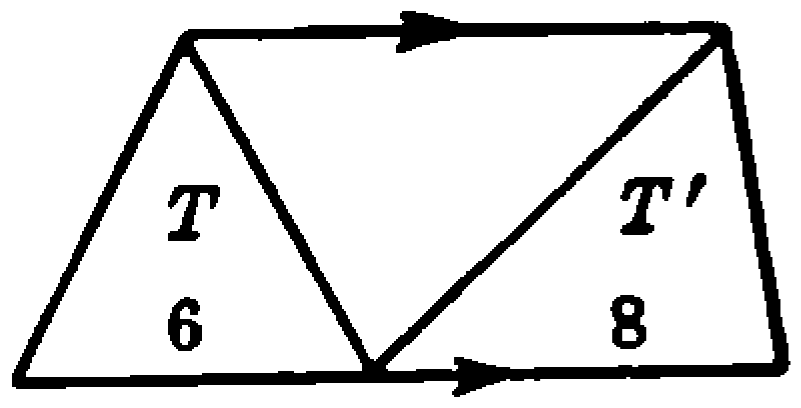


$\dfrac{\triangle ABC}{\triangle BCD} =$ —?—

6. $\dfrac{\triangle ABD}{\triangle ABC} =$ —?—

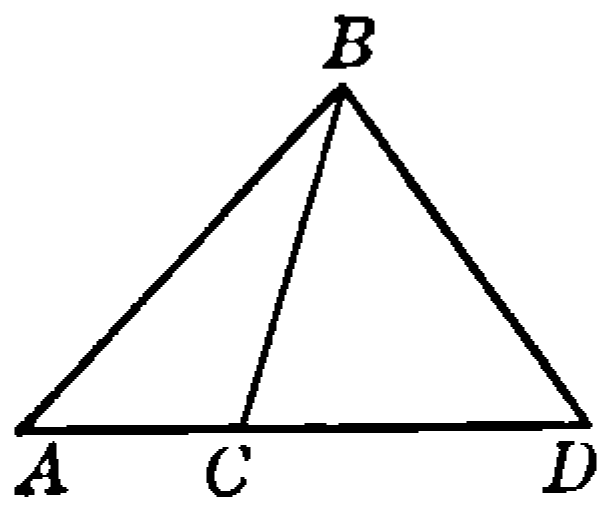
7. $\dfrac{\triangle ABD}{\triangle BCD} =$ —?—

8.



$\dfrac{T}{T'} =$ —?—

9.

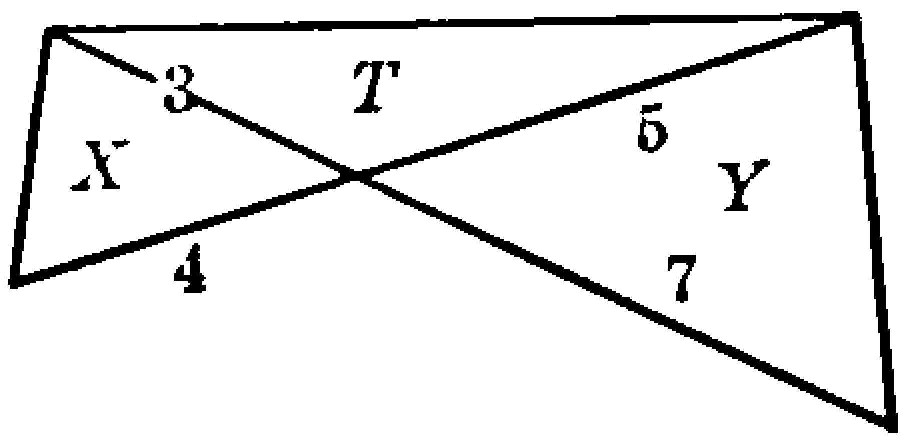


$\dfrac{\triangle ABC}{\triangle BCD} =$ —?—

10. $\dfrac{\triangle ABC}{\triangle ABD} =$ —?—

11. $\dfrac{\triangle BCD}{\triangle ABD} =$ —?—

12.



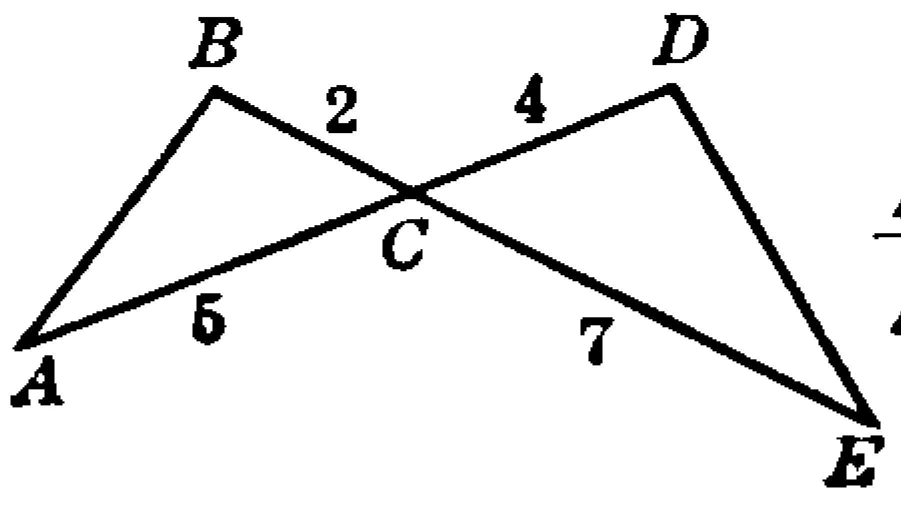
$\dfrac{X}{T} =$ —?—

13. $\dfrac{T}{Y} =$ —?—

14. $\dfrac{X}{Y} =$ —?—

Hint: Eliminate T from Nos. 12 and 13.

15.



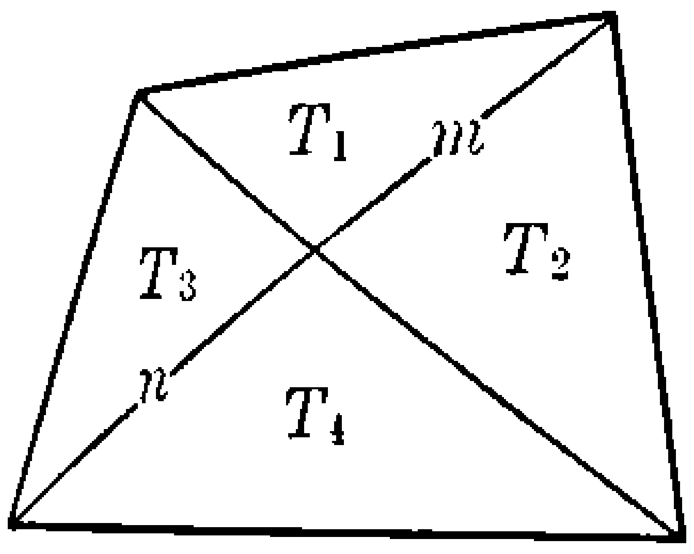
$\dfrac{\triangle ABC}{\triangle CDE} =$ —?—

16. (*a*) Draw the figure in Ex. 15 so that the two triangles are apart from each other. Mark the equal angles.

(*b*) Complete the proportion as in Ex. 15 but use the names of the line segments instead of their numerical values.

SUPERIOR WORK

S1.

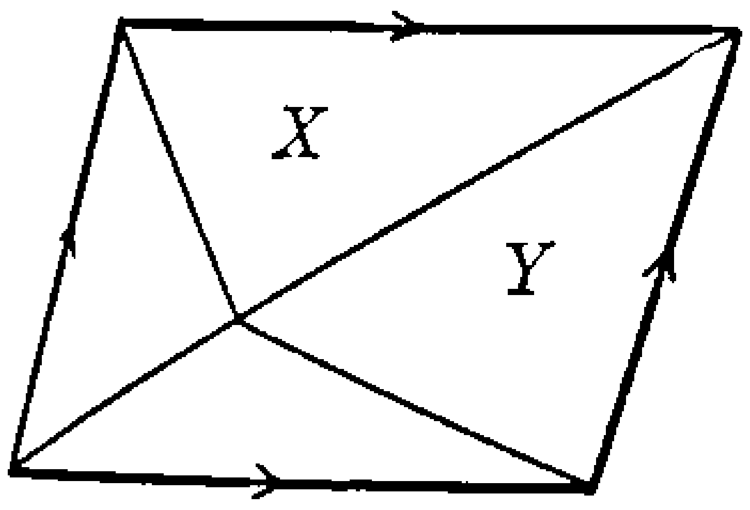


Prove:

(*a*) $\frac{T_1}{T_3} = \frac{T_2}{T_4}$

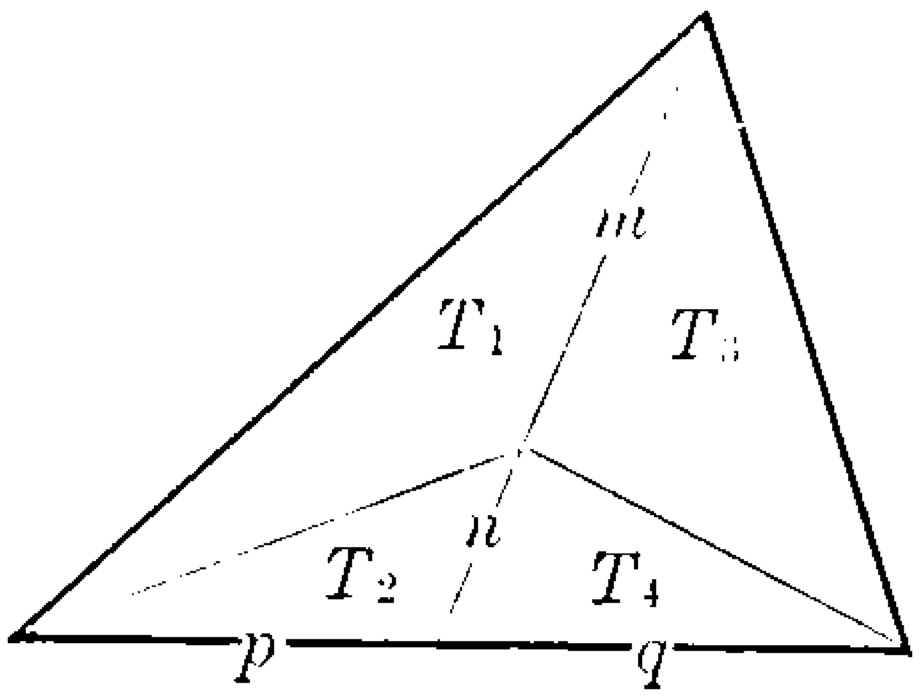
(*b*) $\frac{T_1 + T_2}{T_3 + T_4} = \frac{m}{n}$

S2.



Prove:
$\triangle X = \triangle Y$
in the ▱

S3.



Prove:

(*a*) $\frac{T_1}{T_2} = \frac{T_3}{T_4}$

(*b*) $\frac{T_1 + T_3}{T_2 + T_4} = \frac{m}{n}$

(*c*) $\frac{T_1}{T_3} = \frac{p}{q}$

S4. Theorem § 291 was stated as a static proposition. Complete the following dynamic form of it as a variation:

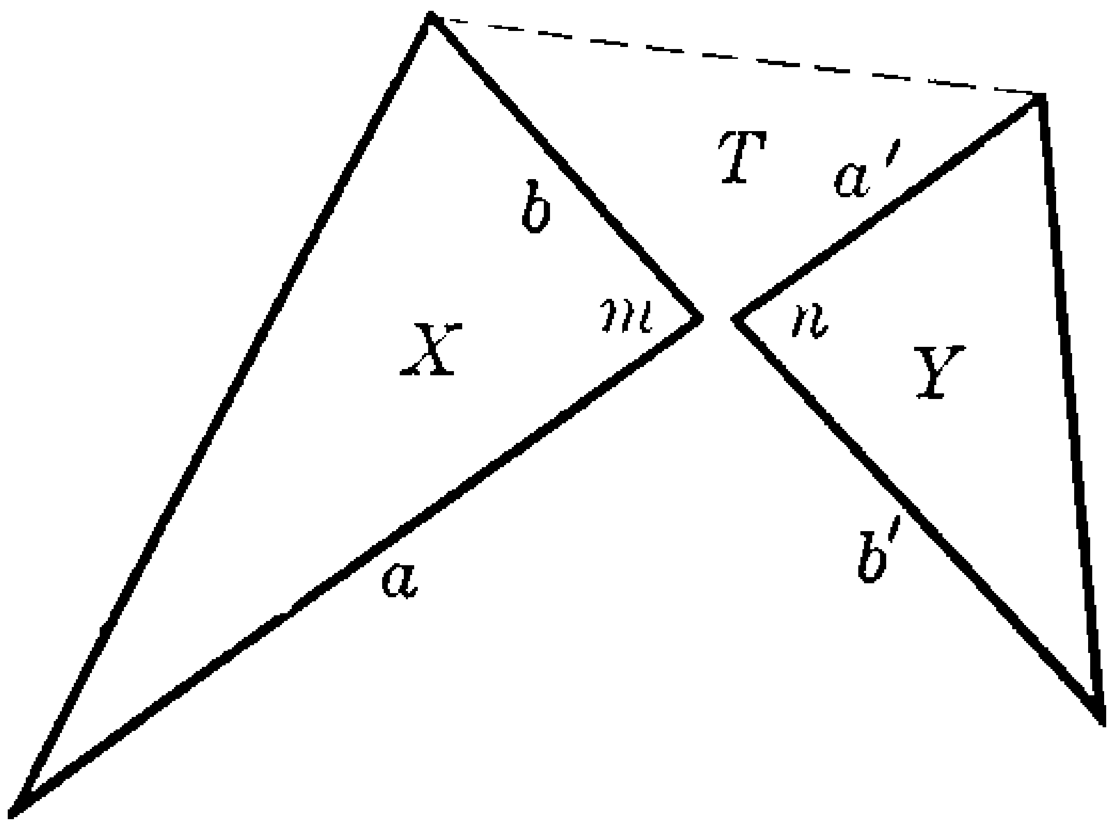
If an altitude of a triangle is constant, then . . . .

The equation for this variation is $T = kb$.

Triangles that Agree in an Angle

If two triangles agree in one angle, we shall prove that their areas are proportional to the *products* of the sides that form the equal angles.

292. THEOREM. If two triangles have an angle of one equal to an angle of the other, they are to each other as the products of the sides forming the equal angles.



Given: $\triangle X$ and $\triangle Y$ with vertical ∠s m and n

Prove: $\frac{X}{Y} = \frac{ab}{a'b'}$

Cons.: Draw the line forming $\triangle T$.

(We have placed the triangles so that the equal angles are vertical.)

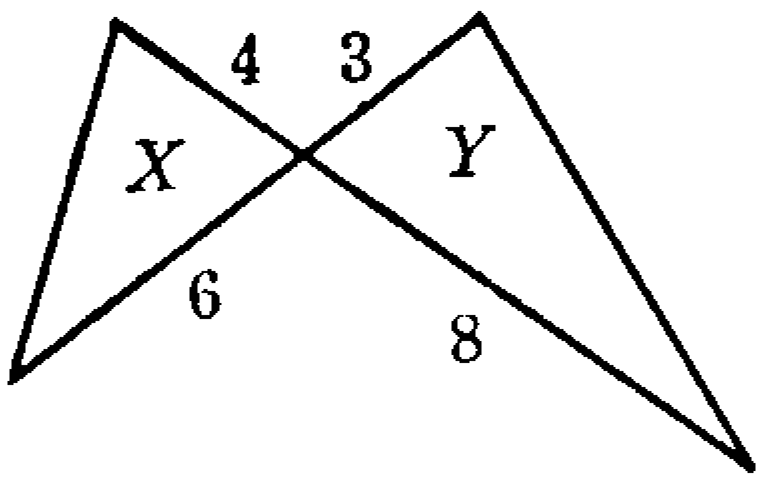
PROOF

Statements	*Reasons*
1. $\frac{X}{T} = \frac{a}{a'}$ $\frac{T}{Y} = \frac{b}{b'}$	1. If 2 △s have same alt., their areas are proportional to their bases (§ 291).
2. $\frac{X}{Y} = \frac{ab}{a'b'}$	2. Mult. Ax.

EXERCISES

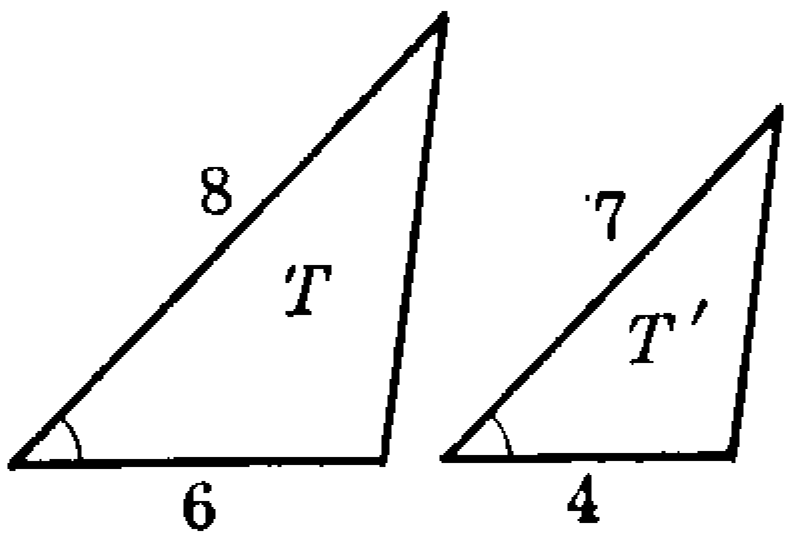
Complete the following proportions:

1.



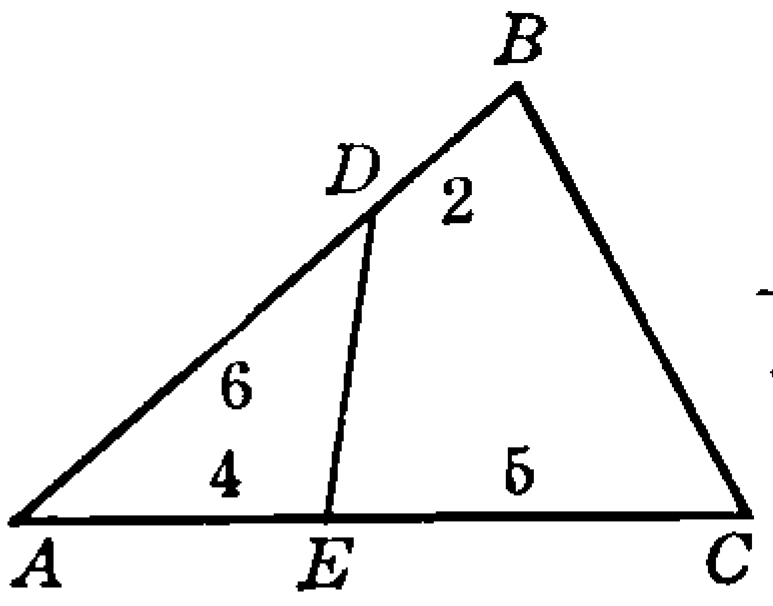
$\frac{X}{Y} = -?-$

2.



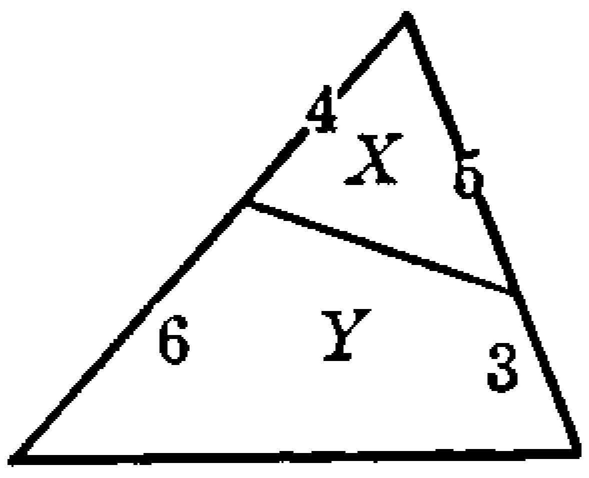
$\frac{T}{T'} = -?-$

3.



$\frac{\triangle ABC}{\triangle ADE} = -?-$

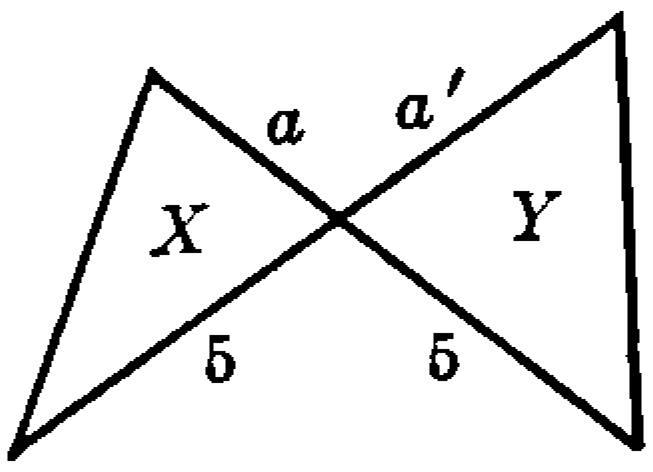
4.



(a) $\frac{X+Y}{X} = -?-$

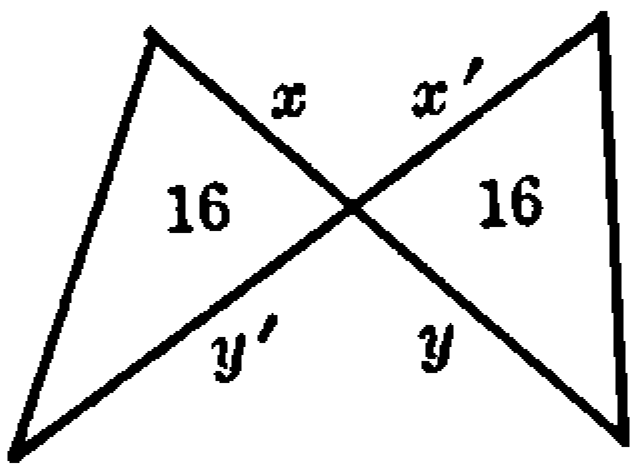
(b) $\frac{X}{Y} = -?-$

5.



$\frac{X}{Y} = -?-$

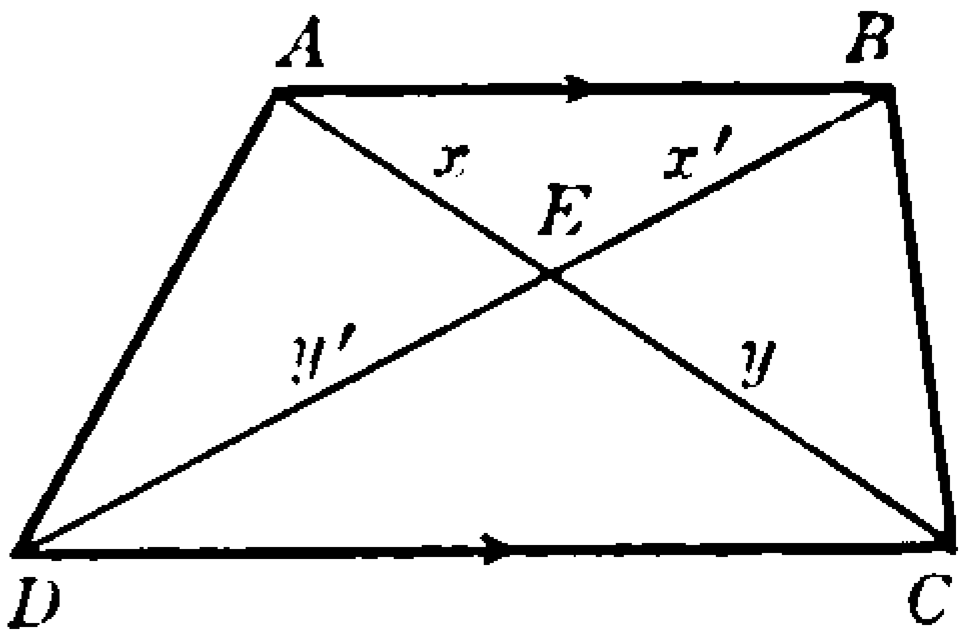
6.



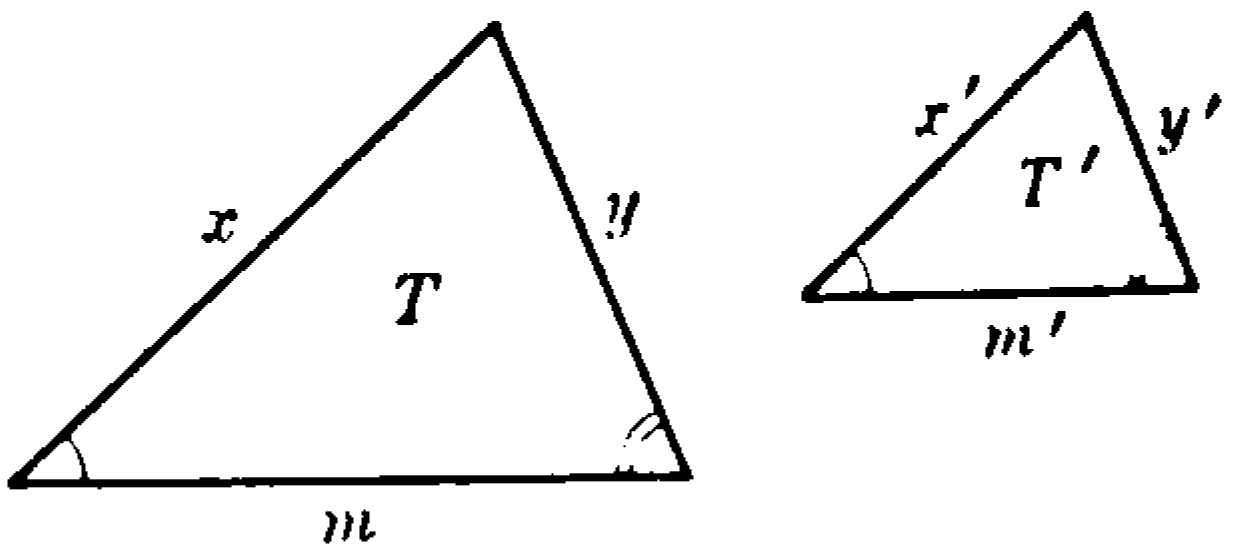
$\frac{x}{y} = -?-$

SUPERIOR WORK

S1. Prove: The diagonals of a trapezoid cut each other into four segments that are proportional. *Hint:* In Ex. 8, page 179, you proved $\triangle AED = \triangle BEC$.



S2. (*a*) Complete the following proportions: $\frac{xm}{x'm'} = -?-$ $\frac{ym}{y'm'} = -?-$

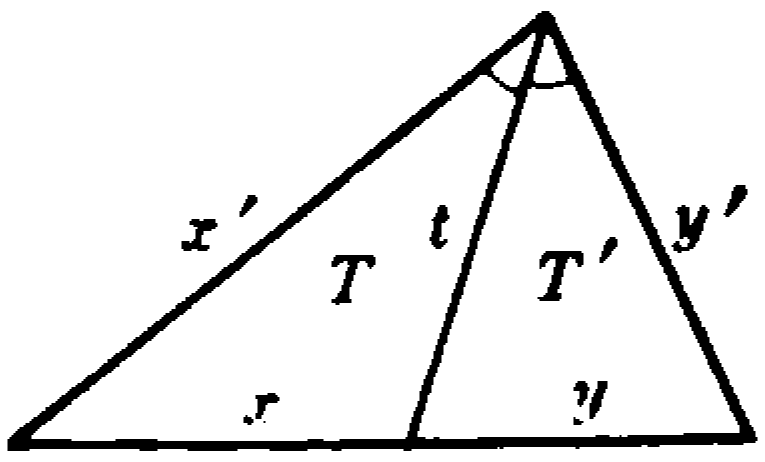


(*b*) Now derive a proportion connecting x, y, x', and y'.

(*c*) State this new proportion as a general proposition about the triangles.

S3. (*a*) From the figure, complete the following proportions:

$\frac{x}{y} = -?-$ $\frac{x't}{y't} = -?-$



(*b*) From these derive a proportion connecting x, y, x', and y'.

You have now proved the following theorem:

Theorem. **The bisector of an angle of a triangle divides the opposite side into segments proportional to the other two sides.**

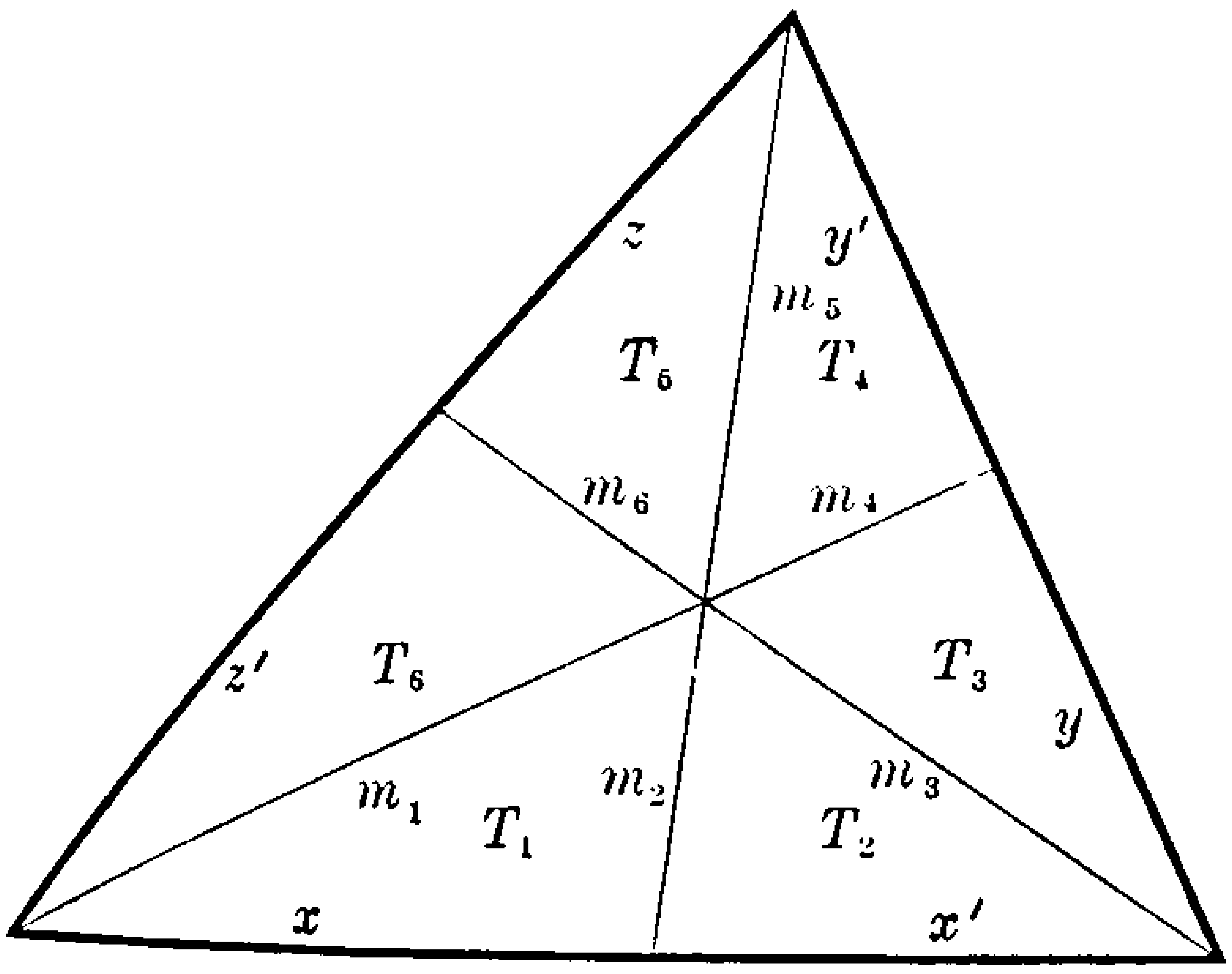
S4. The sides of a triangle are 8, 12, and 15 units. Find the segments on the 15-unit side made by the bisector of the opposite angle. *Hint:* Let one segment $= x$, then the other segment $= 15 - x$.

S5. If a median of a triangle is also an angle bisector, then the triangle is isosceles. Prove this by using the theorem in S3.

S6. *Ceva's Theorem.* If three lines from the vertices of a triangle pass through a common point within the triangle, then the six segments cut on the sides of the triangle are such that the product of three alternate segments equals the product of the other three.

Given: Three concurrent lines (lines passing through the same point) from the vertices

Prove: $xyz = x'y'z'$



Plan of Proof. The figure is divided into six triangles. Some of these triangles agree in an altitude and some agree in an angle. We therefore use § 291 and § 292.

PROOF

Statements	*Reasons*
1. $\frac{T_1}{T_2} = \frac{x}{x'}$ $\frac{T_3}{T_4} = \frac{y}{y'}$ $\frac{T_5}{T_6} = \frac{z}{z'}$	1. If 2 △ have same alt., they are proportional to their bases (§ 291).
2. $\frac{T_2}{T_5} = \frac{m_2m_3}{m_5m_6}$ $\frac{T_4}{T_1} = \frac{m_4m_5}{m_1m_2}$ $\frac{T_6}{T_3} = \frac{m_6m_1}{m_3m_4}$	2. If 2 △ agree in one ∠, they are proportional to products of sides forming the = ∠ (§ 292).
3. $1 = \frac{xyz}{x'y'z'}$ (Or, $xyz = x'y'z'$.	3. Mult. Ax. (Multiply the six equations.)

Using Ceva's Theorem, prove that if two medians of a triangle meet, then a line through the point of intersection and the third vertex will be the third median.

S7. Using Ceva's Theorem, prove that two lines drawn through any point in the median of a triangle and the other two vertices will divide the sides of the triangle proportionally.

S8. The theorem about two triangles that agree in one angle was stated as a proportion and is therefore in static form. Complete the following dynamic form of this theorem as a variation:

If one angle of a triangle is constant, then the triangle . . . .

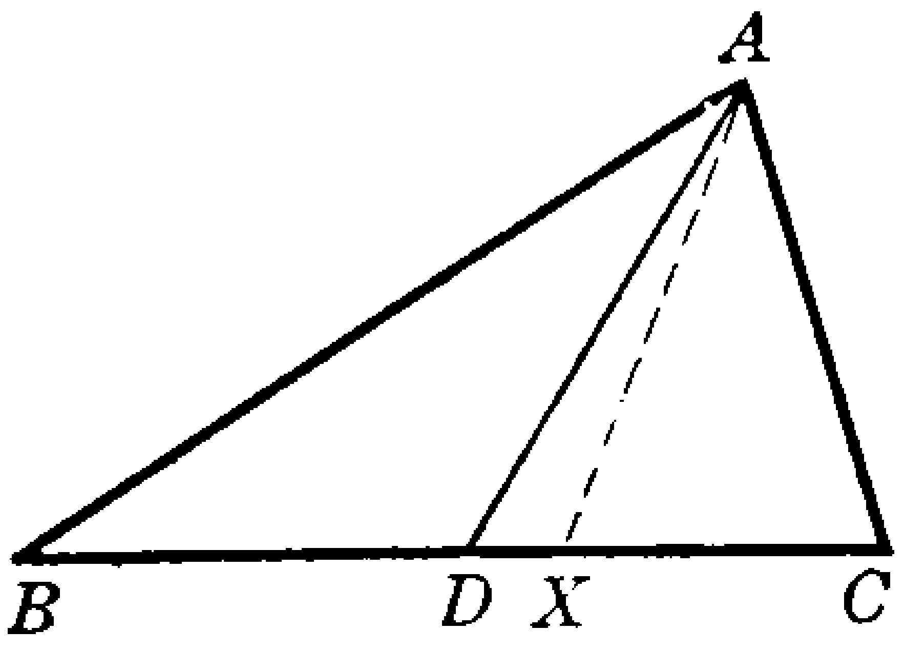
The equation for this variation is $T = kab$.

HONOR WORK

H1. *Converse Theorem.* If a line from the vertex of a triangle divides the opposite side into segments that are proportional to the other two sides of the triangle, then it bisects the angle of the triangle.

Given: $\frac{BD}{DC} = \frac{BA}{AC}$

Prove: AD bisects $\angle A$



PROOF

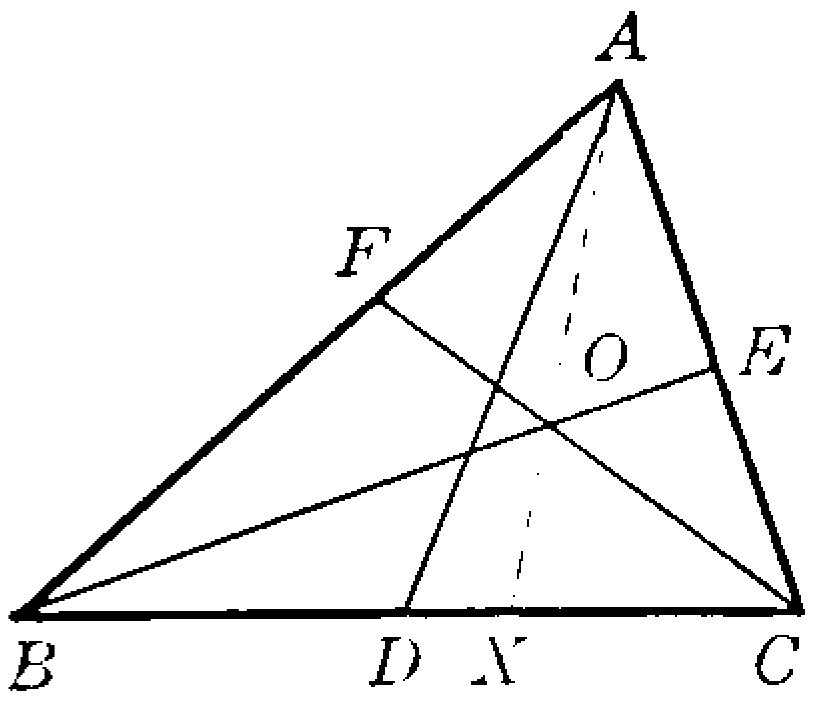
1. Either AD bisects $\angle A$ or not	
2. *Assume* that AD does not bisect $\angle A$, then draw *another* line AX to bisect $\angle A$	
3. $\frac{BX}{XC} = \frac{BA}{AC}$	3. ∠ bisector of △ divides opp. side into segments proportional to other sides (page 192, S3).

4. $\frac{BD}{DC} = \frac{BA}{AC}$	4. Given.
5. $\frac{BX}{XC} = \frac{BD}{DC}$	5. Subs. Ax.
6. $\frac{BC}{BX} = \frac{BC}{BD}$	6. Addition Transformation (§ 286).
7. $BC = BC$	7. Identity.
8. $BX = BD$ or X coincides with D	8. Div. Ax.
9. Hence, AX and AD are not separate lines, and AD bisects $\angle A$	9. Only one st. line can be drawn through 2 pts. (§ 11).

Converse of Ceva's Theorem. If three lines from the vertices of a triangle cut six segments on the sides, such that the product of three alternate segments equals the product of the other three, then the lines pass through a common point.

Given: $BD \times CE \times AF = DC \times EA \times FB$

Prove: AD, BE, and CF meet in a point



Let O be the intersection of BE and CF. Assume that AD does not pass through O. Then draw *another* line AX through O. Following the method above, show that AX and AD cannot be separate lines.

H2. Using the converse of Ceva's Theorem, prove that the medians of a triangle are concurrent.

H3. If a circle is inscribed in a triangle, then the lines joining the points of tangency to the opposite vertices are concurrent.

H4. Using the proportional segments made by the angle bisectors of a triangle, prove that the angle bisectors are concurrent.

SUMMARY

293. The size of many things can be told only by using several measures, called *dimensions.* The size of a surface, called its *area,* depends on two dimensions. The unit of area is a square whose side is the unit of length. It is assumed that the number of unit squares contained in a rectangle is the product of the number of linear units in its length and width.

If a trapezoid and two rectangles all have equal altitudes, and the bases of the rectangles are equal respectively to the bases of the trapezoid, then the trapezoid is the average of the two rectangles. This also means that the area of the trapezoid equals the average of its bases times the altitude.

In a dynamic sense, a trapezoid becomes a parallelogram when the two bases become equal. In that case, the average of the two bases is the same as either base alone. Therefore, the area of a parallelogram is the base times its altitude.

In a dynamic sense, again, a trapezoid becomes a triangle when one base becomes zero. Therefore, the area of a triangle is one-half the base times the altitude.

Proportion is a convenient way of comparing dimensional magnitudes. If the ratio (quotient) of two quantities equals the ratio of two other quantities, the four quantities are said to be in proportion, or they are said to be proportional. In any proportion the product of the means (the two middle terms) equals the product of the extremes (the two end terms). Conversely, if the product of one pair of numbers equals the product of another pair of numbers, either pair may be made the means of a proportion in which the other pair are the extremes.

Any two proportions that lead to the same equation between the product of the means and the product of the extremes are equally valid. In particular, we may invert the terms of a proportion, we may alternate them, or we may replace each numerator (or denominator) by the sum (or the difference) of the numerator and its denominator. In a series of equal ratios, the sum of any number of numerators is to the sum of their denominators as any numerator is to its denominator.

If the means of a proportion are equal, each mean is called the mean proportional between the extremes.

If two triangles agree in an altitude, they are proportional to the corresponding bases. If two triangles agree in an angle, they are proportional to the products of the sides forming the equal angles.

RECOGNIZING NEW TERMS

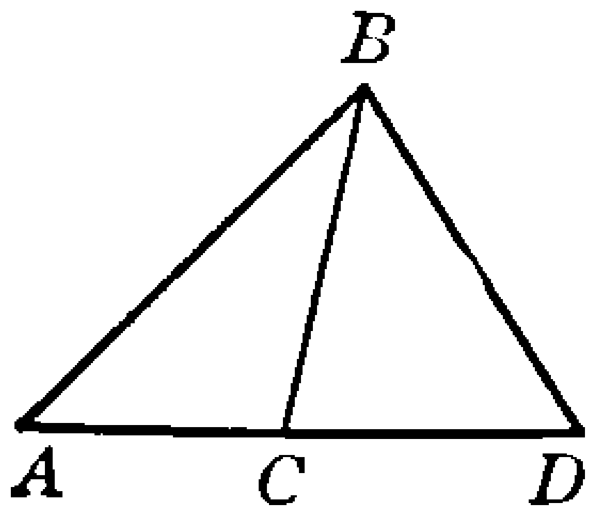
Can you describe each word or term below? If not, go back to the section indicated.

dimension	§ 259
area	§ 261
significant digits	§ 277
proportion	§ 279
means, extremes	§ 280
mean proportional	§ 283
transformation of proportion	§ 284
inversion	§ 285
alternation	§ 285
addition	§ 286
subtraction	§ 287
transforming a geometric figure	§ 275

REVIEW EXERCISES

1. A quadrilateral is formed by a diameter of a circle and three tangents to the circle. Two of the tangents pass through the ends of the diameter. Prove that the area of the quadrilateral equals half the diameter times the side opposite the diameter.
2. The side of an equilateral triangle is three times as long as the side of another equilateral triangle whose area is one acre. What is the area of the larger triangle (§ 292)?
3. (*a*) Since the triangles ABC and ABD have $\angle A$ in common, complete the proportion:

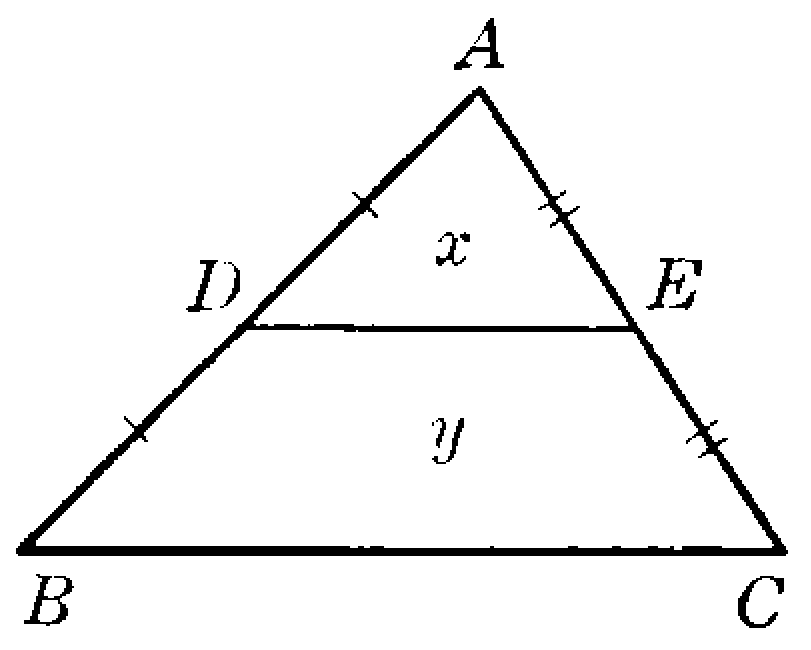
$$\frac{\triangle ABC}{\triangle ABD} = \text{—?—}$$



(*b*) Since the same triangles have a common altitude from the vertex B, complete the proportion:

$$\frac{\triangle ABC}{\triangle ABD} = \text{—?—}$$

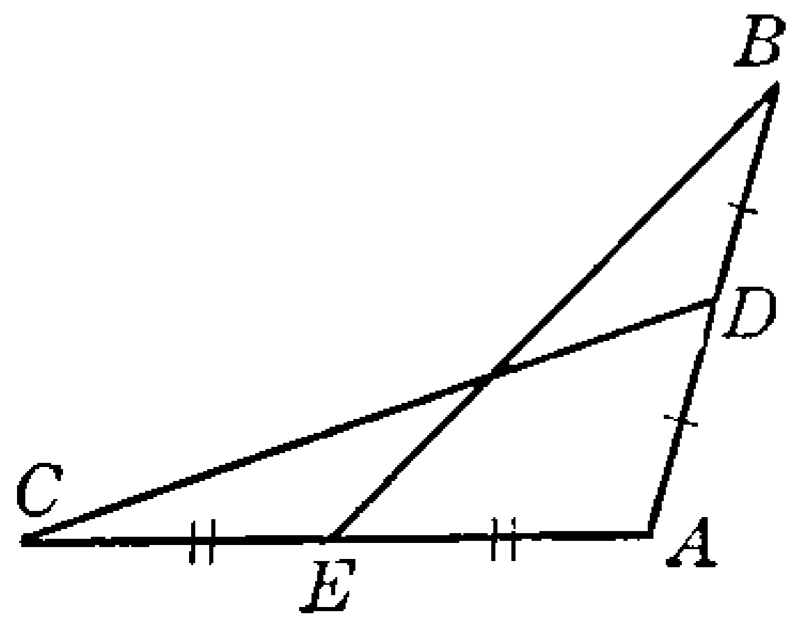
4. In the figure, $AD = DB$, and $AE = EC$. $x = \triangle ADE$ and $y = \text{quad. } DECB$



Find the numerical values of the ratios:

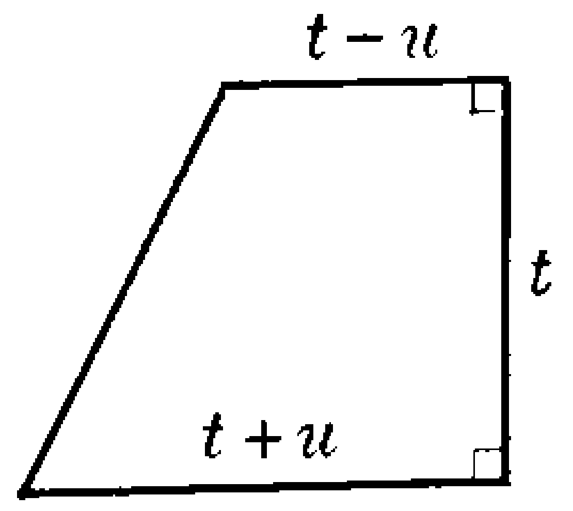
$$\frac{x+y}{x} \quad \text{and} \quad \frac{x}{y}$$

5.



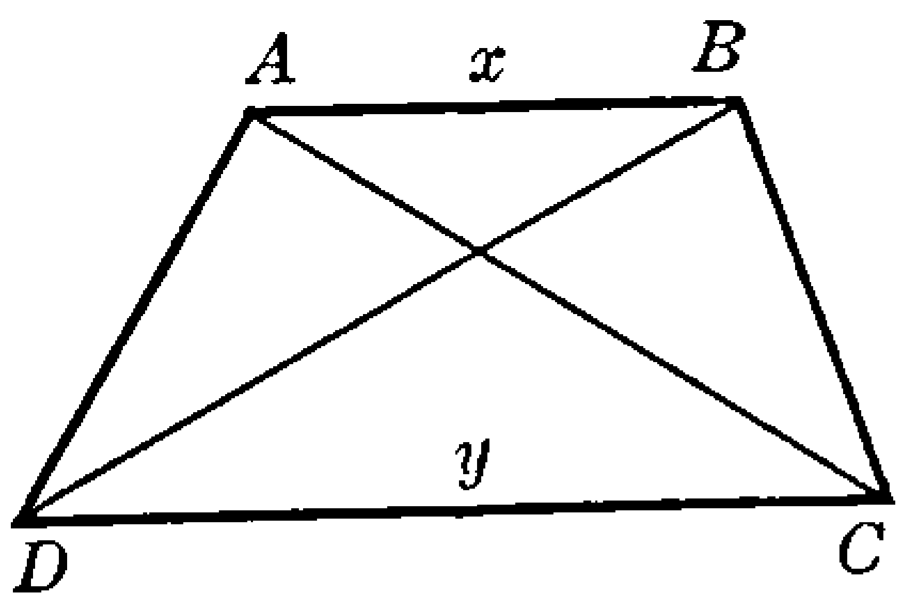
Given:
$BD = DA$
$AE = EC$
Prove:
$\triangle ABE = \triangle ACD$

6.



Find the area of the trapezoid.

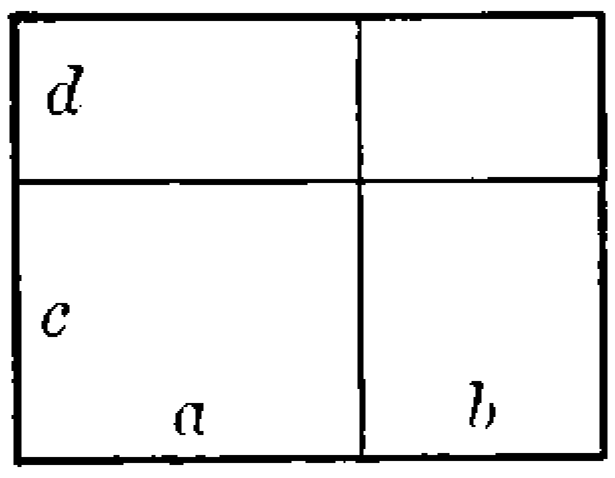
7.



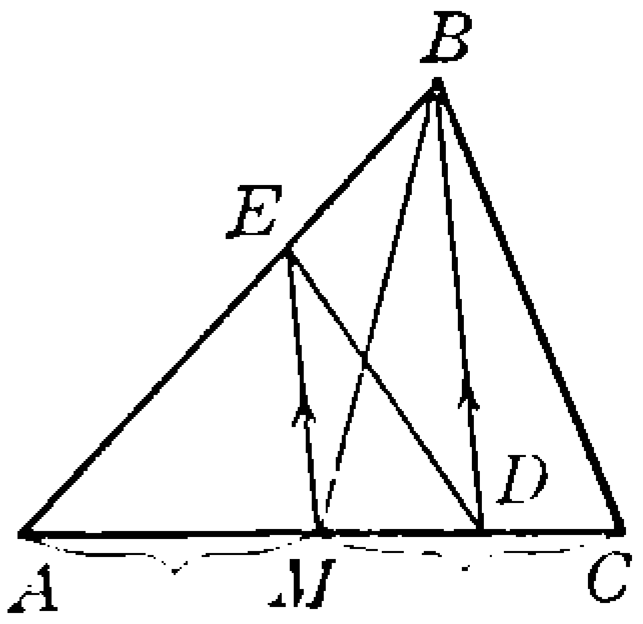
Given:
$\triangle ADC = \triangle BDC$
Prove:
$AB \parallel DC$

8. By expressing the area of the triangle twice with different sides used as the base, prove that if two altitudes of a triangle are equal, the triangle is isosceles.

9. By finding the area of each rectangle in the figure, prove the algebraic method of multiplying $(a + b)(c + d)$.



10.



Given:
In $\triangle ABC$
BM a median
$BD \parallel ME$
Prove:
$\triangle ADE = \frac{1}{2}\triangle ABC$

SUPERIOR WORK

S1. Are the following statements true or false?

(*a*) If x increases as y increases, then x varies as y.

(*b*) The converse of (*a*)

(*c*) The statement that "x is proportional to y" refers to a static situation.

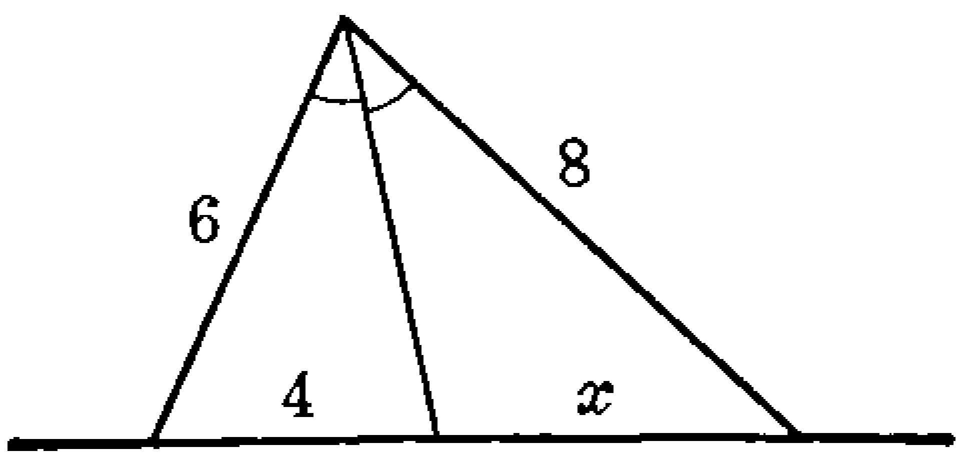
Complete the following statements:

(*d*) If the base of a triangle is constant, the area varies as —?—.

(*e*) If one angle of a triangle is constant and the area of the triangle is also constant, then . . . .

(*f*) If one angle of a triangle is constant and a side adjacent to that angle is constant, then . . . .

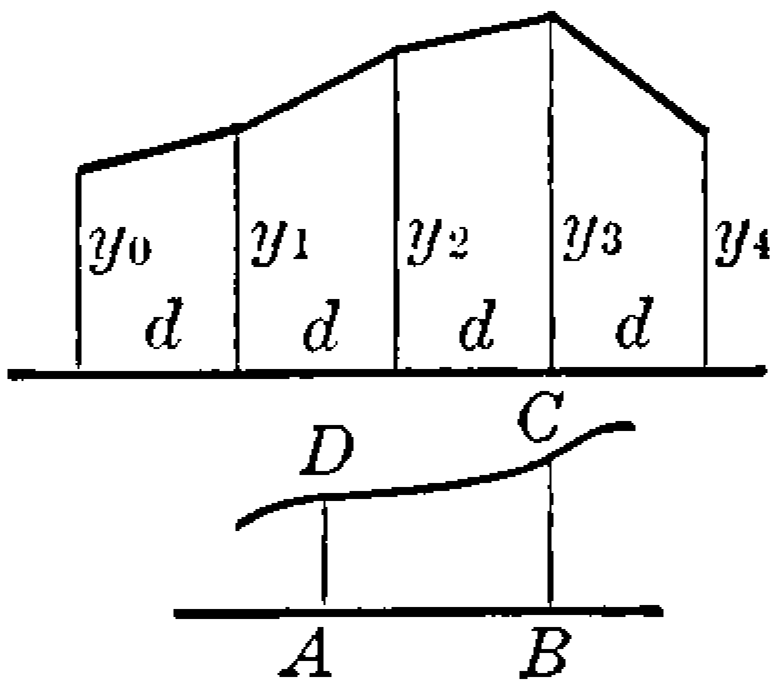
(*g*) In the figure, x = —?—.



(*h*) In the figure, if x were changed to 4, how long would the side marked 8 have to be? Could you have reasoned this out from congruent triangles?

S2. Prove that the sum of the trapezoids in the figure is

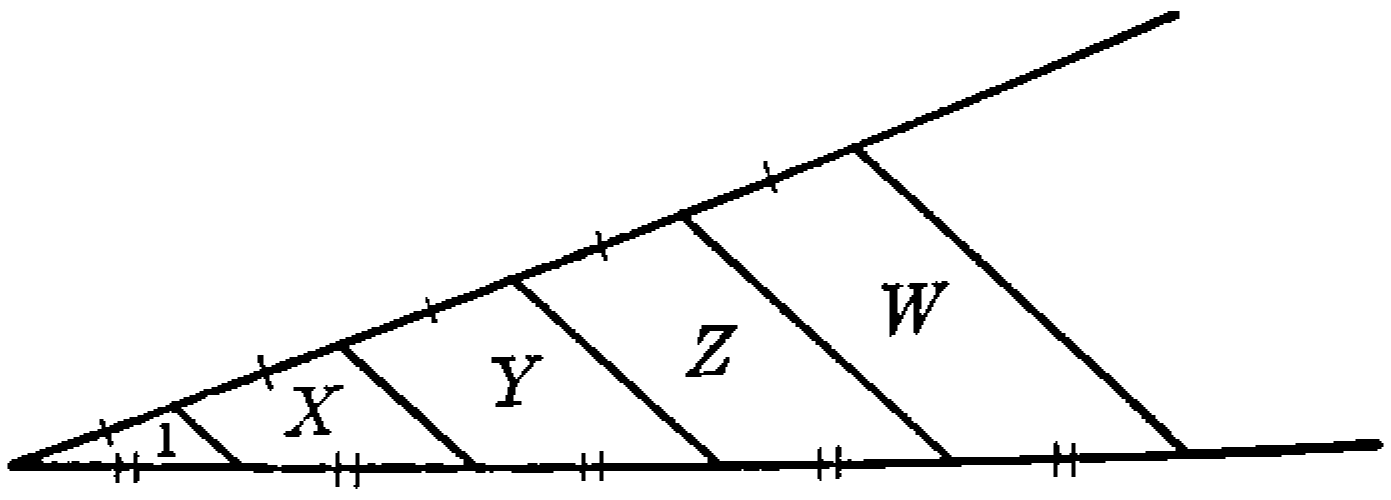
$$d\left(\frac{y_0 + y_4}{2} + y_1 + y_2 + y_3\right)$$



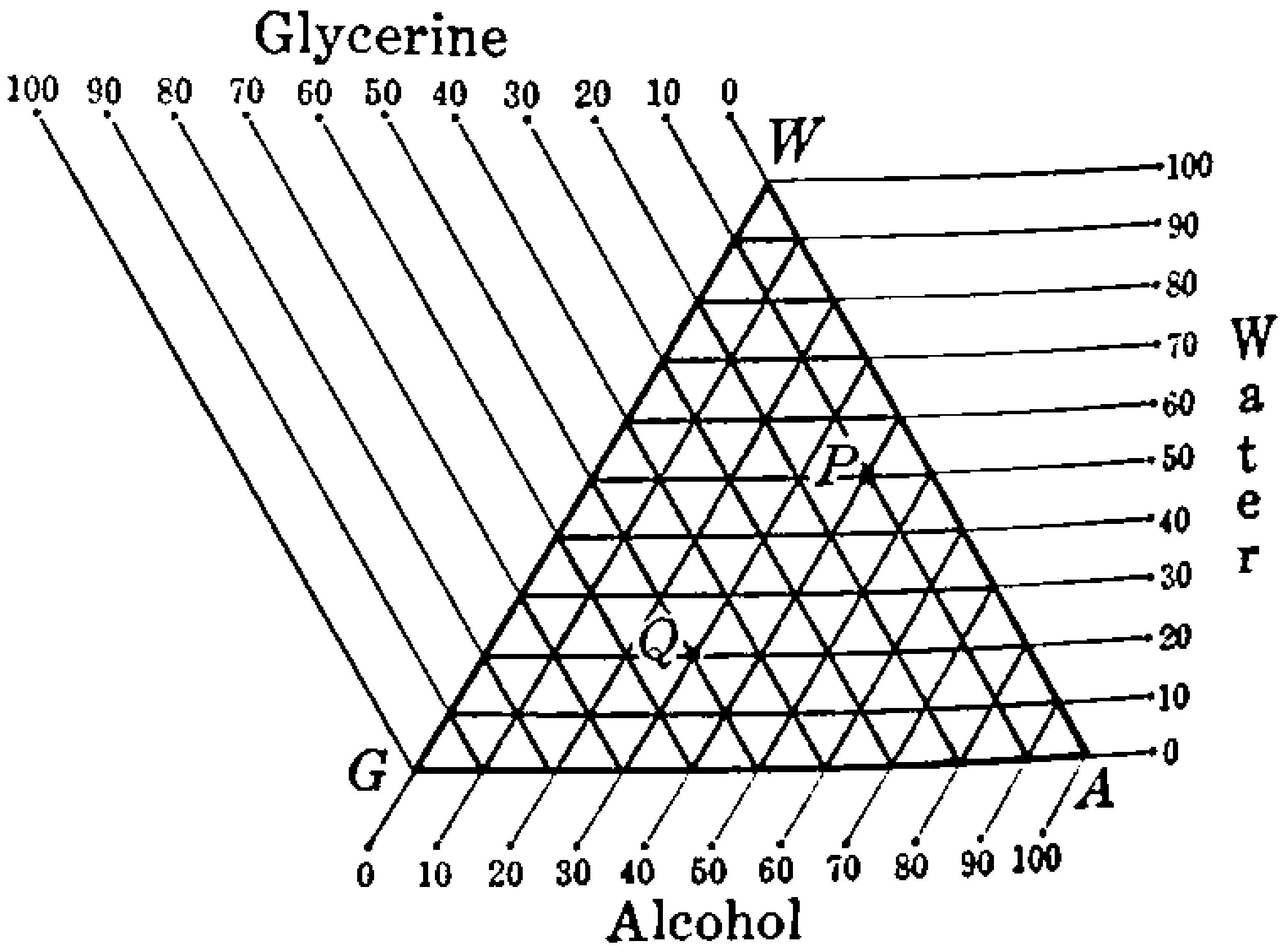
Can you write the formula for n trapezoids?

Note: This is known as the *trapezoidal rule.* It is often used to approximate the area $ABCD$ under a curve. AB is divided into a convenient number of equal parts and the perpendiculars, called offsets, are measured.

S3. In the figure, equal distances were laid off on each side of the angle. If the area of the small triangle is 1 unit find the areas of the successive quadrilaterals X, Y, Z, $\cdots$



S4. (*a*) The point P in the figure represents a solution that is 50% water, 40% alcohol, and 10% glycerine. What solution does Q represent?



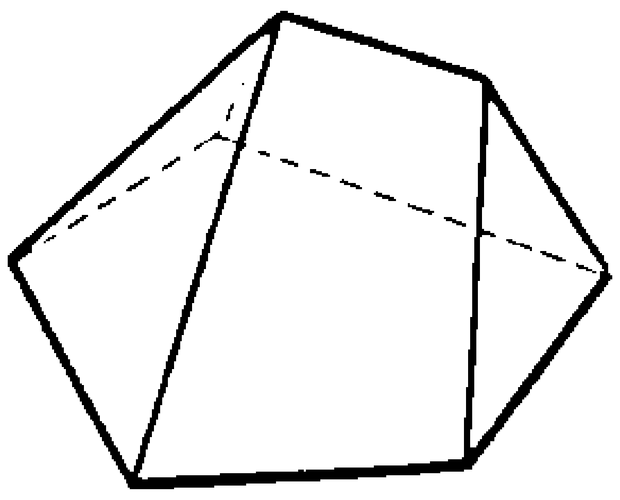
(*b*) Why do these "tri-co-ordinate" axes work? (Page 180, Ex. 11.)

(*c*) What kind of solutions would be represented by a line parallel to AW? by the bisector of $\angle W$? by the intersection of two altitudes?

Three Dimensions

S5. A *polyhedron* is a solid bounded by plane polygons.

The bounding polygons are called the *faces*, the intersections of the faces are called the *edges*, and the vertices of the polygons are also the vertices of the polyhedron.



Polyhedron

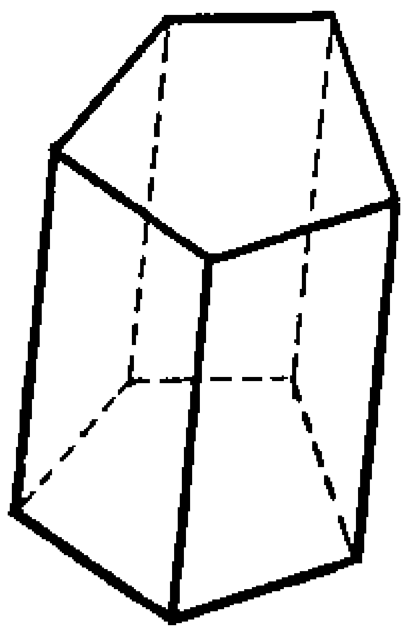
(*a*) What is the smallest number of faces that a polyhedron may have? Such a polyhedron is called a *tetrahedron*. Draw a tetrahedron.

(*b*) How many edges and vertices does a tetrahedron have?

(*c*) What kind of figures are the faces of a tetrahedron?

S6. A *prism* is a polyhedron in which two faces are congruent and parallel, and the other faces are parallelograms.

The congruent and parallel faces are called the *bases;* the other faces are called the *lateral faces.*



Prism

(*a*) A *right prism* is one whose lateral edges are perpendicular to the bases. What kind of figures are the lateral faces of a right prism?

(*b*) Prisms are classified according to their bases. In a *triangular prism* the bases are triangles, in a *quadrangular prism* the bases are quadrilaterals, etc.

How many faces does a triangular prism have? a quadrangular prism?

(*c*) The *altitude* of a prism is a line segment between the bases of the prism and perpendicular to the bases. In a right prism, how does the altitude compare with the lateral edges?

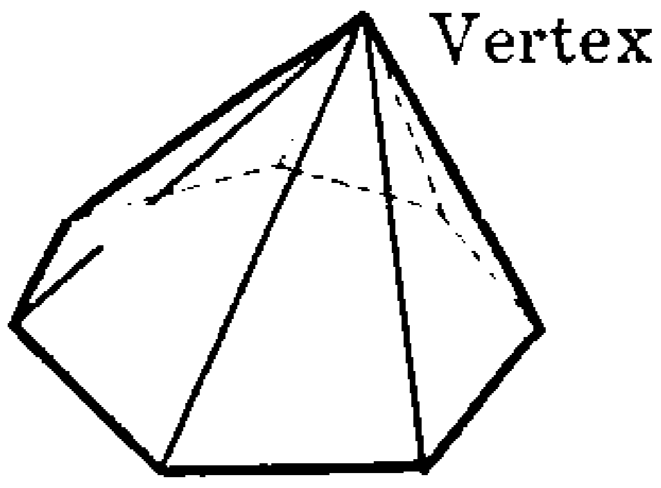
(*d*) A rectangular solid is a polyhedron in which all the faces are rectangles. How many faces does it have? Is it a prism?

S7. Find the lateral area of a right prism in which the bases are triangles with sides 3, 4, and 5 units, and the altitude of the prism is 7 units.

S8. Find the lateral area of a right prism in which the perimeter of the base is 12 units and the altitude is 7 units.

S9. The bases of a right prism are squares 5 units on a side, and the altitude is 8 units. Find the total area of the prism.

S10. A *pyramid* is a polyhedron in which one face is a polygon, called the *base*, and the other faces are triangles, called *lateral faces*, meeting in a common *vertex*.



Pyramid

Pyramids are classified according to their bases, such as triangular, quadrangular, etc.

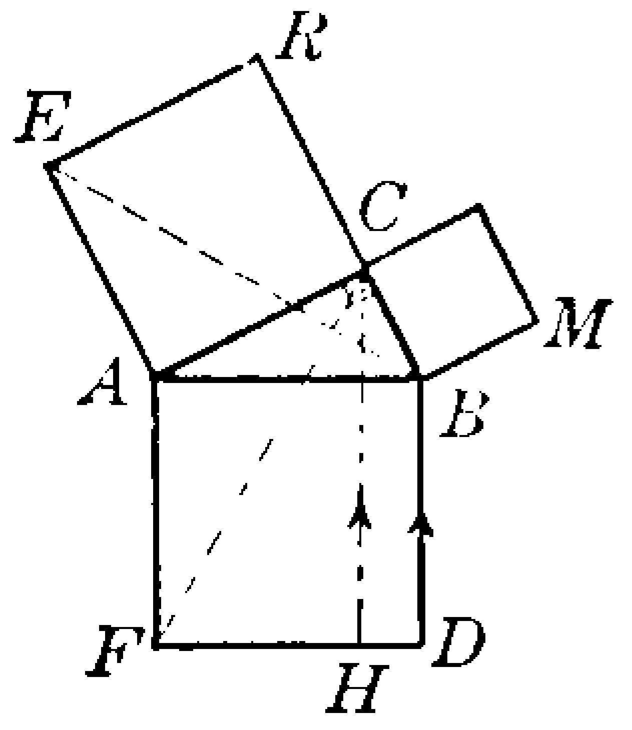
(*a*) What is another name for a triangular pyramid?

(*b*) Each lateral face of a pyramid is an isosceles triangle with 6 inches as the base and 8 inches as the altitude to the base. The base of the pyramid is a square 6 inches on each side. Find the total area.

(*c*) In a *regular pyramid* all the faces are isosceles triangles. If the altitude to the base of each triangle is 12 and the perimeter of the base of the pyramid is 35, find the lateral area of the pyramid.

HONOR WORK

H1. The following theorem, named after Pythagoras, is one of the most useful: The sum of the squares of the legs of a right triangle equals the square of the hypotenuse.



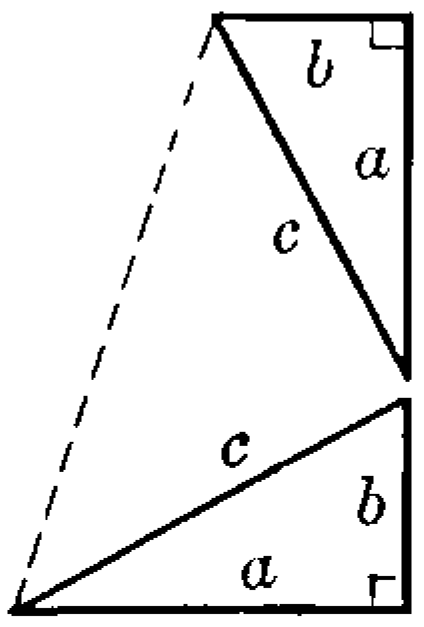
One way (although not the easiest) of proving this is to construct squares on the sides of the right triangle and prove that the sum of the areas of the two small squares equals the area of the large square. The outline of the proof follows. Write it out completely.

Because RCB is a straight line, square EC and $\triangle EAB$ have the same base and equal altitudes.

$$\begin{aligned} \text{Sq. } EC &= 2 \cdot \triangle EAB. \\ \text{Similarly, } \square AH &= 2 \cdot \triangle CAF. \\ \triangle EAB &\cong \triangle CAF. \\ \text{Sq. } EC &= \square AH. \end{aligned}$$

In the same way, Sq. $CM = \square BH$.

H2. An interesting and simple proof of the Pythagorean Theorem was given by President James A. Garfield. He placed two identical right triangles as shown in the figure. This forms a trapezoid whose bases are a and b and whose altitude is $a + b$. It can easily be shown that the trapezoid is made up of three right triangles. Find the area of these right triangles and add them. This is the same as the area of the trapezoid when found directly from its bases and altitude. The equation then reduces to $a^2 + b^2 = c^2$.



Invariants in a Changing World

It is often important to determine what remains unchanged in a changing situation. In the case of a motorist traveling at uniform speed, the ratio of his changing distance to his changing time remains invariant (unchanged). If a gas expands under constant temperature, the product of the volume and the pressure remains invariant (page 188, S10). Can you give other examples of invariants in the world of science?

We often wonder what is invariant in an ever-changing world. Some people have no hope of finding any constant. Others feel that such constants exist and search for them. Still others believe that they have found these invariants. Do you know of any?

H3. (*a*) Our form of government has undergone a number of changes. What are the important invariants throughout these changes?

(*b*) What proposition about the angles remains invariant as two given lines are cut by various transversals?

(*c*) What are some important invariants that you would like to preserve in your own life?

H4. Can you tell the difference between propaganda and education by recognizing the invariants of each?

Is it true that the invariant in propaganda is a total disregard of the means and an all-important concern for the end in view? Is propaganda intended to make people think a certain way? Is the information that is given or withheld not important in itself but only if it helps people reach a desired conclusion?

Is it true, on the other hand, that the invariant in education is a jealous regard for the means and an impartial concern for the end—except that it shall be the truth? Is it true that the object of education is to enable people to arrive at the truth (insofar as it is possible for man to know the truth) through sound and independent thinking?

(*a*) State the purpose of propaganda and the purpose of education.

(*b*) Which of these purposes can be achieved in less time?

(*c*) Is the purpose of propaganda always harmful?

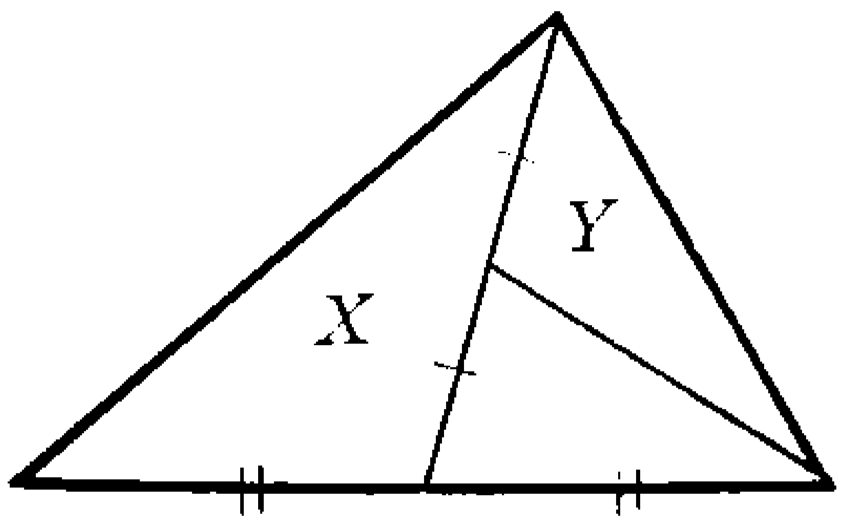
(*d*) What is a half-truth? Do half-truths have any place in education? In propaganda?

(*e*) What is meant by a "jealous regard" for the means in education? What is meant by saying that we are "jealous of our American rights as individuals"?

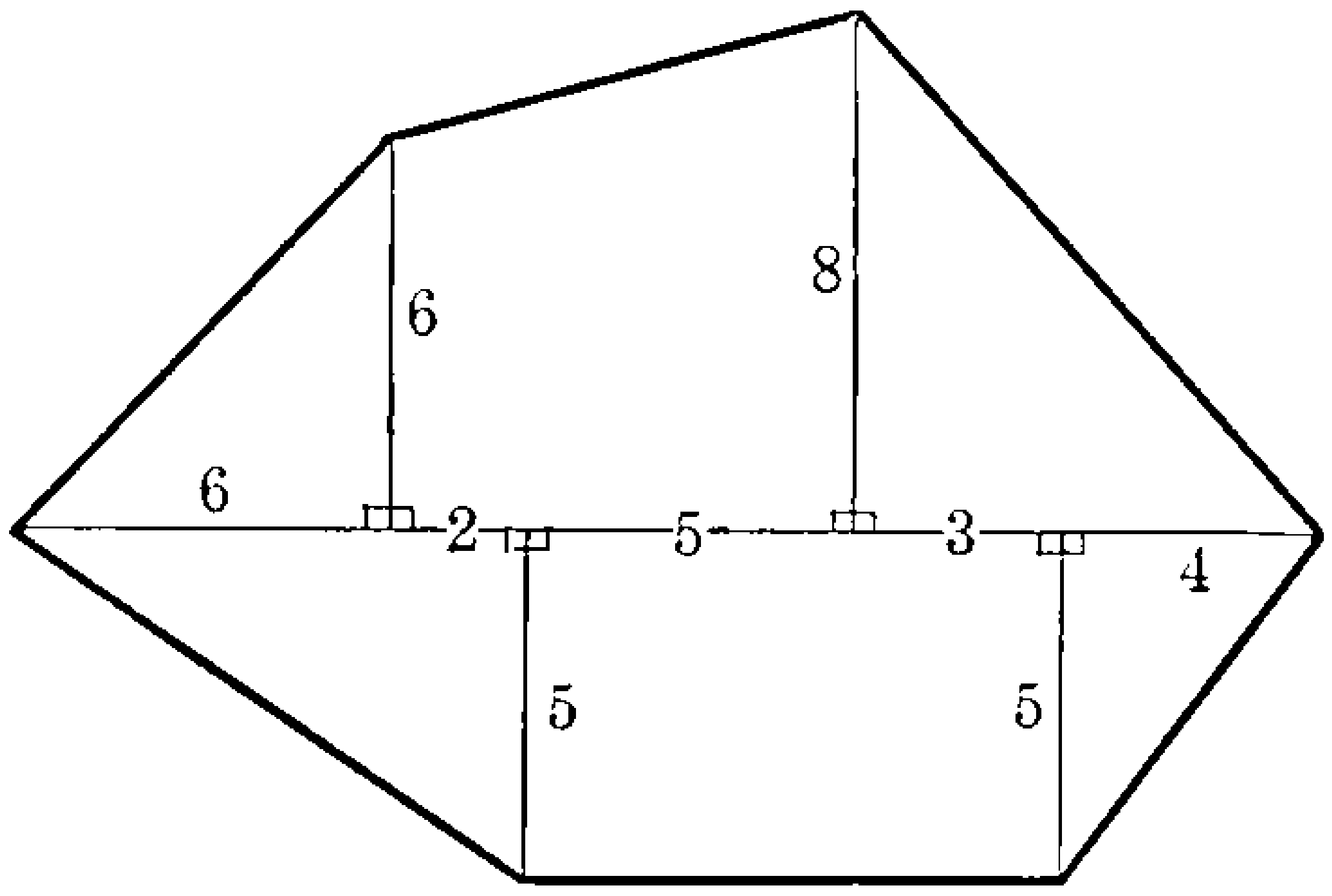
(*f*) Find an article that you think is propaganda and point out the invariants of propaganda that it contains. Tell how it differs from education.

TEST

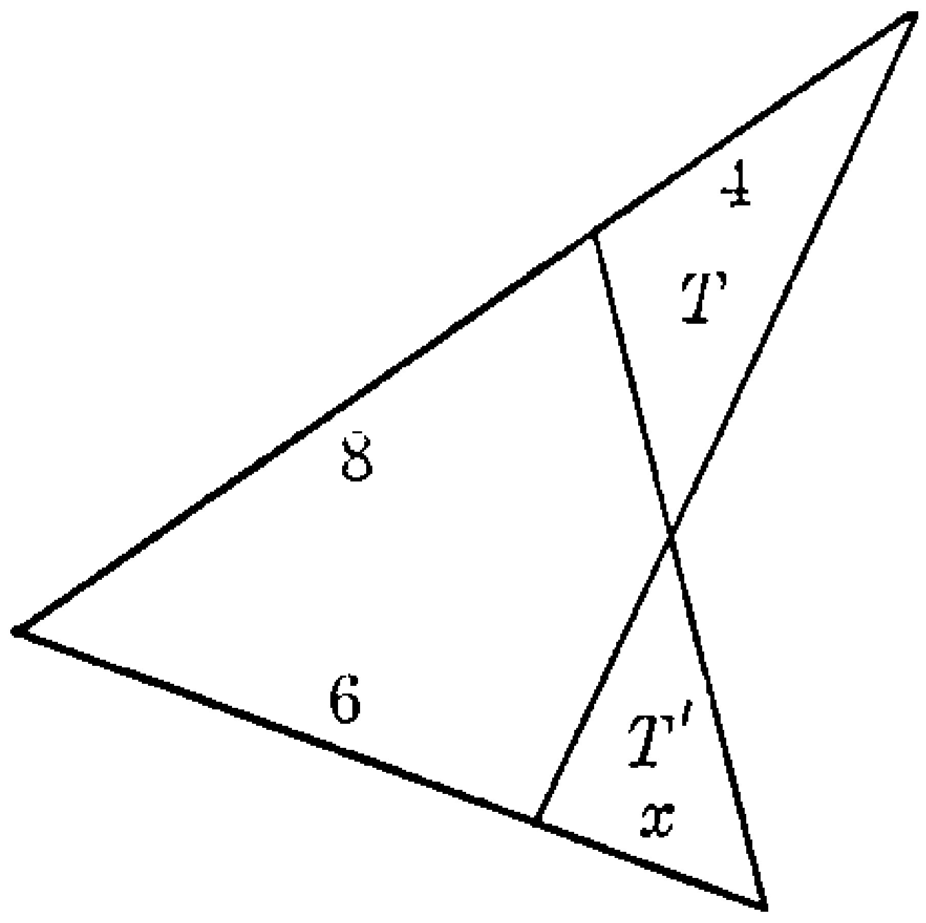
1. Alternate the proportion $\frac{x}{y} = \frac{3}{4}$, and then transform the new proportion by addition.
2. Write the simplest proportion that you can derive from: $\frac{x - 3}{y - 4} = \frac{6}{8}$
3. Write the simplest proportion that you can derive from: $\frac{a + b}{a - b} = \frac{c + d}{c - d}$
4. In the figure, the large triangle has a median that is bisected. What is the relationship between the triangles X and Y?



5. A rectangle 3 feet by 5 feet has a 6-inch border on the outside. Find the area of the border.
6. A polygon is circumscribed about a circle. The sides of the polygon are 6, 8, 9, 5, and 12 units. The area of the polygon is 100 square units. Find the radius of the circle.
7. Two cement walks cross each other to form a parallelogram whose sides are 8 feet and 6 feet. If the broader walk is 4 feet wide, how wide is the other walk?
8. Find the area of the polygon shown.



9. The sides of two squares are 12 feet and 7 feet. Find the side of the square whose area is the sum of these two squares.
10. In the figure, $T = T'$. Find the side x.



8

Analogy

Similar Triangles

PROPORTIONS FROM SIMILAR TRIANGLES

Comparison by Analogy

294. Complete the following comparisons:

1. Hammer is to nail as pencil is to —?—.
2. Friend is to foe as hero is to —?—.
3. Democracy is to dictatorship as freedom is to —?—.
4. Wholesaler is to retailer as retailer is to —?—.
5. 2 is to 10 as 3 is to —?—.

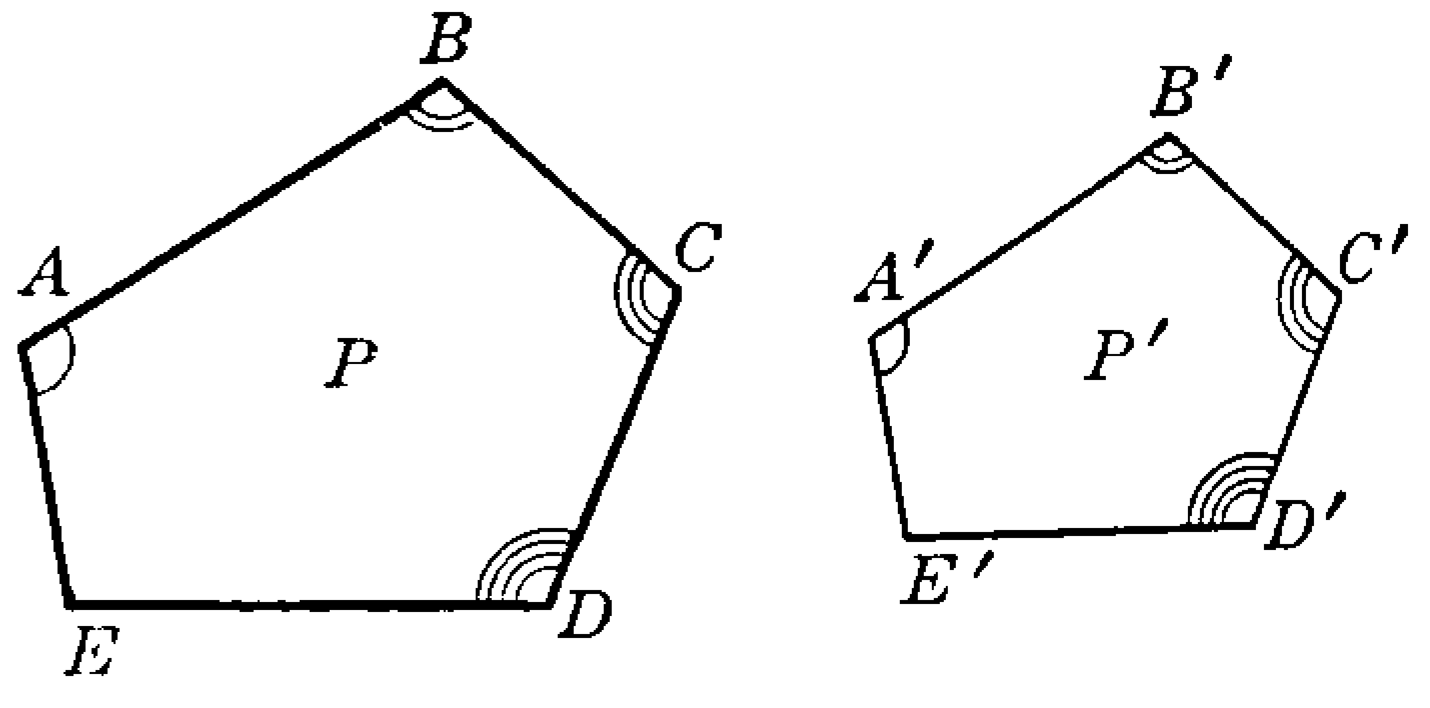
The first four comparisons illustrate what is meant by *analogy*. If two situations are alike in certain ways, then they are said to be analogous. The fifth comparison is a *mathematical analogy*. We recognize it as a proportion. In other words, the couple 2 and 10 is analogous or proportional to the couple 3 and 15. In mathematics, proportion is an important type of analogy.

Similar Polygons

295. Another important kind of analogy consists of two figures that have the same shape, like big John and little John. Two things are said to have the same shape if the parts of one are proportional to the parts of the other.

Two polygons that have the same shape are called *similar*. In similar polygons, then, the sides of one polygon are proportional to the sides of the other, and the same is true about the angles. It is easy to prove that if the angles of one polygon are proportional to the angles of another polygon, they are equal in pairs (since the sum of the angles of a polygon of a given number of sides is constant). We therefore have the following definition of similar polygons.

296. Definition. Two polygons are similar if the angles of one are equal respectively to the angles of the other, and the sides of one are proportional to the corresponding sides of the other.



Thus, to say that polygon $P \sim$ (is similar to) polygon P' means:

(1) $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$, $\cdots$

(2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{CD}{C'D'} = \cdots$

Again, a map is similar to the territory it represents because:

(1) The distances on the map are proportional to the actual distances on

the land. This makes it possible to have a scale showing how many miles are represented by an inch.

(2) The angles on the map are equal respectively to the angles on the land. This makes the directions on the map the same as the true directions on the land.

If the scale of a map is 1:64,000, how many miles are represented by one inch?

Similar Triangles

297. We have learned that two triangles are congruent if they satisfy three conditions of a certain type. Similarity is a weaker relationship than congruence. Two triangles are similar if they satisfy but two conditions of a certain kind. We shall now prove that if two triangles agree in two angles, they are similar.

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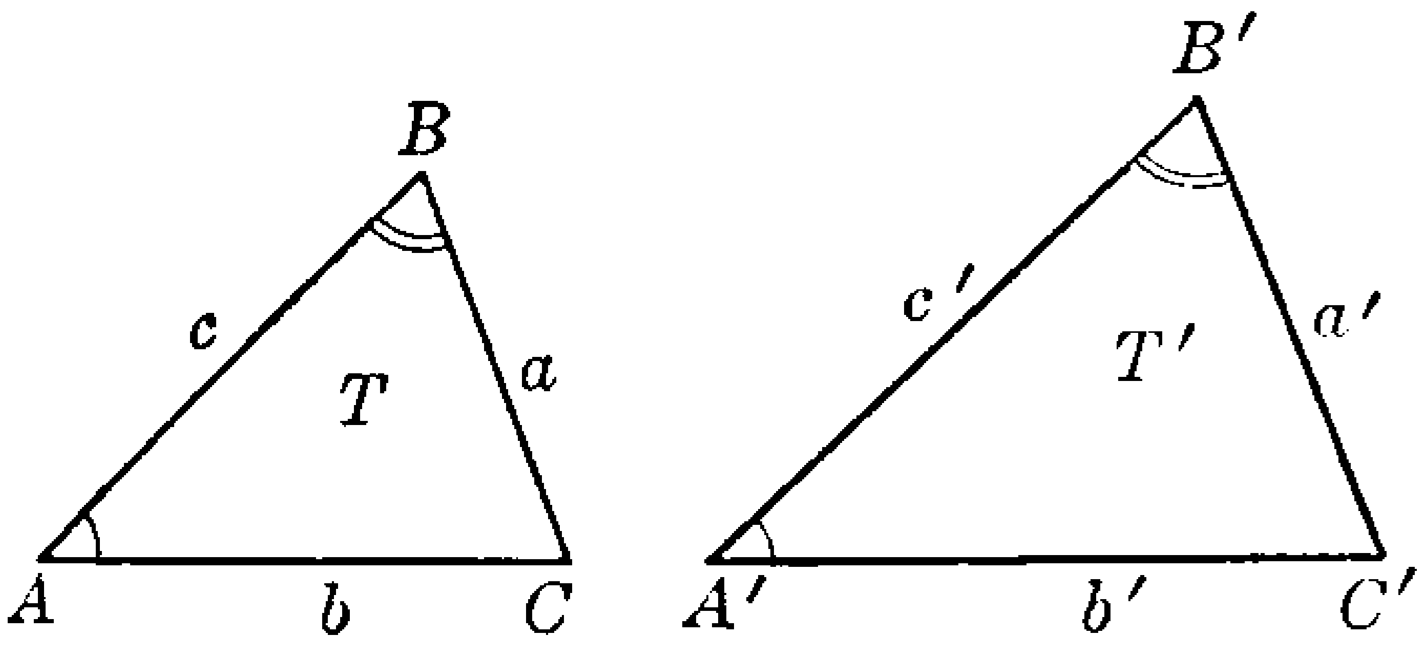
298. THEOREM. If two angles of one triangle are equal respectively to two angles of another triangle, the triangles are similar.



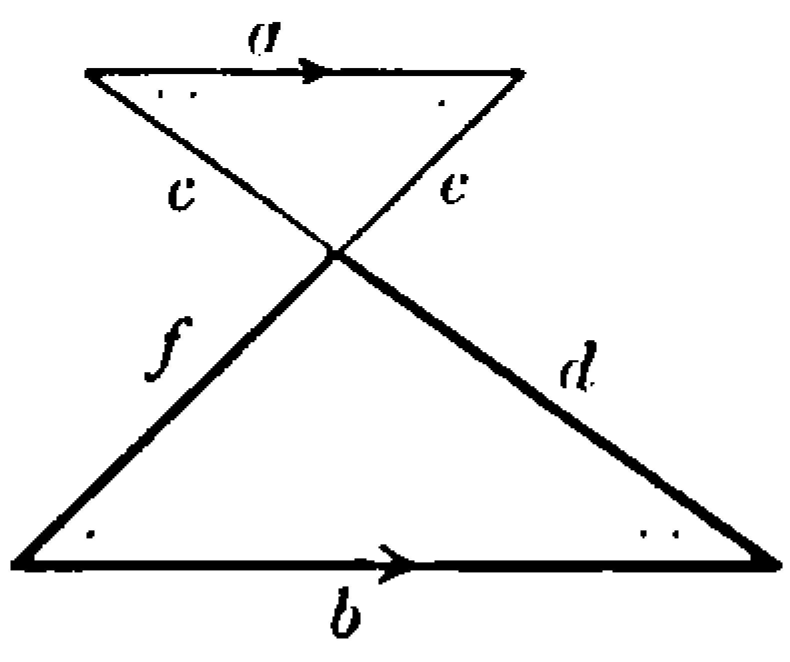
Given: In $\triangle$ T and T'
$\angle A = \angle A'$
$\angle B = \angle B'$
Prove: $T \sim T'$

PROOF

Statements	*Reasons*
1. $\angle C = \angle C'$	1. If $\triangle$ agree in 2 $\angle$s, they agree in all the $\angle$s.
2. $\frac{ac}{a'c'} = \frac{T}{T'}$ $\frac{bc}{b'c'} = \frac{T}{T'}$	2. If 2 $\triangle$ agree in one $\angle$, they are proportional to products of sides of the = $\angle$s (§ 292).
3. $\frac{ac}{a'c'} = \frac{bc}{b'c'}$	3. Subs. Ax.
4. $\frac{a}{a'} = \frac{b}{b'}$	4. Div. Ax.
Similarly, we can prove $\frac{b}{b'} = \frac{c}{c'}$.	
5. $T \sim T'$	5. Def. $\sim$ polygons.

We have found that a common way of proving two line segments equal was to prove that they are the corresponding sides of two congruent triangles. Similarly, we shall see that a common way of proving two line segments proportional to two other line segments is to show that they are the corresponding sides of two similar triangles. This will now be illustrated.

Illustrative Example. Because a is parallel to b, it is easy to prove by our theorem that the two triangles in the figure are similar.



From the definition of similar polygons, we can then set up a number of proportions.

Now, in the case of congruent triangles, we found it easy to select the corresponding sides. But in two similar triangles this may be troublesome—for three reasons: (1) the eye is not so good a judge of proportional magnitudes as it is of equal magnitudes; (2) it is necessary to work with four sides at one time instead of two; and (3) even if the four sides are selected correctly, it is still possible to make errors in arranging them. For these reasons, it is a good idea to adopt some scheme that will insure the correct proportions. One plan is here suggested.

Step 1. If the equal angles are not already so marked in the figure, mark them. In the figure given, single and double dots have been used to show which pairs of angles can be proved equal. The other two angles are easily paired because they are vertical angles. Another method is to letter the angles as m and m', or q and q'.

Step 2. Think of one triangle as the numerator triangle and the other as the denominator triangle. In the figure given, think of the upper triangle as the numerator and the lower triangle as the denominator.

Step 3. The proportions may now be set up as follows:

$$\frac{a \text{ (opp. vert. } \angle \text{ in the numerator } \triangle)}{b \text{ (opp. vert. } \angle \text{ in the denominator } \triangle)} = \frac{c \text{ (opp. single dot in numerator } \triangle)}{d \text{ (opp. single dot in denominator } \triangle)}$$

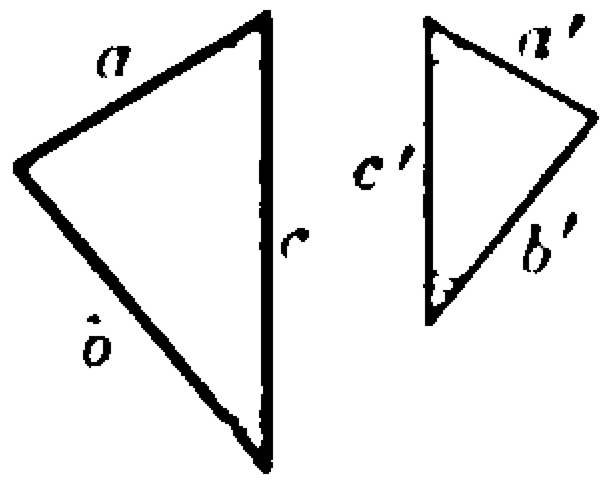
Similarly, $$\frac{c}{d} = \frac{e}{f}$$

and $$\frac{a}{b} = \frac{e}{f}$$

EXERCISES

In each of the first 12 problems, prove two triangles similar. Then write one or two proportions using all the letters marked in the figure.

1.



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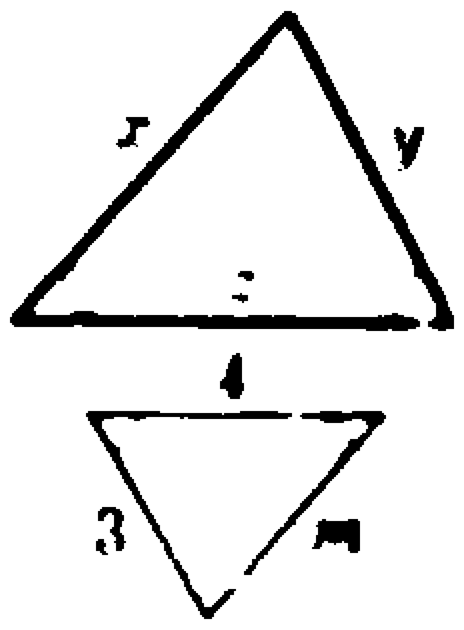

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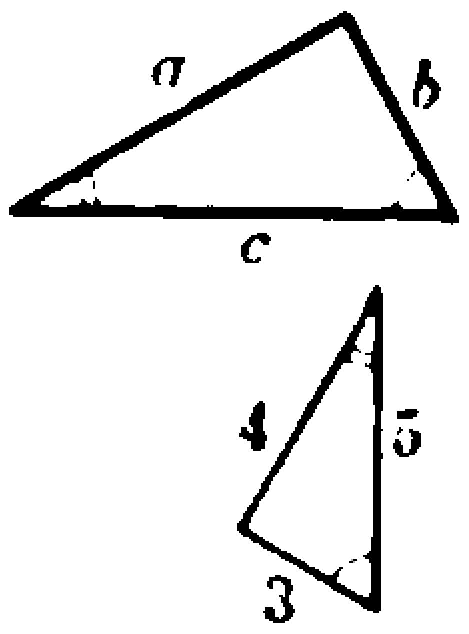

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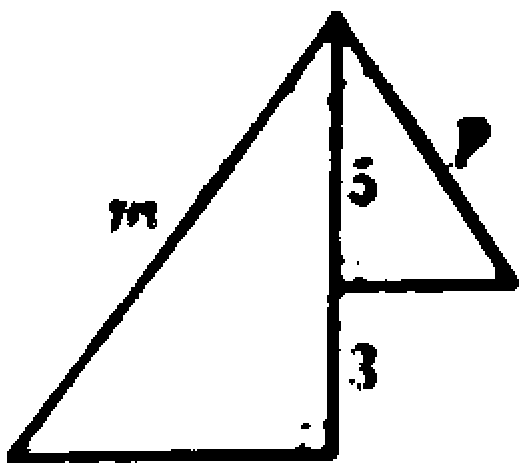

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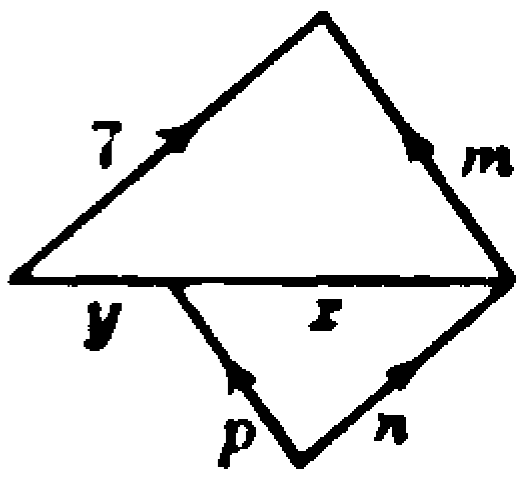

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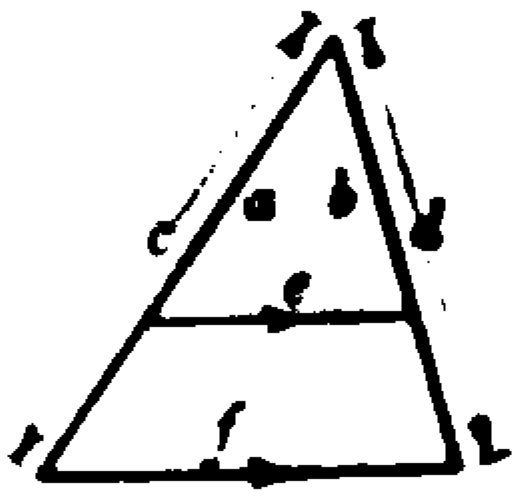

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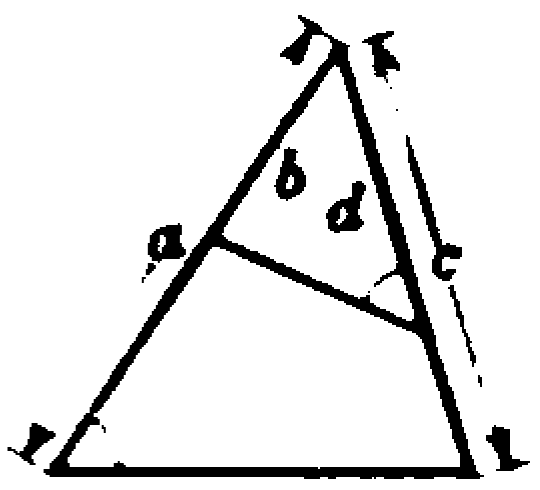

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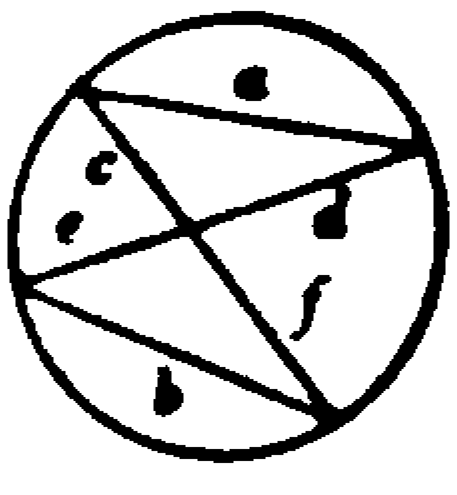

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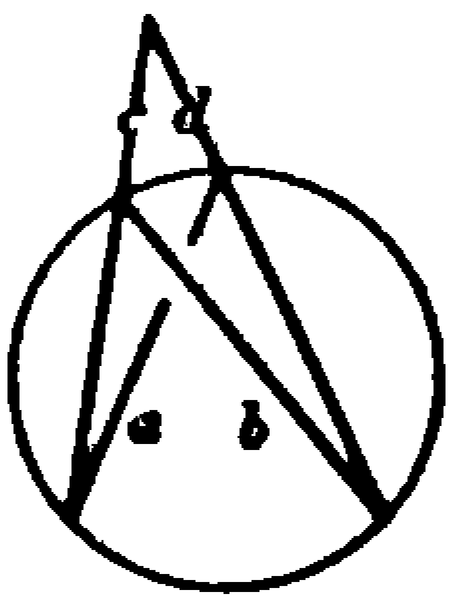

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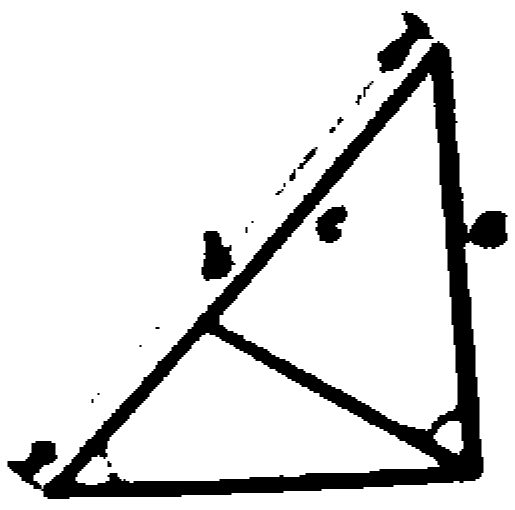
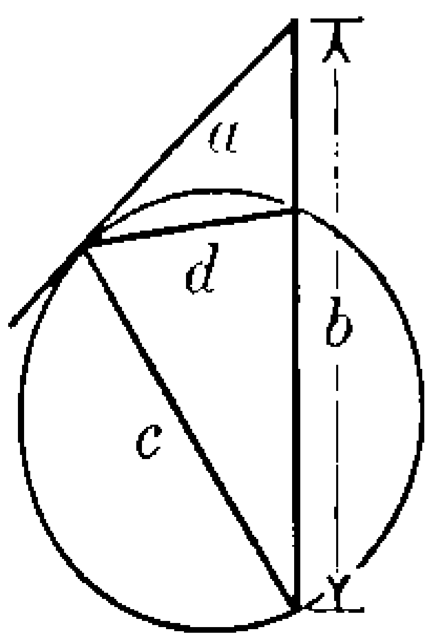
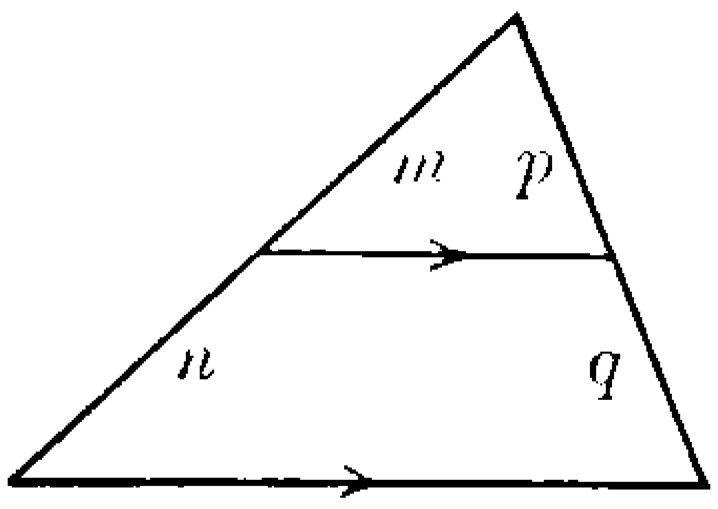


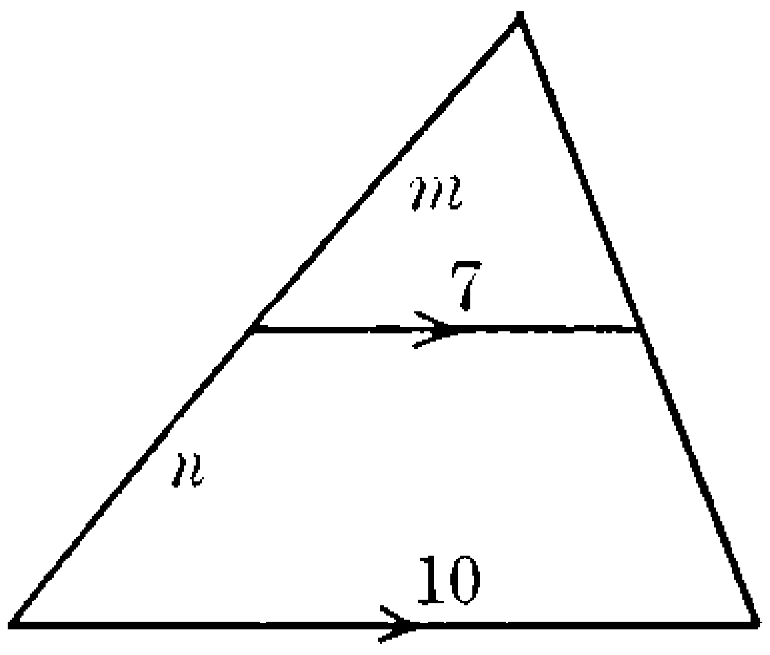
11.



12.



13.



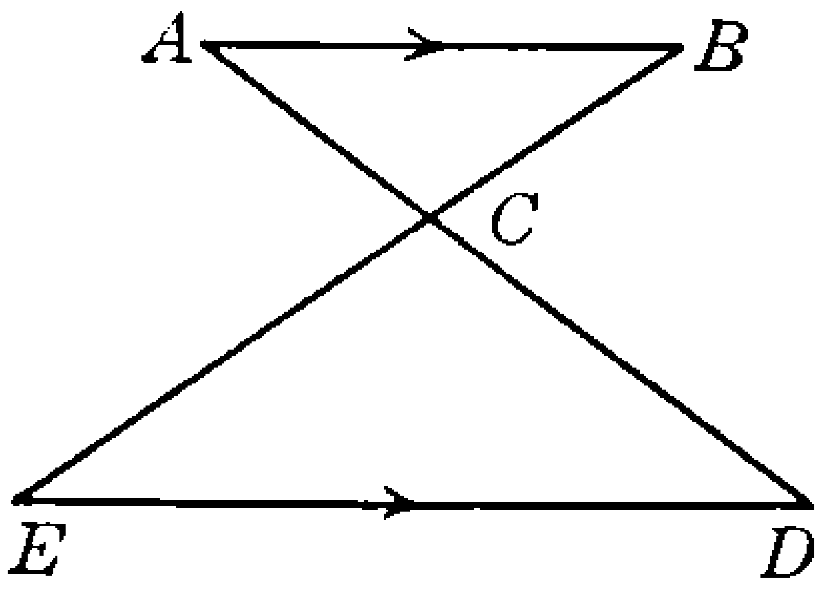
Complete the proportion:

$$\frac{m}{n} = \text{—?—}$$

14. Complete the following equations:

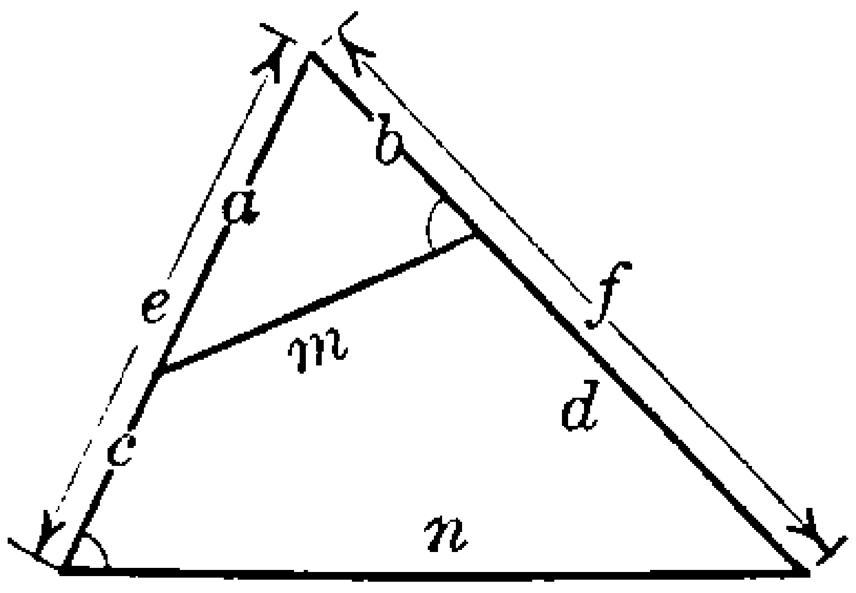
(a) $\frac{AB}{ED} = \frac{BC}{?}$; (b) $AC \cdot CE = BC \cdot ?$,

(c) $\frac{AD}{AC} = \frac{BE}{?}$



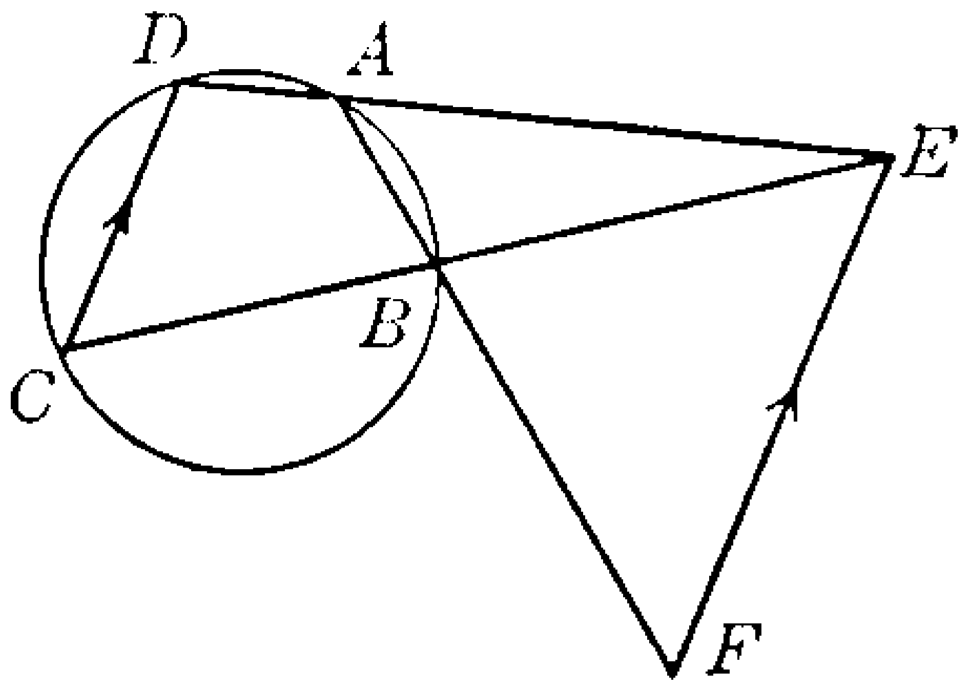
15. Determine which of the following equations are true.

(a) $a \cdot e = b \cdot f$, (b) $\frac{m}{n} = \frac{a}{b}$, (c) $\frac{a}{b} = \frac{d}{c}$

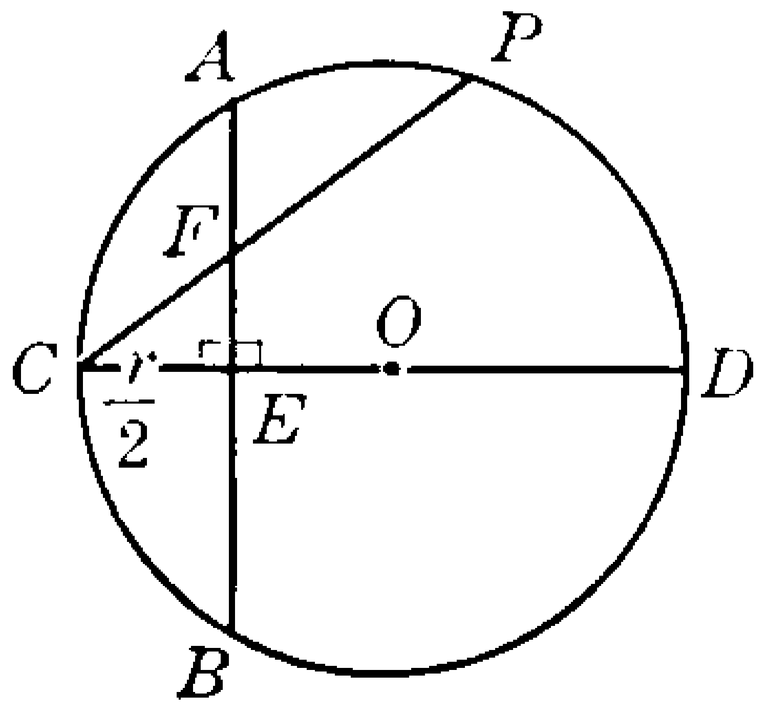


16. In the figure at the top of the next column, the sides DA and CB of the inscribed quadrilateral $ABCD$ are prolonged to meet at E. Through E, EF is drawn parallel to DC, meeting AB prolonged at F.

Prove that EF is the mean proportional between the whole secant FA and its external part FB.

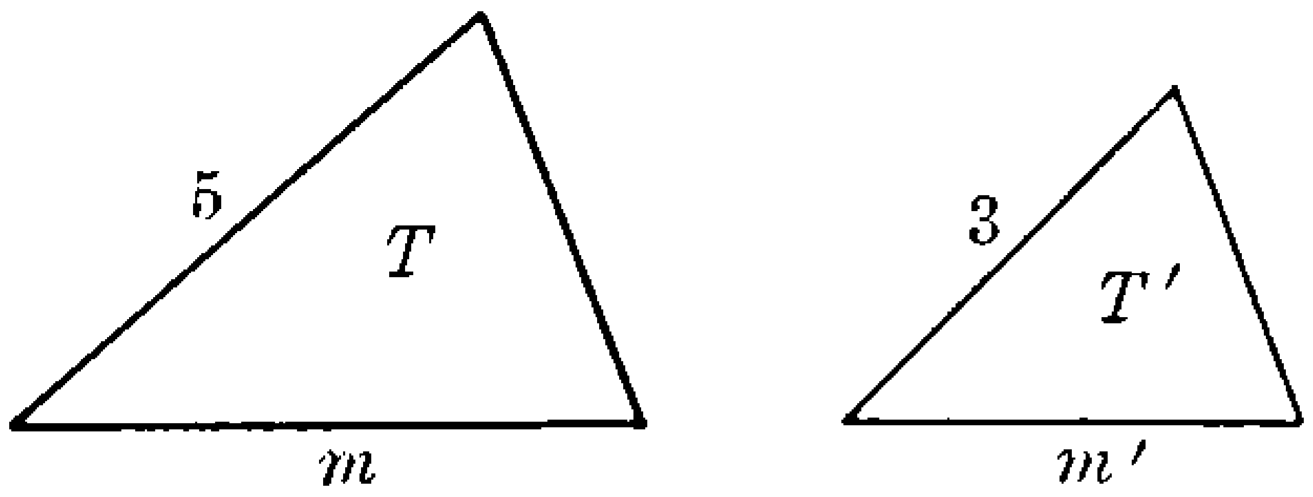


17. CD is a diameter of the circle whose radius is r. $AB \perp CD$ at E, where $CE = \frac{1}{2}r$.



Prove that the product of any chord CP through C and its segment CF equals r^2.

18. $T \sim T'$



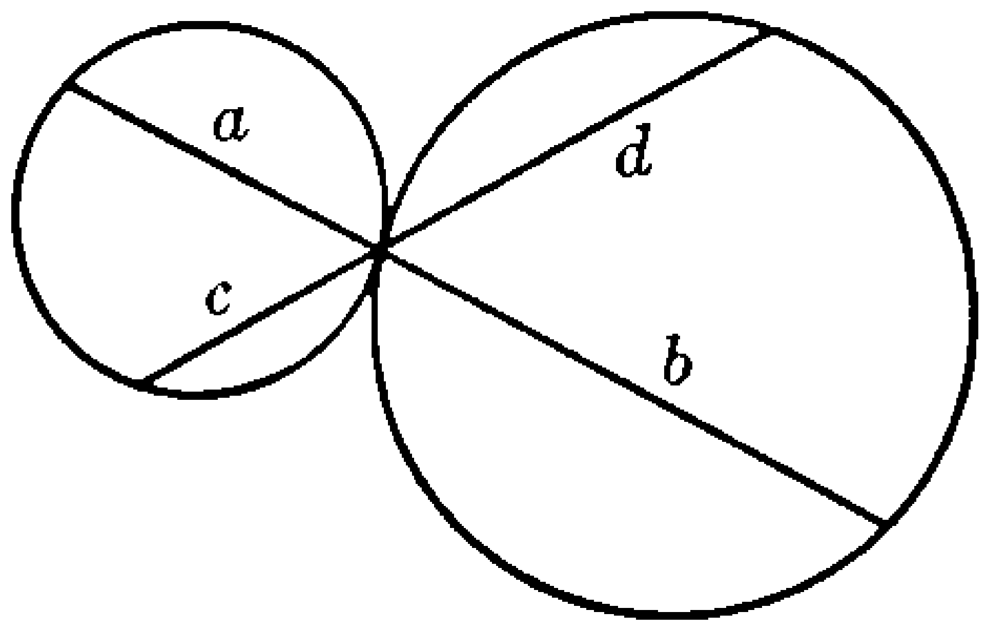
(a) Complete the proportion $\frac{m}{m'} = \text{—?—}$.

(b) Complete the proportion $\frac{T}{T'} = \text{—?—}$.

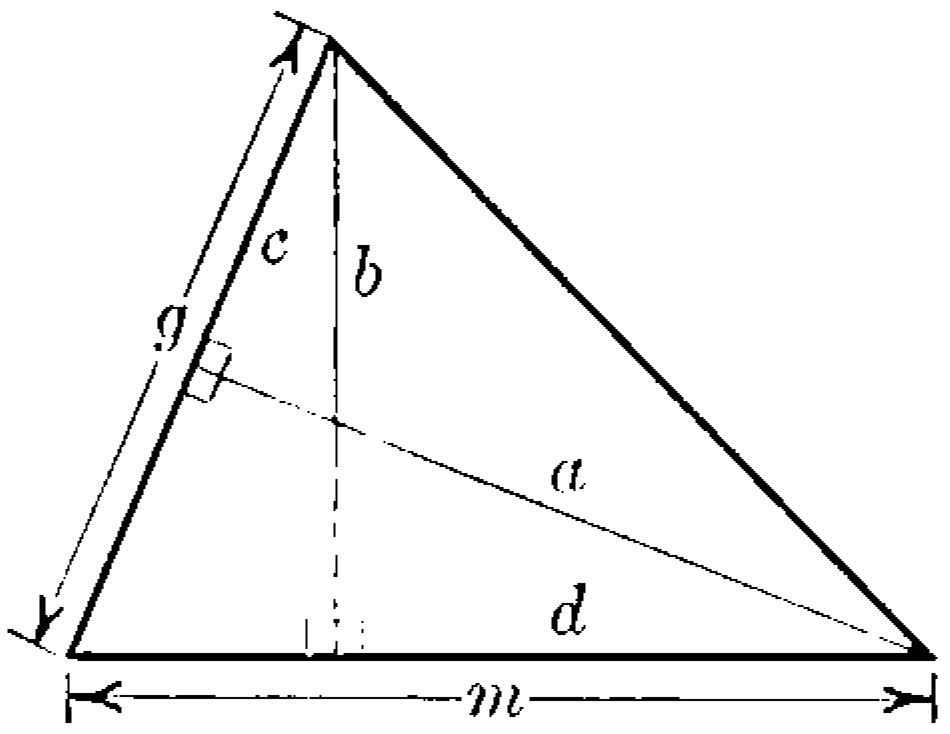
(c) Find the numerical value of $\frac{T}{T'}$

SUPERIOR WORK

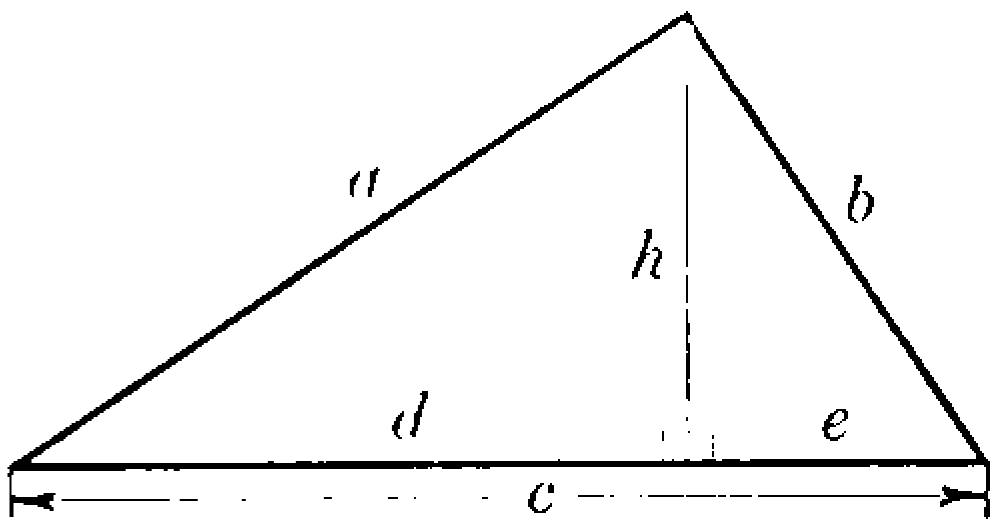
S1. In the two tangent circles, two lines (double chords) are drawn through the point of contact forming the chords a, b, c, and d. Prove that these chords are proportional.



S2. Write two proportions, using the letters given in the figure below.



S3. In the figure, where h is the altitude to the hypotenuse of the right triangle, there are three similar triangles.



(*a*) Prove that h is the mean proportional between two other line segments. Complete the equation $h^2 = \text{—?—}$.

(*b*) Prove that a is the mean proportional between two line segments. $a^2 = \text{—?—}$.

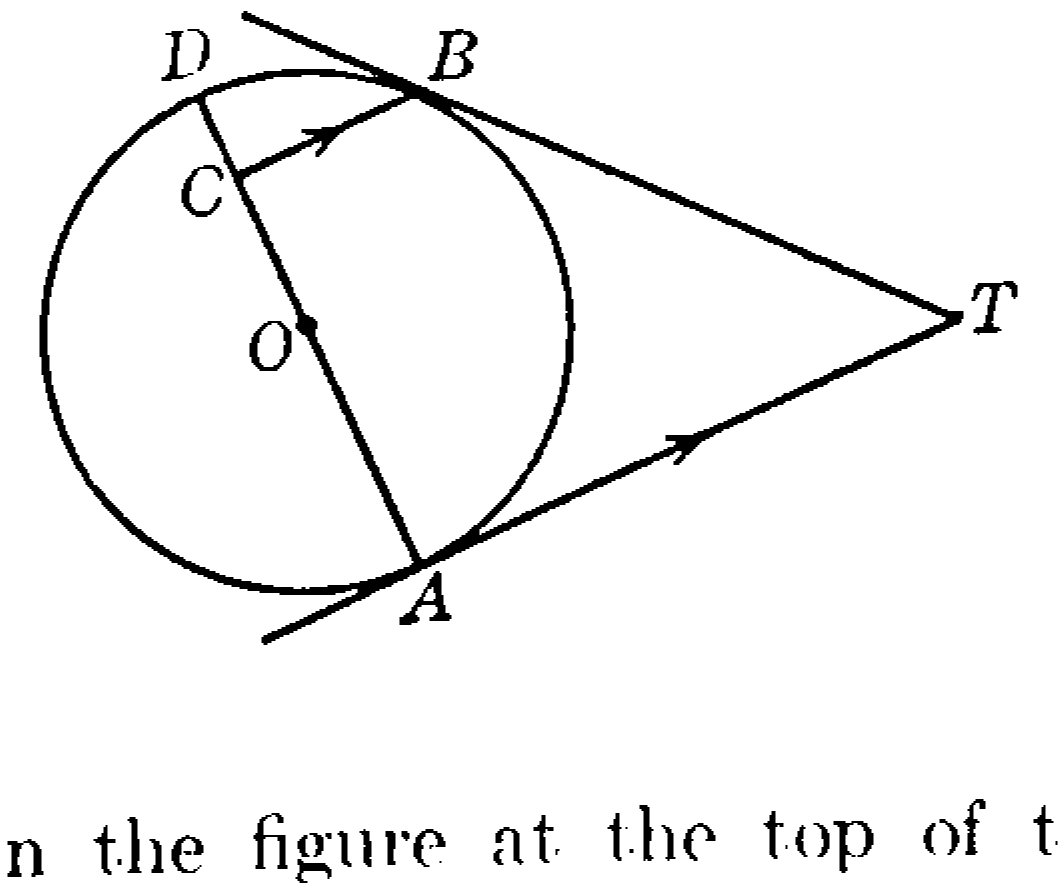
(*c*) Prove that b is the mean proportional between two line segments. $b^2 = \text{—?—}$.

S4. Given: TA and TB tangent to $\odot O$ at A and B

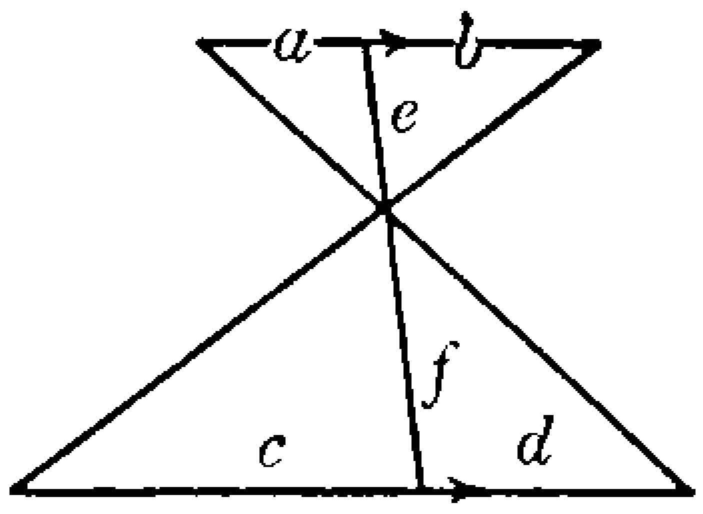
AD a diameter

$BC \parallel TA$

Prove: $TA \cdot BC = AC \cdot OA$



S5. In the figure at the top of the next column, two parallels are cut by three transversals passing through a common point.



(*a*) Complete the proportion $\dfrac{a}{d} = \text{—?—}$.

(*b*) Complete the proportion $\dfrac{b}{c} = \text{—?—}$.

(*c*) Derive a proportion in a, b, c, and d.

S6. The perpendicular bisector of one leg of an isosceles triangle cuts the base (prolonged, if necessary) at a certain point. Prove that the distance between this point and the vertex of the isosceles triangle times the base of the isosceles triangle equals the square of either leg.

S7. Through one end of a diameter, a tangent is drawn to the circle. Beginning at the other end of the diameter, two secants are drawn terminating in the tangent. Prove that one secant times its internal segment (the segment within the circle) equals the other secant times its internal segment.

Nonmathematical Analogy

S299. The following conversation is intended to show that unless analogy is clearly defined it may lead to confusion.

John. I heard a puzzle. If 2 times 5 were 15, what would 3 times 4 be?

Henry. That's easy. 2 times 5 is really 10, and 3 times 4 is really 12. Therefore, 10 is to 12 as 15 is to x. So x is 18.

John. I figure the answer might be 17. Since 2 times 5 is really 10, then the phony answer 15 is 5 more than the true answer. Therefore, I make out the imaginary answer for 3 times 4 to be 5 more than the true answer, or 12 plus 5.

Henry. I think you're wrong, but I can't tell exactly what mistake you've made. Let me explain my answer another way. We have two analogous situations: a real situation where the multiplication table works so that 2 times 5 is 10, 3 times 4 is 12, and so on; and an imaginary situation where 2 times 5 is 15, and 3 times 4 is what not. Now I say that the imaginary answers are proportional to the true answers; thus, $x : 15 = 12 : 10$.

John. I agree that we have two analogous situations. But how do you know that the quotient of the two imaginary answers equals the quotient of the two real answers? I say that the difference between the two fictitious answers may be equal to the difference between the two real answers; thus, $x - 15 = 12 - 10$. Therefore —

Henry. Hold on! Now I see what's wrong with your thinking. You're not using proportion. In mathematics, analogy is usually a proportion.

John. But in this puzzle, the analogy hasn't been defined at all. If the analogy is not defined, then we can't prove anything.

Henry. But people use analogy every day to prove their arguments. For example, they argue that a government ought to do the same for its citizens as a father does for his family, or that a plan that works in a small community will work in a large nation, and so on.

John. Such arguments aren't proof. A clever debater can often use the same analogy to prove the exact opposite conclusion.

S8. Compare your state government with the federal government, pointing out resemblances that make these governments analogous.

S9. Obtain a copy of the United Nations Charter. In what respects is the United Nations Charter analogous to the United States Constitution?

S10. Is the New England town meeting analogous to the United States Congress? Why? In what respects does the analogy fail?

False Analogy

S300. In mathematics, analogy is defined as proportion. Outside mathematics, analogy is based on a number of relationships that are not clearly defined and sometimes only vaguely understood. For this reason, analogy cannot be used as *proof* in such cases. Fallacies in thinking often arise from false or incomplete analogies. Many of these can be classified as (1) unjustified transfer or (2) personalities.

1. *Unjustified Transfer.* In this case, transfer is made from one situation to another without sufficient reason. Two types are common:

(*a*) *Prestige.* This fallacy assumes that because a person is successful in one line of endeavor he must also be an authority in other fields. Example: Motion-picture stars endorsing commercial products.

(*b*) *Halo.* In this argument, the attempt is made to transfer a proposition about well-loved and cherished ideals to other propositions. Example: Most Fourth of July speeches.

2. *Personalities.* The injection of personalities may be illustrated by the following two types:

(*a*) *Argumentum ad Hominem.* This Latin phrase means that the argument is aimed at the man. The implication here is that if an opponent can be attacked on personal grounds, then his argument must be equally weak. Example: Name-calling.

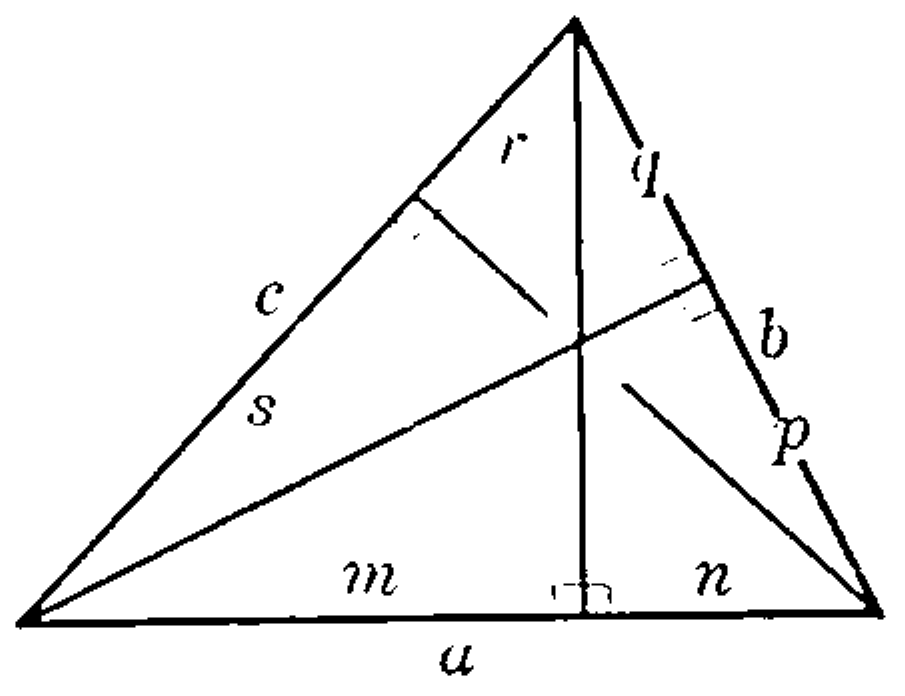
(*b*) *Flattery.* This is based on the psychological fact that some people readily believe things that are favorable to them personally. Such people may then accept other propositions that the flatterer sponsors without examining them too closely. Classic example: Cassius persuading Brutus to join the conspiracy against Caesar.

S11. Find some examples in everyday thinking to illustrate the fallacies just described.

S12. A speaker will sometimes use language that is unnecessarily technical. Which fallacy of analogy does this illustrate? Find examples of it.

HONOR WORK

H1. In the figure, the altitudes of the triangle form certain similar triangles. From these prove:



$$a \cdot m = c \cdot s$$
$$b \cdot p = a \cdot n$$
and
$$c \cdot r = b \cdot q$$

Using these equations and the converse of Ceva's Theorem (page 194, H1), show that the altitudes meet in a point.

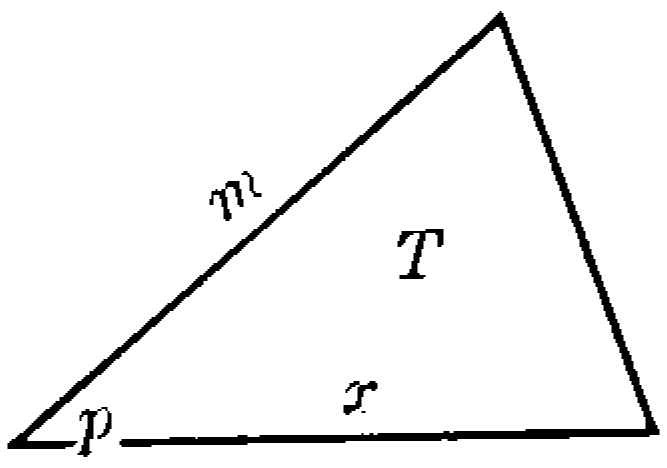
Areas of Similar Triangles

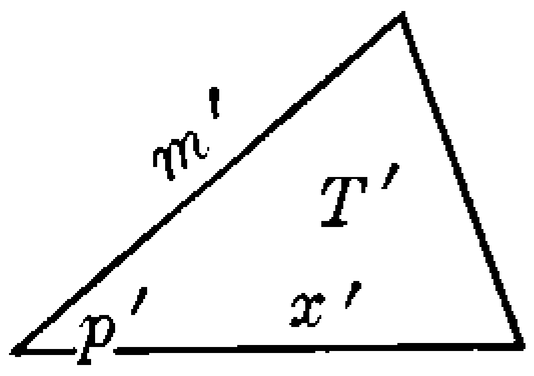
We have seen that if two triangles are similar, their sides may be compared by proportion. We shall now prove that the areas of these triangles may be compared with the squares of their sides.

Similar triangles cannot be compared with their sides. We must use the *squares* of their sides.

It can be proved that the volumes of two similar solid figures are proportional to the *cubes* of corresponding sides.

301. THEOREM. The areas of two similar triangles are to each other as the squares of their corresponding sides.





Given: $T \sim T'$

Prove: $\frac{T}{T'} = \frac{x^2}{x'^2}$

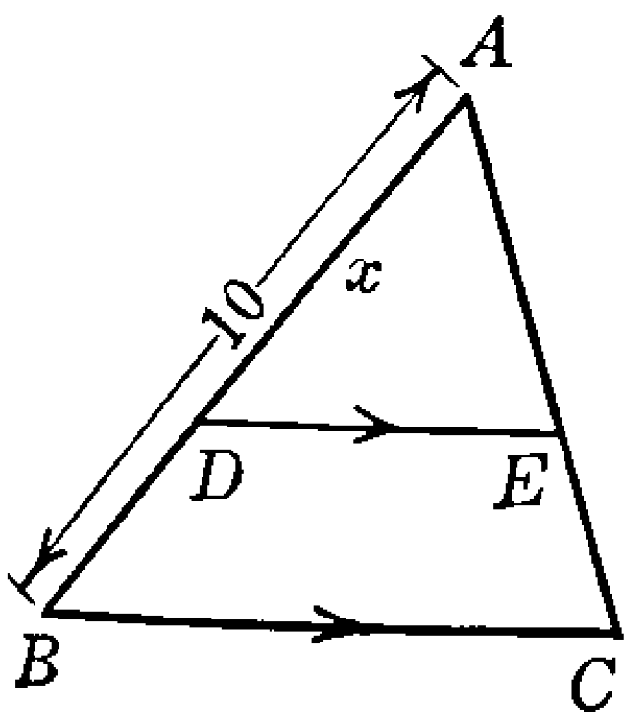
PROOF

Statements	*Reasons*
1. $\angle p = \angle p'$	1. Def. ∼ polygons.
2. $\frac{T}{T'} = \frac{mx}{m'x'} = \frac{m}{m'} \cdot \frac{x}{x'}$	2. If 2 △ agree in one ∠, they are proportional to products of sides of the = ∡ (§ 292).
3. $\frac{m}{m'} = \frac{x}{x'}$	3. Def. ∼ polygons.
4. $\frac{T}{T'} = \frac{x}{x'} \cdot \frac{x}{x'} = \frac{x^2}{x'^2}$	4. Subs. Ax.

EXERCISES

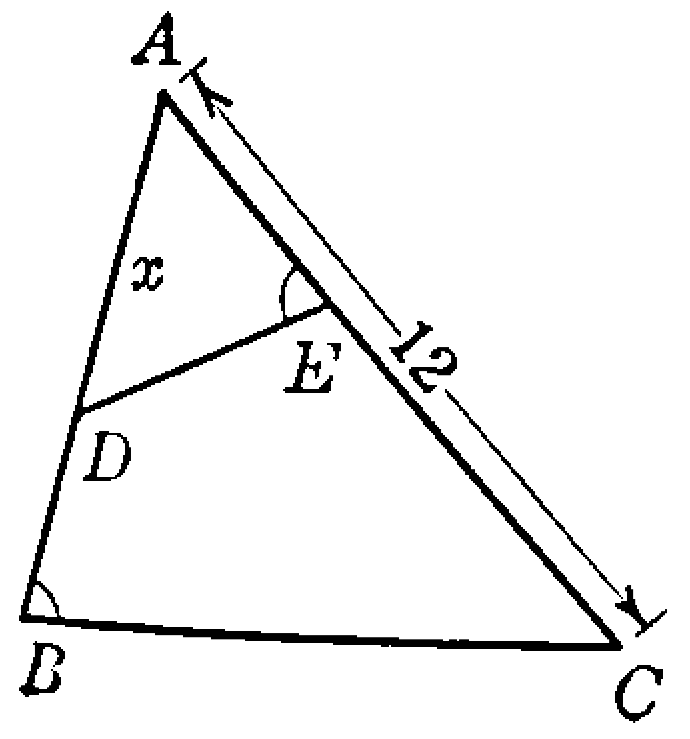
1. Two corresponding sides of similar triangles are 8 units and 3 units. If the area of the large triangle is 14 square units, find the area of the smaller triangle.
2. The areas of two similar triangles are 12 and 3 square units. Find the side of the smaller triangle that corresponds to a 5-unit side of the larger triangle.
3. Two right triangles have an acute angle of one equal to an acute angle of the other. The hypotenuses are 7 and 15. If the area of the smaller triangle is 5, find the area of the larger triangle.
4. The vertex angles of two isosceles triangles are equal. The areas of the two triangles are 10 square inches and 40 square inches. If the base of the large triangle is 6 inches, find the base of the smaller triangle.
5. The areas of two similar triangles are 6 square feet and 150 square feet. The sides of one triangle are 3 feet, 4 feet, and 5 feet. Find the sides of the other triangle.
6. Two corresponding sides of similar triangles are 5 units and $5\sqrt{3}$ units. If the area of the smaller triangle is 6 square units, find the area of the larger triangle.

7.



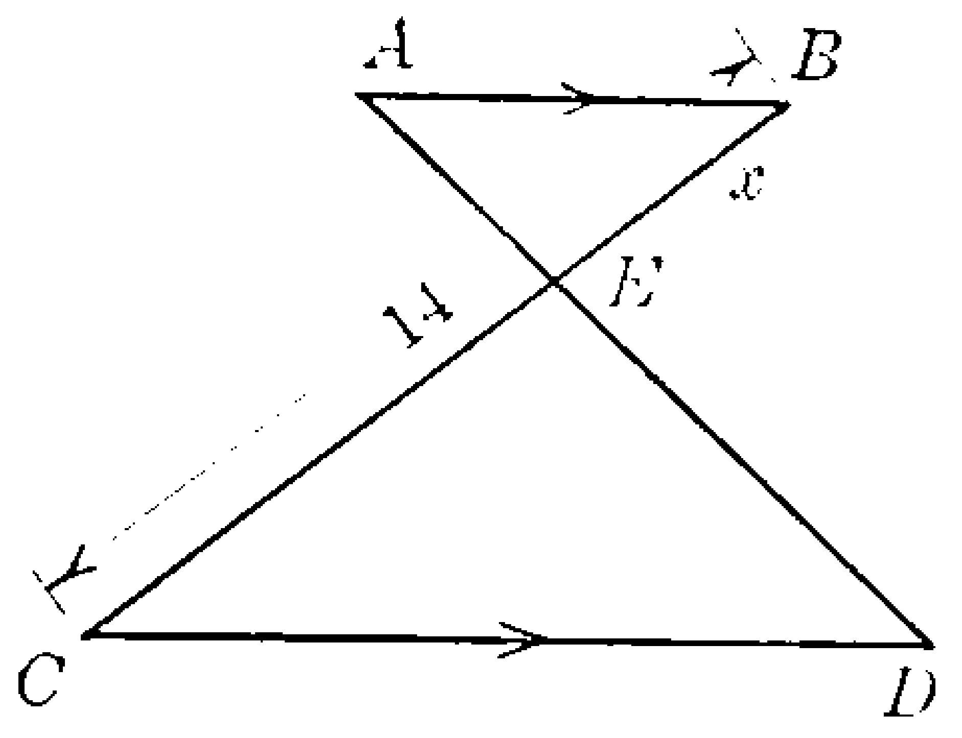
Given:
$DE \parallel BC$
$AB = 10$
$\triangle ADE =$ trap. $DECB$
Find:
AD

8.



Given:
$\angle B = \angle AED$
$AC = 12$
$\triangle ABC = 3 \cdot \triangle ADE$
Find:
AD

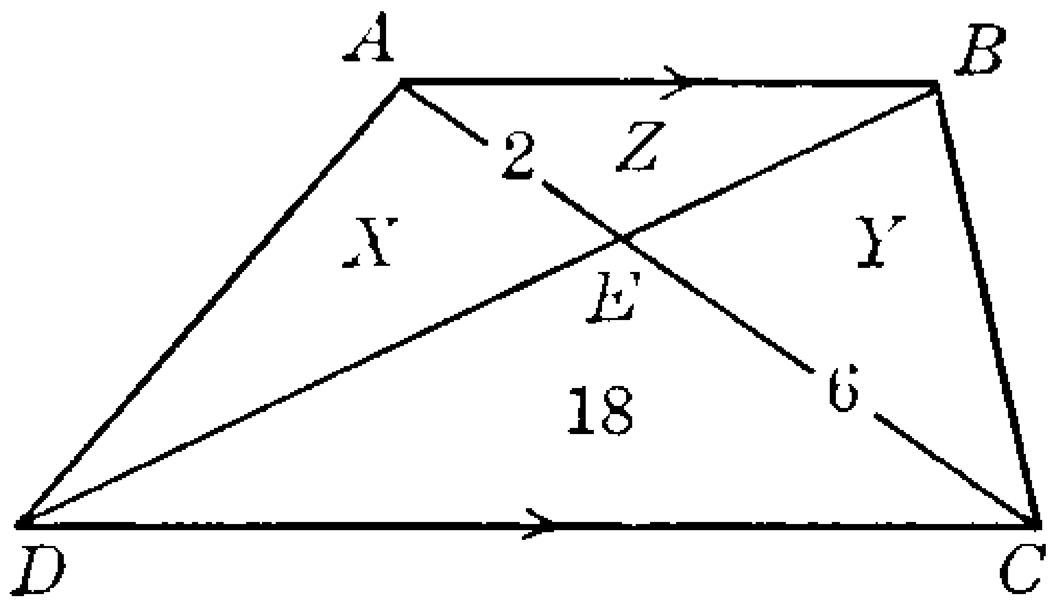
9.



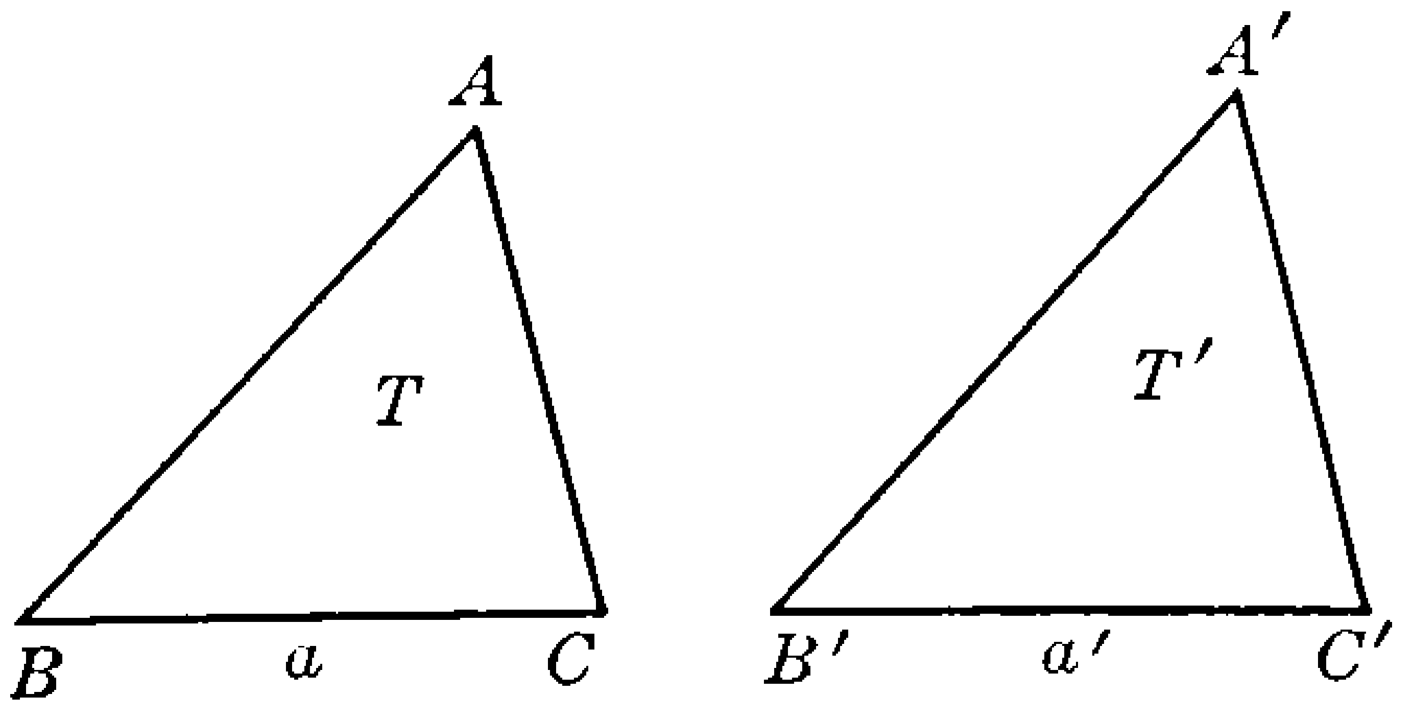
Given:
$AB \parallel CD$
$CB = 14$
$\triangle CDE = 2 \cdot \triangle ABE$
Find:
BE correct to 3 digit places

Help: $\left(\frac{x}{14 - x}\right)^2 = \frac{\triangle ABE}{\triangle CDE} = \frac{1}{2}$

10. Given: In trap. $ABCD$
$AE = 2$
$EC = 6$
Area $\triangle CDE = 18$
Find: Areas of the ▵ X, Y, and Z



11. Using the theorem about the areas of two similar triangles, prove the following: If two triangles are similar and equal, they are congruent.
Given: $T \sim T'$
$T = T'$
Prove: $T \cong T'$



PROOF

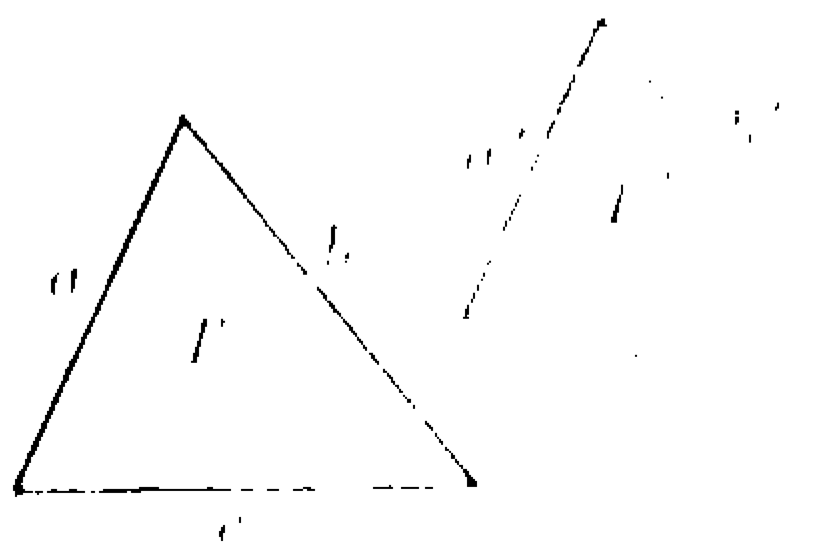
Statements	*Reasons*
1. $\frac{T}{T'} = \frac{a^2}{a'^2}$	1. Why?

Complete the proof.

12. The sides of a triangle are 7, 8, and 12 units. In another triangle similar to it, the side corresponding to 8 units is

[illegible] units [illegible] and [illegible]
second triangle.

13. Prove: The perimeters [illegible] triangles are proportional [illegible] sponding sides.



Given: [illegible]

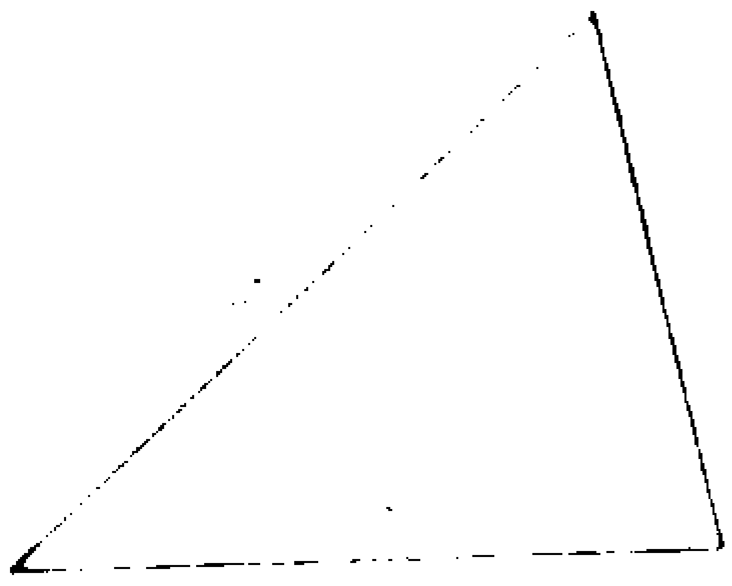
Prove: [illegible]

14. The areas of two [illegible] 27 and 3 units. [illegible] smaller triangle [illegible] perimeter of the larger [illegible]

SUPERIOR WORK

S1. Our fundamental [illegible] if two angles of one triangle [illegible] tively to two angles of another [illegible] the triangles are similar. [illegible] form because it is about [illegible]

S2. [illegible]



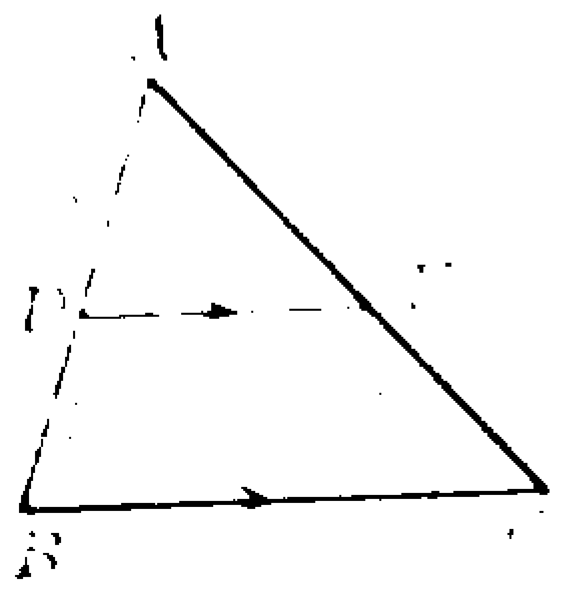
S3. [illegible]

PROPORTIONS FROM PARALLEL LINES

Segments Made by Two Parallels

If a number of parallel [illegible] by transversals, certain [illegible]

302. THEOREM. **If a line is parallel to one side of a triangle [illegible] other two sides proportionally.**



PROOF

Statements	*Reasons*
1. $\angle ADE = \angle B$ $\angle AED = \angle C$	1. ∥s, corr. ∠s are =.
2. $\triangle ABC \sim \triangle ADE$	2. If 2 △s agree in 2 ∠s, they are ∼.
3. $\dfrac{x+y}{x} = \dfrac{x'+y'}{x'}$	3. Def. ∼ polygons.
4. $\dfrac{x}{y} = \dfrac{x'}{y'}$	4. Transformation by subtraction and inversion.

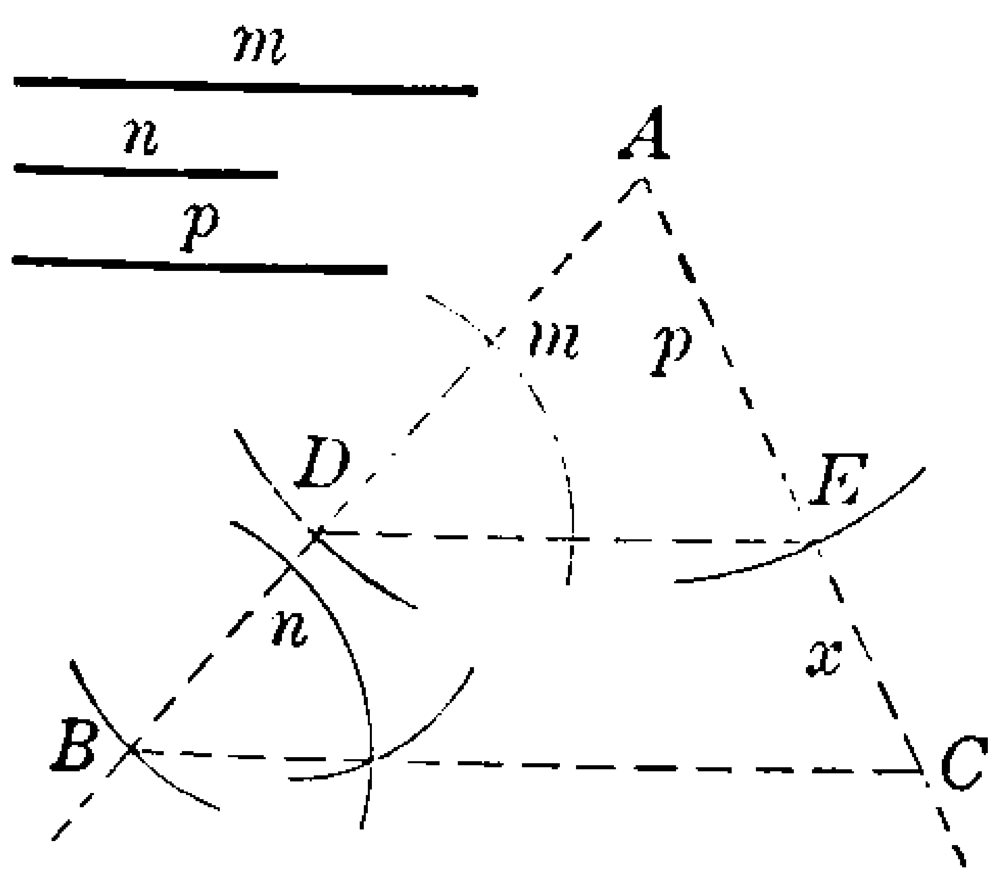
---

303. Corollary. If a line bisects one side of a triangle and is parallel to another side, then it bisects the third side.

If we have three line segments as the terms of a proportion, we can construct the fourth term, or the *fourth proportional*, by making the given line segments parts of a triangle cut by a line parallel to the base.

---

304. PROBLEM. Construct the fourth proportional to three given line segments.



Given: Line segments m, n, and p

Req.: Cons. line segment x so that $\dfrac{m}{n} = \dfrac{p}{x}$

Construction

1. Draw any angle A.
2. On the sides of $\angle A$, lay off $AD = m$, $DB = n$, and $AE = p$. Draw DE.
3. Cons. $BC \parallel DE$.

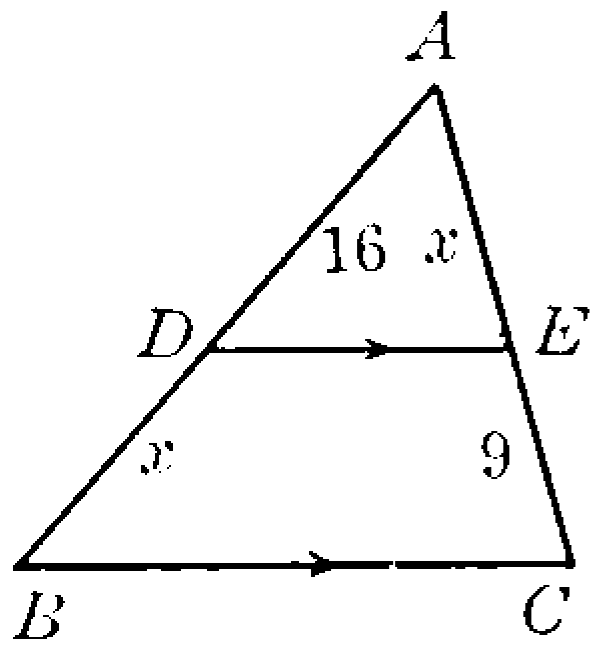
Conclusion: $\dfrac{m}{n} = \dfrac{p}{x}$.

PROOF

Statements	*Reasons*
1. $DE \parallel BC$	1. Cons.
2. $\dfrac{m}{n} = \dfrac{p}{x}$	2. A line ∥ to one side of a △ divides the other 2 sides proportionally (§ 302).

EXERCISES

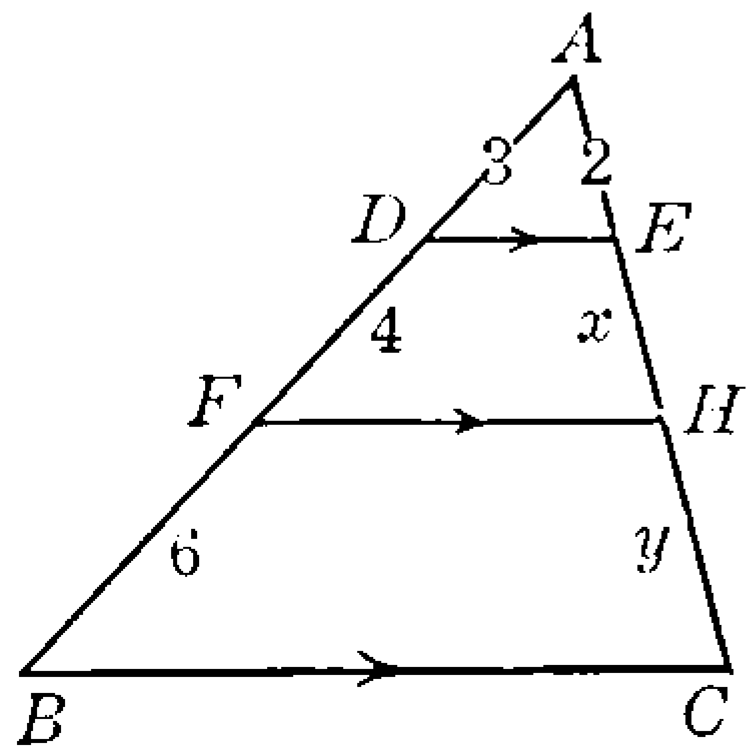
1. (*a*)



Given:
$DE \parallel BC$
$AD = 16$
$EC = 9$
$DB = AE$
Find:
DB

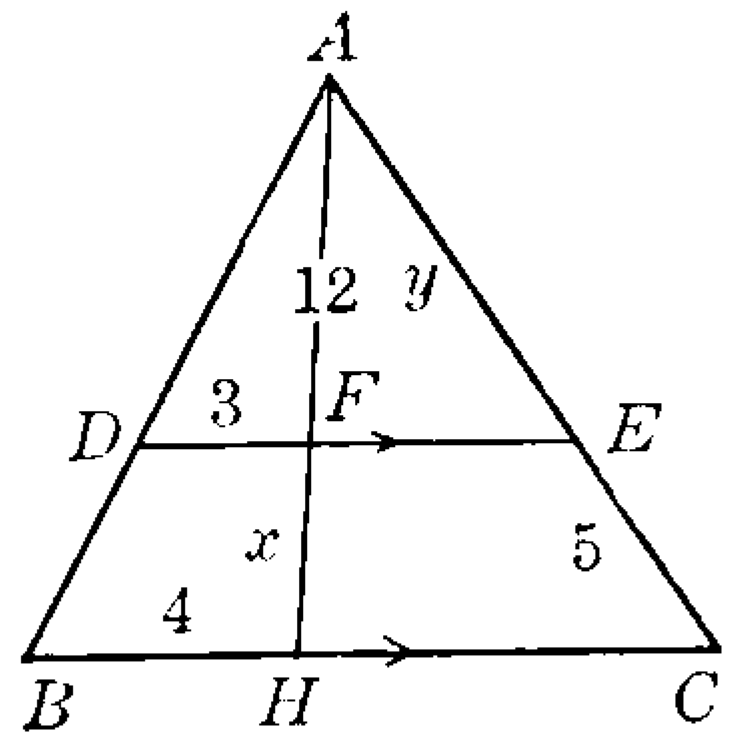
(*b*) A line parallel to one side of a triangle divides another side into segments having a ratio of 3 : 4. Find the segments on the third side, if they differ by 2.

2. (*a*)



Given:
$DE \parallel FH \parallel BC$
$AD = 3$
$DF = 4$
$FB = 6$
$AE = 2$
Find:
EH and HC

(*b*)



Given:
$DE \parallel BC$
$DF = 3$
$BH = 4$
$AF = 12$
$EC = 5$
Find:
FH and AE

3. If a line bisects one leg of a trapezoid and is parallel to the bases, prove that it bisects the other leg and both diagonals.

4. (*a*) Draw three line segments m, n, and p, and construct the fourth proportional to them.
   (*b*) Construct the fourth proportional to the same three line segments, but in the order m, p, and n. Using your compass, compare the results of (*a*) and (*b*).

5. Given line segments a, b, and c. Construct a line segment x so that $x = \frac{ab}{c}$.

   Hint: First set up a proportion in which x is the last term.

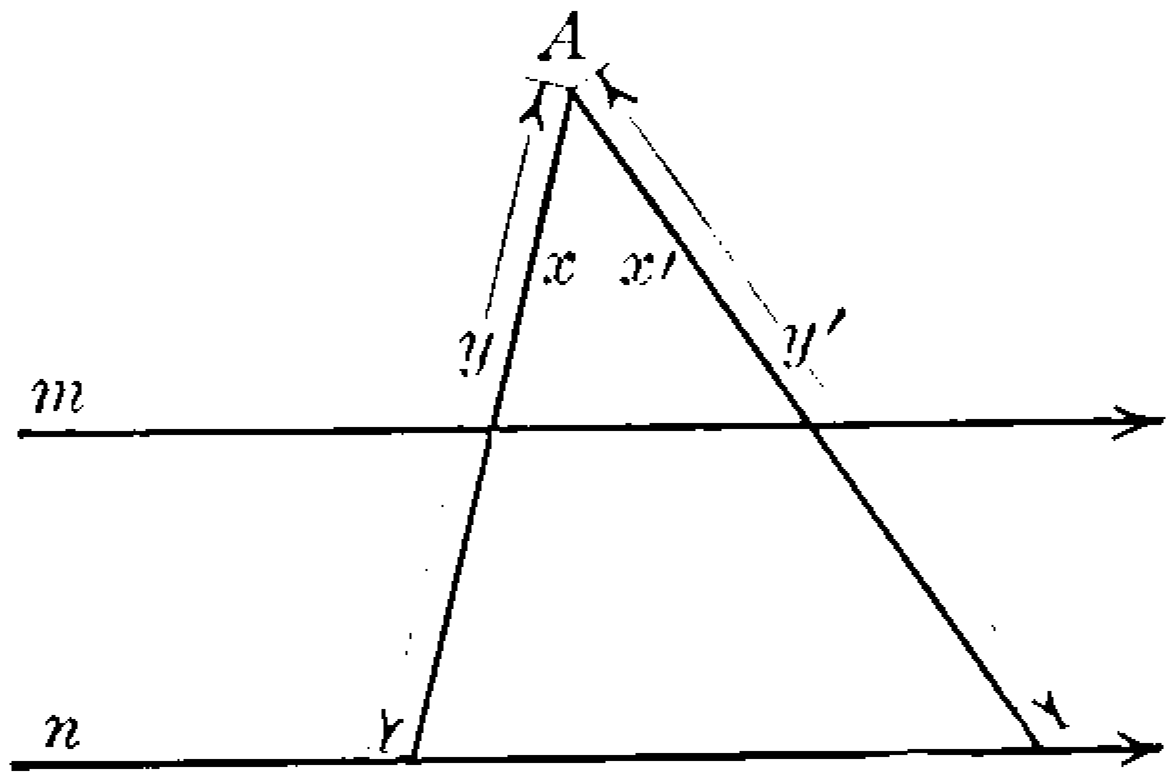
6. Construct a line segment x so that $x = \frac{a^2}{b}$, where a and b are two given line segments.

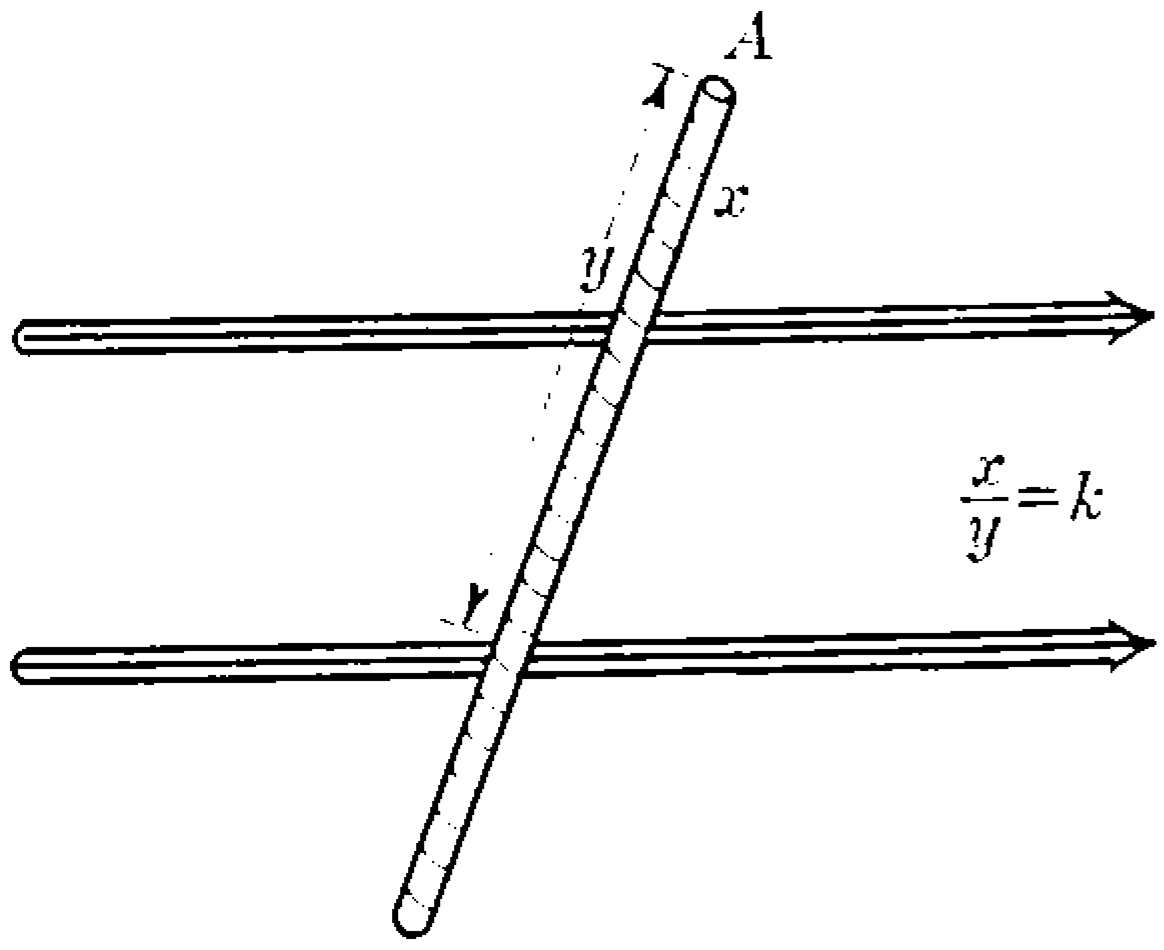
7. Given its base, construct a rectangle that is equal to a given square. (First construct the given square in full lines.)

8. Given its base, construct a rectangle that is equal to a given triangle.

SUPERIOR WORK

S1. In the figure, m is parallel to n. From our theorem about a line parallel to one side of a triangle, we know that $\frac{x}{y} = \frac{x'}{y'}$. This is a static situation as long as we think of the whole figure as fixed. However, we may think of it dynamically in the following manner.





Pivoted at A, and swinging from it, would be a pendulum. The line segments x and y from the point A to each of the parallels, measured along the pendulum, are called *collinear*, because they are in a straight line. Since the ratio of these two line segments would be the same in any position of the pendulum, the variation $\frac{x}{y} = k$ results.

Our static theorem can therefore be stated in the following dynamic form: The ratio of the collinear segments from a given point to two given parallel lines is constant.

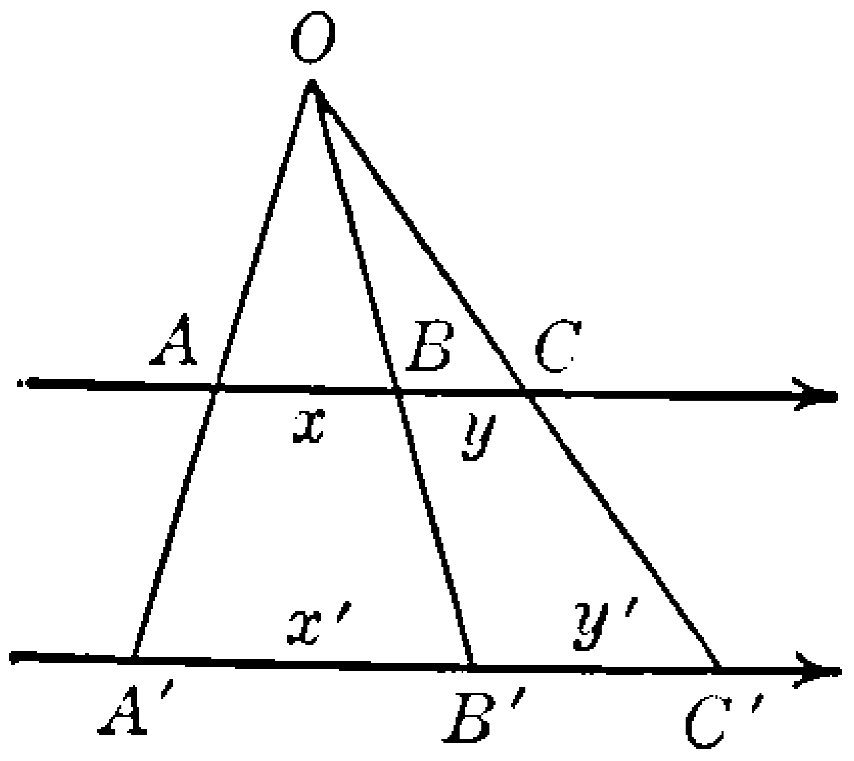
Prove that this proposition is also true if the given point is *between* the two parallels.

S2. (*a*) Prove this theorem: If two parallels are cut by three concurrent transversals, then the segments on the parallels are proportional.

Given: $AC \parallel A'C'$

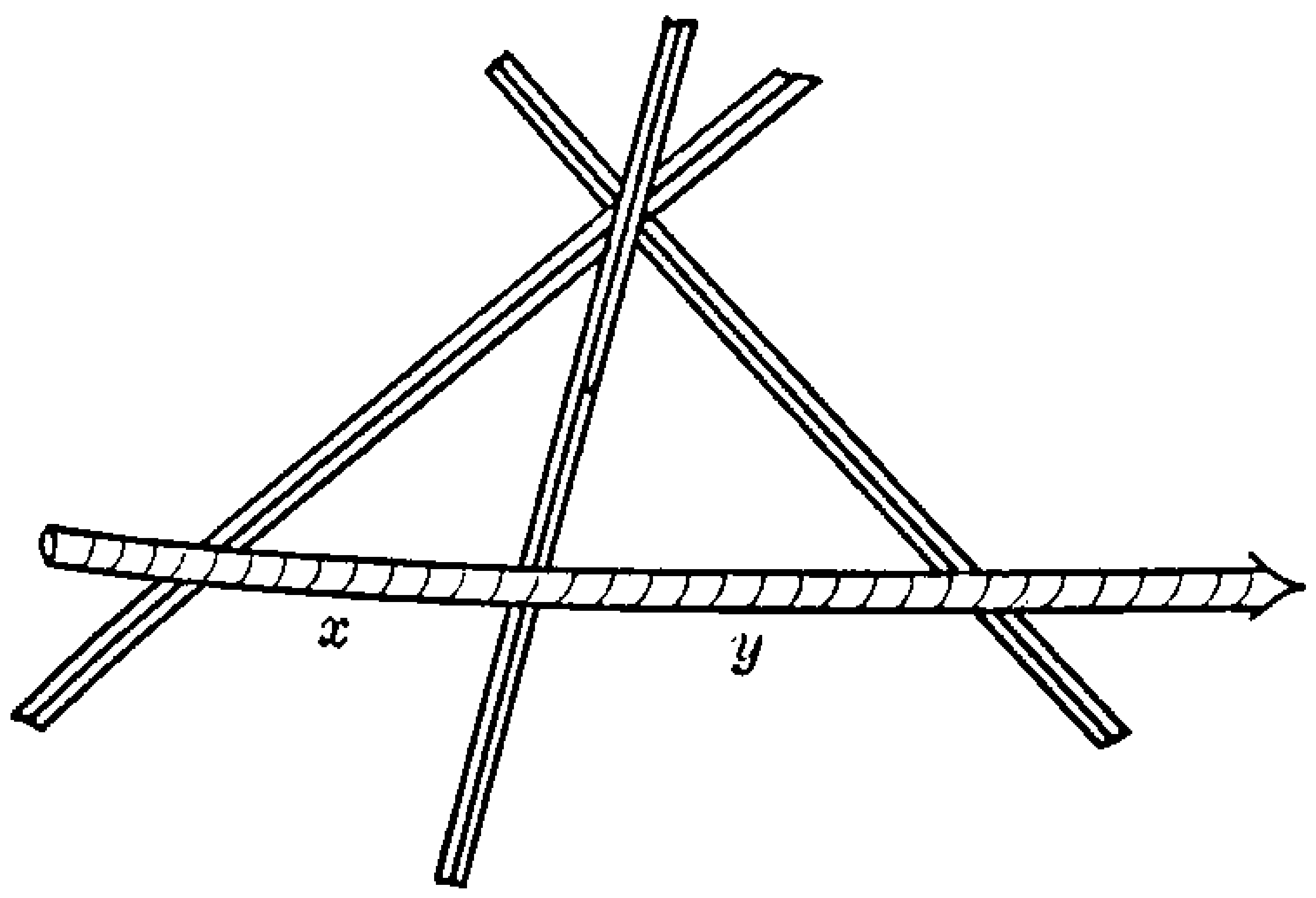
Prove: $\frac{x}{y} = \frac{x'}{y'}$

Hint: $\frac{x}{x'} = \frac{OB}{OB'}$ and $\frac{y}{y'} = \frac{OB}{OB'}$



(*b*) Draw a figure in which O is between the parallel lines, and letter it so that the same proof applies.

S3. The theorem in the preceding exercise is in static form, because it deals with a figure that is completely fixed. Instead of two fixed parallels, think of one line AC as *moving* so that it remains parallel to itself; that is, as being translated (§ 229). The diagram represents this dynamic situation.



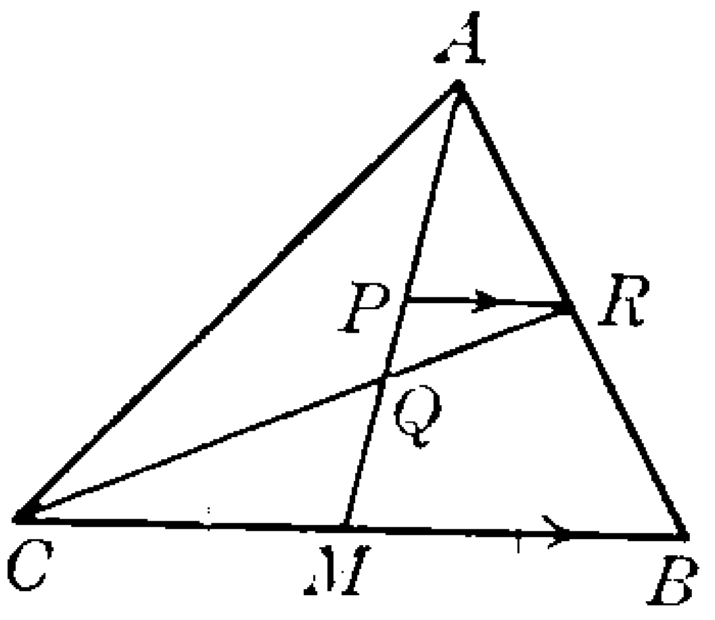
(*a*) Write the equation of variation.

(*b*) State the dynamic proposition in words.

S4. Given: Through any pt. P on the median AM of $\triangle ABC$, PR is drawn $\parallel CB$, cutting AB at R

CR cuts AM in Q

Prove: $\frac{PA}{AM} = \frac{PQ}{QM}$

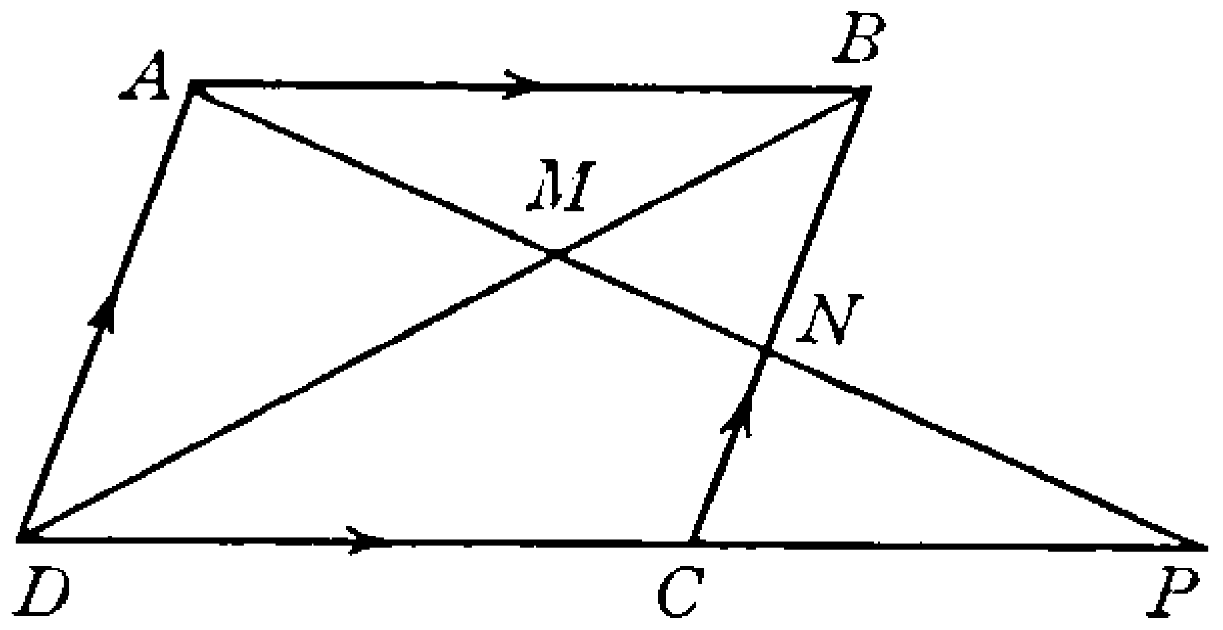


S5. Given: ▱ $ABCD$

P is any pt. on the prolongation of DC

AP cuts diagonal BD in M and side BC in N

Prove: AM the mean proportional between MN and MP



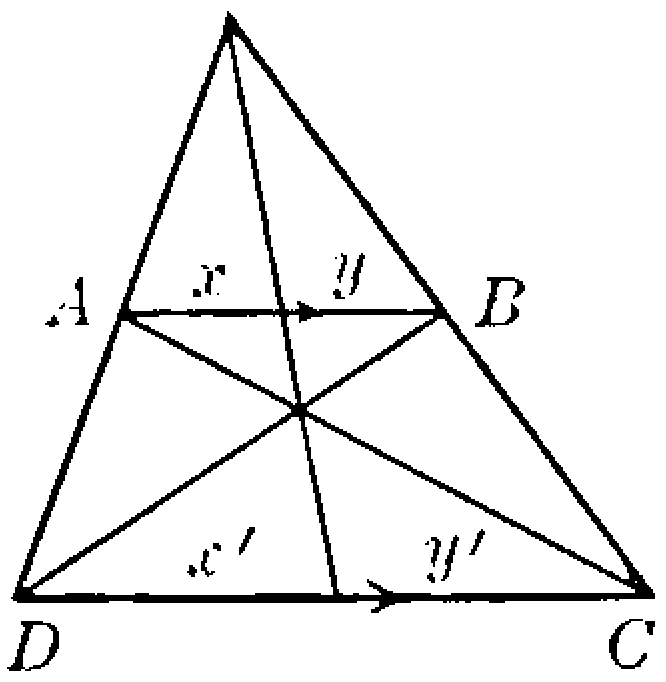
S6. One side of a triangle is extended. From the end of the extension, a line is drawn bisecting another side and cutting the third side into unequal segments. Prove that the extended side and its extension are proportional to the unequal segments on the third side.

Help: Two cases are possible, depending on which side is bisected. Draw a line parallel to the bisected side through the opposite vertex of the original triangle.

S7. If the legs of a trapezoid are prolonged to meet, then a line passing through their point of intersection and through the point of intersection of the diagonals of the trapezoid bisects the bases.

(*a*) Prove this by using (*a*) and (*b*) of S2 on this page.

(*b*) Also prove it by using Ceva's Theorem (page 193, S6).



Given:
$AB \parallel DC$
Prove:
$x = y$
$x' = y'$

HONOR WORK

H1. Prove indirectly that if a line divides two sides of a triangle proportionally, it is parallel to the third side. (See page 193, H1.)

H2. *Substitutes for Analogy.* If a speaker uses an analogy that is obviously weak or farfetched, he will often use various means to cover up such weakness. The following two devices are typical.

(*a*) *Humor.* This can be looked for in many after-dinner speeches where the speaker tells a story. The faint similarity between the subject under discussion and the (generally homely) analogy is of little importance. What *is* important is that the story be funny. If the story is humorous enough, the listener may be counted on to overlook the incomplete or even false analogy.

If you cannot cite examples of this trick in analogy, ask your librarian for a book of after-dinner speeches.

(*b*) *Emotional Language.* A forceful speaker can often cover up weaknesses in argumentation in general, and particularly in analogy, by the use of strong and emotional language. Examples: "Our constructive program is certainly much superior to their quack panaceas," or "We advance cogent arguments while the obstructionists make only lame excuses."

Find examples of such language in speeches, articles, or editorials.

Segments Made by Three Parallels

The Theorem in § 302 may be stated: two parallel lines cut proportional segments on the sides of an angle. This will now be extended to three or more parallels.

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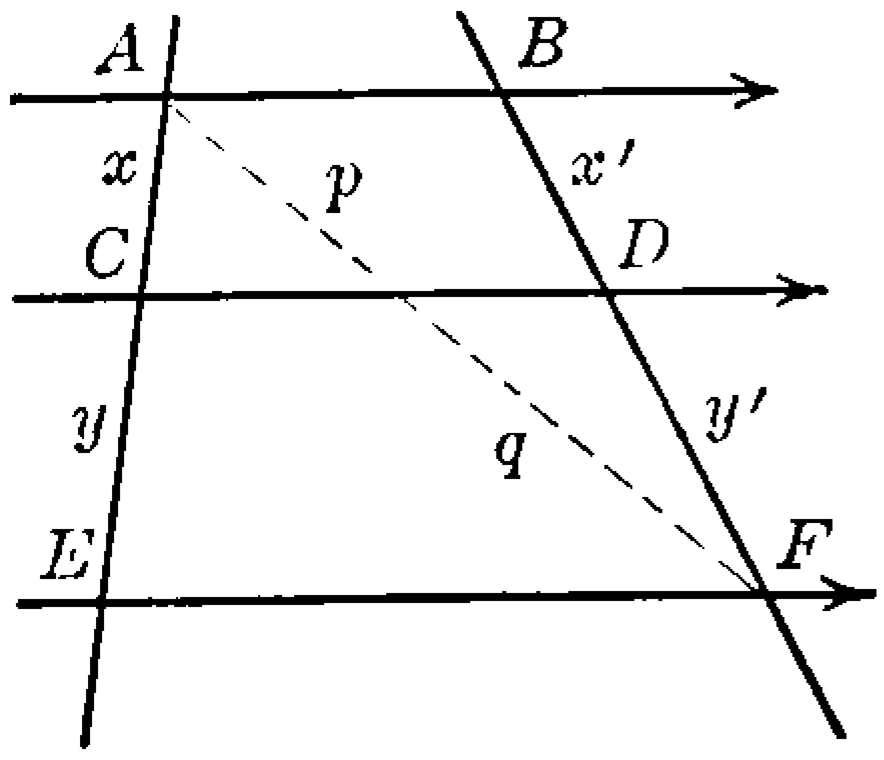
305. THEOREM. Three or more parallel lines cut proportional segments on any two transversals.



Given: $AB \parallel CD \parallel EF$

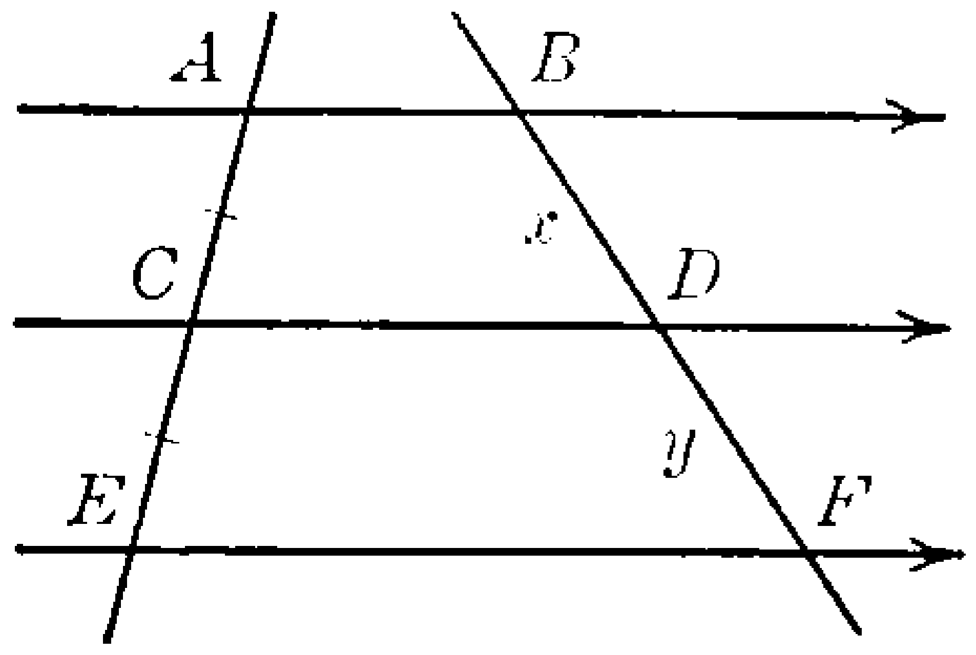
Prove: $\frac{x}{y} = \frac{x'}{y'}$

Cons.: Draw AF.

PROOF

Statements	*Reasons*
1. $\frac{x}{y} = \frac{p}{q}$ $\frac{x'}{y'} = \frac{p}{q}$	1. A line ∥ to one side of a △ divides other sides proportionally (§ 302).
2. $\frac{x}{y} = \frac{x'}{y'}$	2. Subs. Ax.

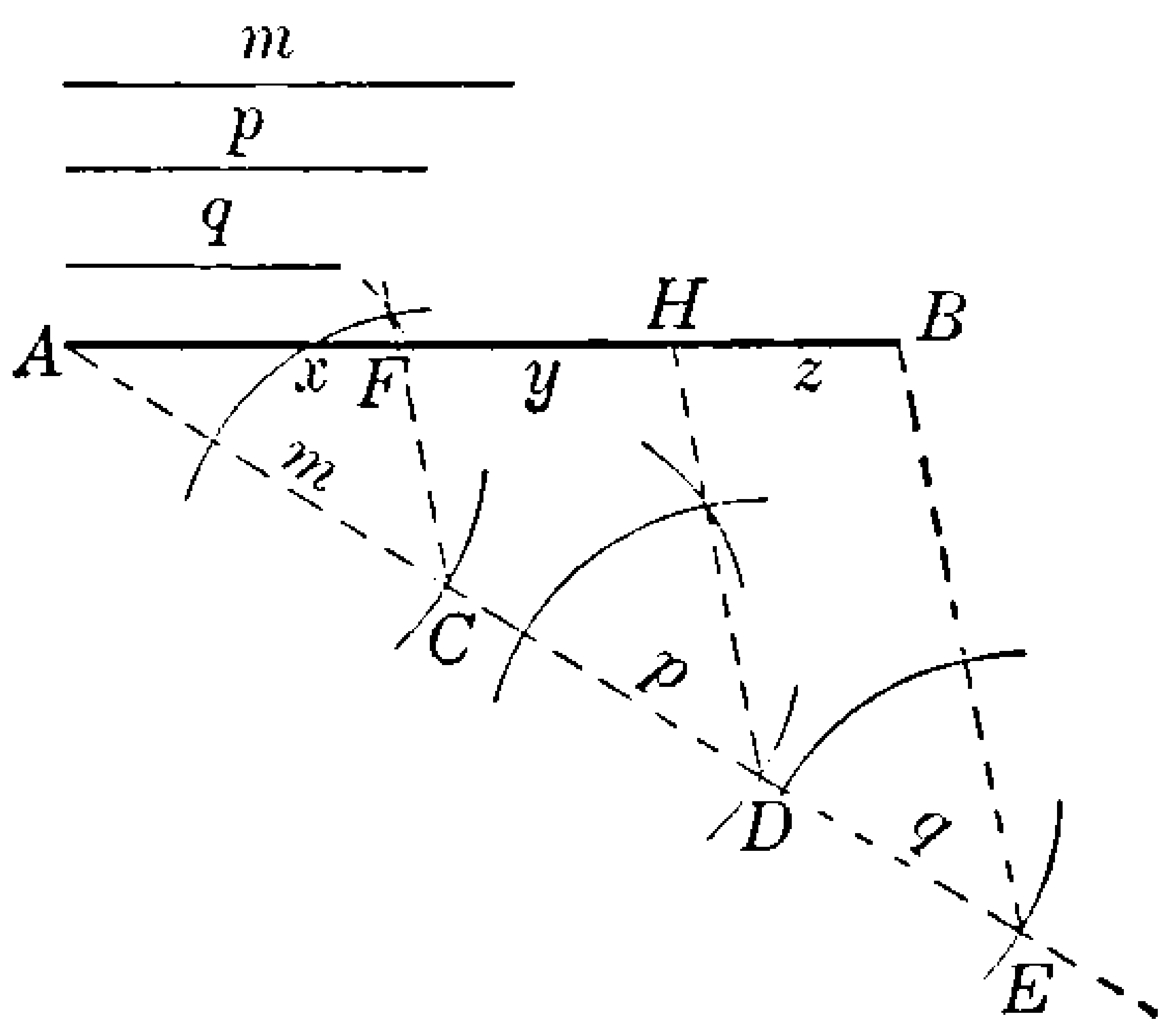
306. Corollary. **If three or more parallels cut equal parts on one transversal, they cut equal parts on any other transversal.**



Given:
$AB \parallel CD \parallel EF$
$AC = CE$
Prove:
$x = y$

---

307. PROBLEM. Divide a given line segment into three parts proportional to three given line segments.



Given: Line segments AB, m, p, and q
Req.: To divide AB into three parts proportional to m, p, and q

Construction

1. Through A, draw any line AE. On it lay off $AC = m$, $CD = p$, and $DE = q$. Draw EB.
2. Cons. CF and DH parallel to EB.

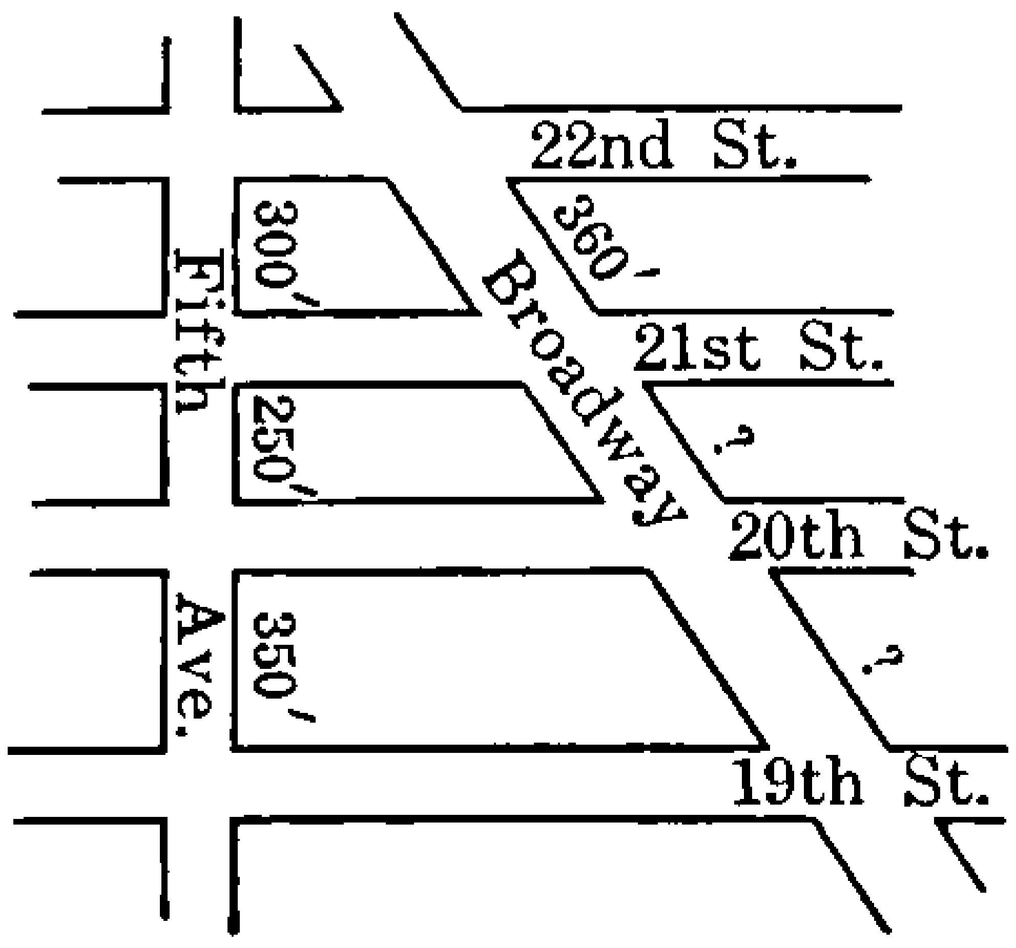
Conclusion: $\frac{x}{m} = \frac{y}{p} = \frac{z}{q}$.

PROOF

Statements	*Reasons*
1. $\frac{x}{m} = \frac{y}{p}$	1. A line $\parallel$ to one side of $\triangle$ divides other sides proportionally (§ 302).
2. $\frac{y}{p} = \frac{z}{q}$	2. Three $\parallel$s cut two transversals proportionally (§ 305).
3. $\frac{x}{m} = \frac{y}{p} = \frac{z}{q}$	3. Subs. Ax.

EXERCISES

1. (*a*) In the street map shown, find the distance along Broadway between 19th and 20th Streets; also, between 20th and 21st Streets.
   (*b*) Two boys walking along Broadway and Fifth Avenue cross the same east-west streets simultaneously. Find the ratio of their speeds. Will they meet?



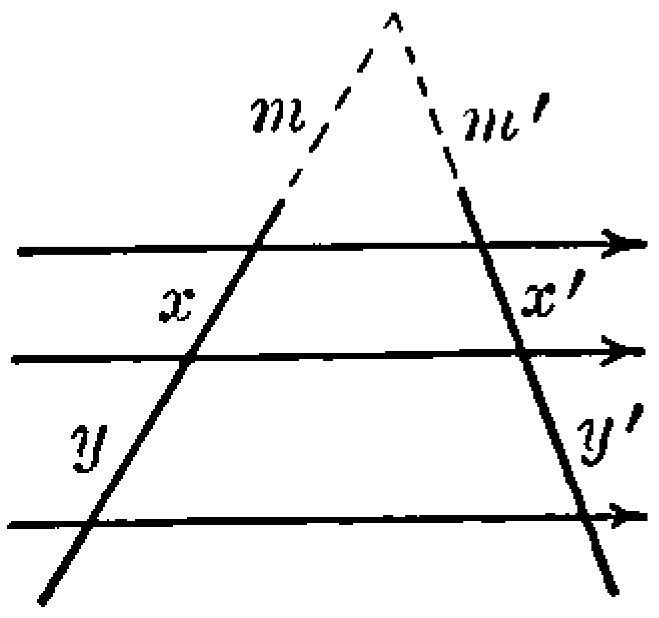
2. Divide a line segment into three parts that have the ratios 2 : 3 : 5.
3. Trisect a given line segment.
4. Show how you can divide the edge of a card into a number of equal parts by laying it across a sheet of ruled paper.
5. Divide a triangle into three equal parts by constructing lines through one vertex.
6. Prove: In a semicircle, perpendiculars to a chord at its ends will cut off equal segments from the ends of the diameter.

SUPERIOR WORK

S1. Divide a parallelogram into three equal parts by constructing lines through one vertex. (Construct the given parallelogram.) *Hint:* First divide the parallelogram into six equal parts.

S2. Prove the theorem about the three parallels (§ 305) by prolonging the two transversals to meet. *Help:*

$$\frac{x + y}{m} = \frac{x' + y'}{m'}, \qquad \frac{m}{x} = \frac{m'}{x'}$$



S3. State the theorem about the three parallels (§ 305) dynamically. Draw a figure and write the variation. (See page 210, S1.)

S4. One leg of a trapezoid is extended equal lengths in both directions. With the extended leg as a diameter, a semicircle is drawn cutting the bases of the trapezoid (extended, if necessary). Prove that the area of the trapezoid equals the average of the bases times the distance between the points where the semicircle cuts the bases.

S5. Given the perimeter of a triangle and two of its angles, construct the triangle. *Help:* First construct any triangle having the two given angles. How is this triangle related to the required triangle?

HONOR WORK

H308. *Valid Analogy.* We have seen that nonmathematical analogy cannot be used as proof because the relationships are not usually defined. However, analogy does have many applications in nonmathematical thinking. The chief uses of analogy that are valid are: (1) explanation, (2) discovery, and (3) embellishment.

Analogy is used to show that certain facts which are little understood are related to each other in the same way as certain other facts with which we are well acquainted. For example, an electric current may be explained by analogy with the flow of water in a pipe. Give some other examples in which analogy is used for the purpose of explanation.

If two situations resemble each other in certain respects, they *may* do so in other ways. If the new resemblance is borne out, then it is the first step of an induction (§ 104), which may lead to a discovery. For example, since the conditions on the planet Mars are in some ways similar to those on the earth, we are led to wonder if Mars is inhabited. Give other examples in which analogy is an aid to discovery.

Analogy is used to make language more picturesque and dynamic. A situation is compared with another more appealing or more forceful situation that it resembles in some respects. If the comparison is expressed, the analogy, or figure of speech, is called a *simile;* if it is implied, it is called a *metaphor.* Such use of analogy is frequent in poetry. Here are a few examples: His purpose was as constant as the North Star (simile). A flatterer is like a distorted mirror (simile). The ship plows the sea (metaphor). "Fog comes on little cat feet" (metaphor).

H1. Find word analogies like "hands of a clock" and "pipe elbow." Form metaphors using the terms *spartan*, *platonic*, *stoic*, *laconic*, and *quisling*.

H2. Illustrate the use of analogy in poetry, including rhyme, figures of speech, and so on.

H3. Read an account of the discovery of the planet Neptune. Explain how this is an example of analogy.

H4. Illustrate analogy from your study of history.

H5. If two parallel lines are cut by a transversal, the corresponding angles are equal. State the analogous proposition in solid geometry, using three planes. Is it valid? Is its converse valid?

SUMMARY

309. In everyday thinking, two things that are alike in some ways but not in others are called similar or analogous. Thus, two figures that have the same shape but not the same size are called similar. In mathematics, most analogies are based on proportion. Similar polygons are polygons that are mutually equiangular and have their sides proportional. It was proved that two triangles are similar if they agree in two angles.

An easy way to prove line segments proportional is by showing that they are the corresponding sides of similar triangles. Proportions may also be set up by comparing the areas of similar triangles with the *squares* of their sides.

If two or more parallels cut segments from the sides of an angle, then these segments are proportional. This fact enables us to construct the fourth proportional to three given line segments, and to divide a given line segment into parts having specified ratios.

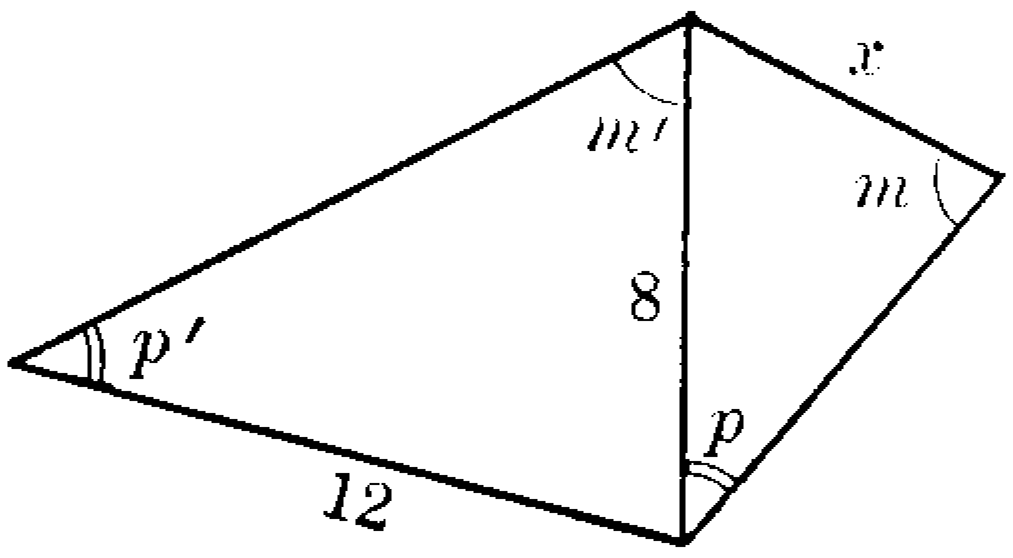
RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

analogy § 294
similar polygons ($\sim$ polygons) § 296
fourth proportional § 303

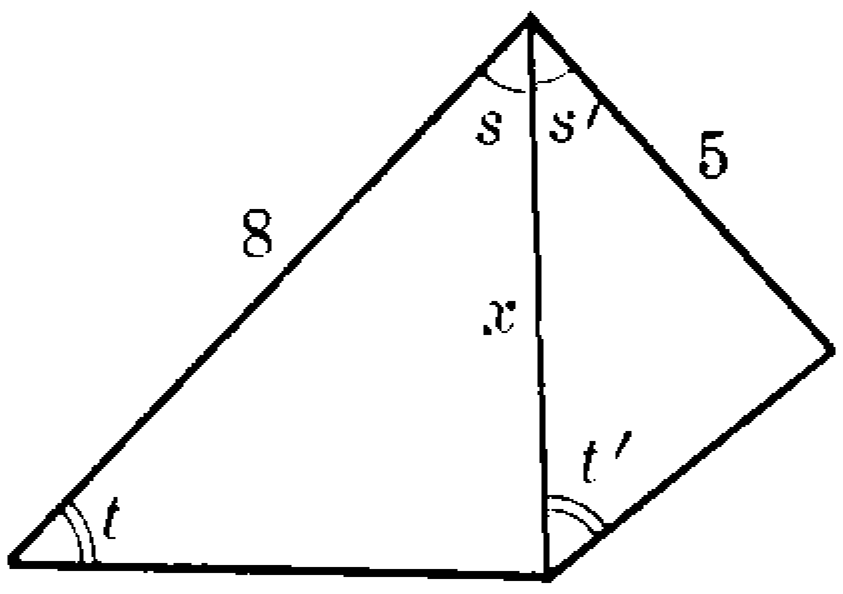
REVIEW EXERCISES

1.



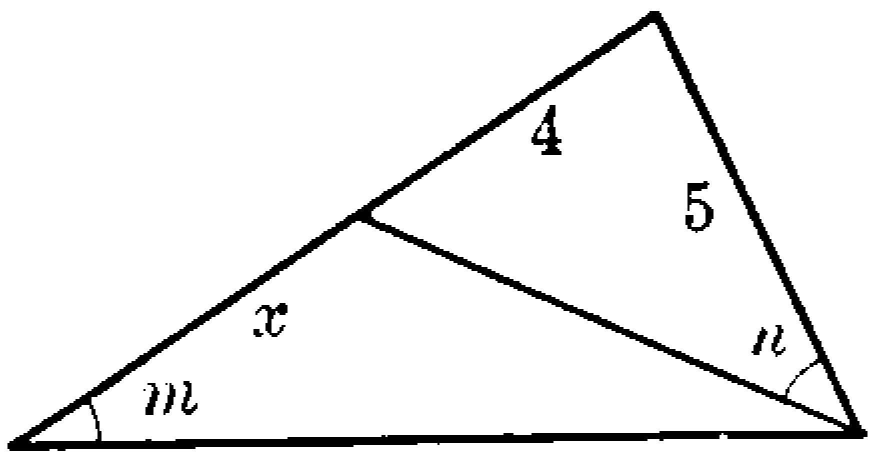
Given:
$\angle m = \angle m'$
$\angle p = \angle p'$
Find:
x

2.



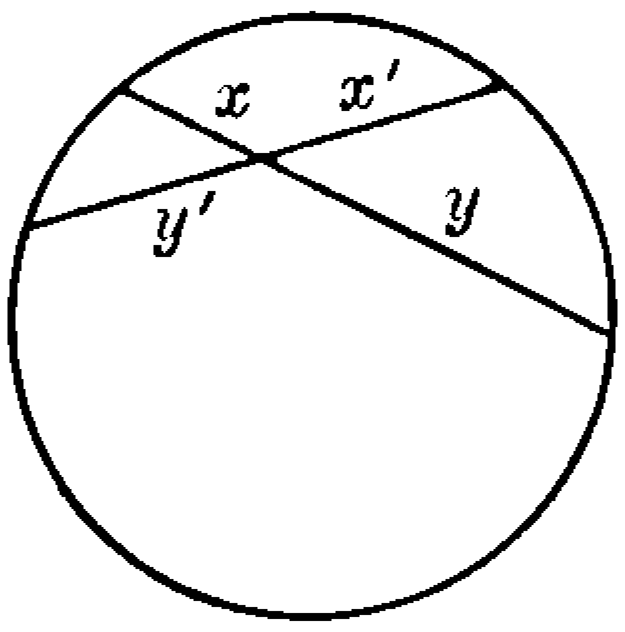
Given:
$\angle s = \angle s'$
$\angle t = \angle t'$
Find:
x

3.



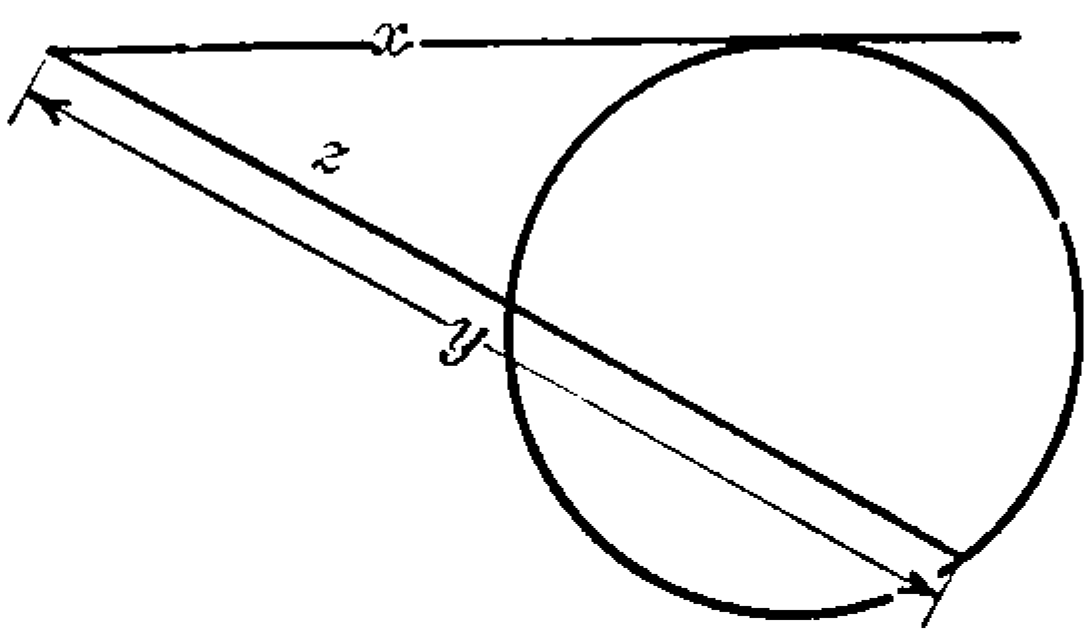
Given:
$\angle m = \angle n$
Find:
x

4.



Prove:
$x\,y = x'y'$
Hint:
Draw two chords forming $\sim\triangle$s.

5. The figure has a tangent x, a secant y from the same point, and the external part of the secant z.
Prove that $x^2 = y \cdot z$.



6. Prove that the diagonals of a trapezoid divide each other proportionally. (Use similar triangles.)

7. Prove that if two altitudes of a triangle meet within the triangle, then the product of the segments of one altitude equals the product of the segments of the other altitude.

8. Prove that the areas of two similar triangles are proportional to the squares of the corresponding altitudes.

9. (*a*) Compute the fourth proportional to 2, 6, and 6. Your answer is called the "third proportional" to 2 and 6.
   (*b*) Construct the third proportional to the diagonal of a given square and its side. With your compass, compare the length of the third proportional with the diagonal.

10. On a given base, construct a rectangle twice as large as a given square.

11. Prove that if a line segment between two sides of a triangle bisects one of these sides, and is parallel to the base, then it is equal to one-half the base.

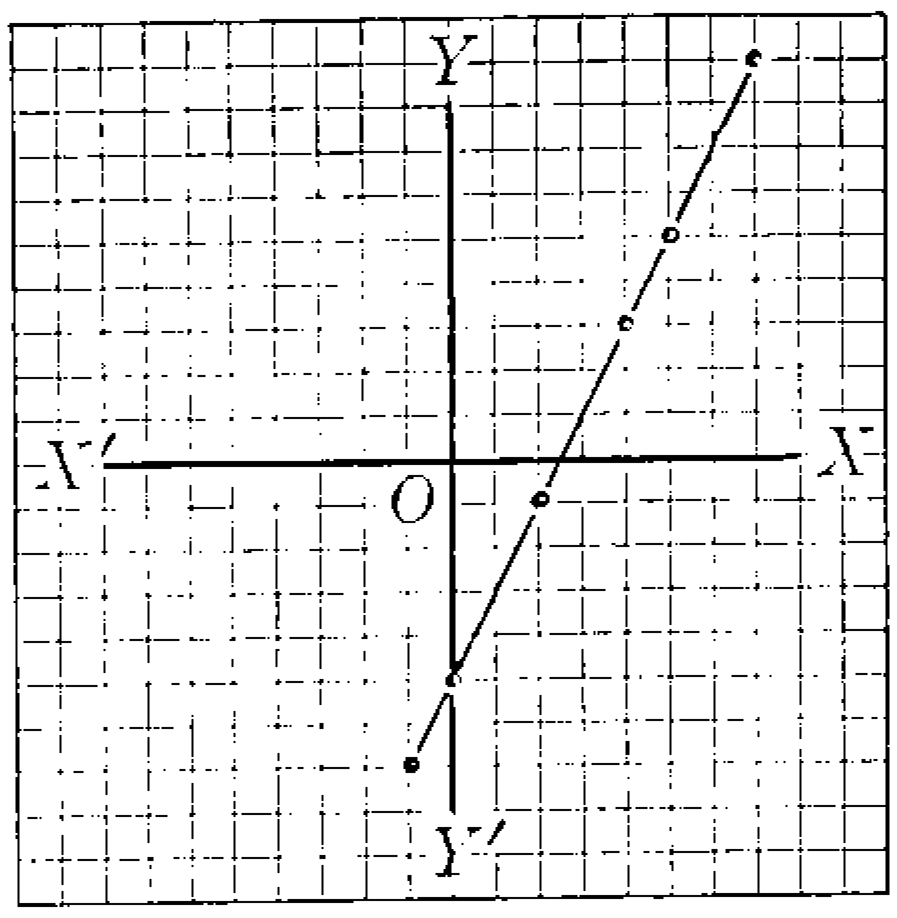
12. (*a*) Prove that if the sides of one triangle are parallel respectively to the sides of another triangle, then the triangles are similar.
    (*b*) Prove the same about two triangles whose sides are respectively perpendicular to each other.

SUPERIOR WORK

S310. *Analogy between Algebra and Geometry.* An important analogy between algebra and geometry was set up by the French mathematician Descartes. It correlates the locus of points in space with the equation that the coordinates of these points must satisfy. You will recall from your algebra that the space locus is called the *graph* of the equation.

Illustrative Example. Find the locus of $P(x, y)$ such that $y = 2x - 5$.

By substituting values of x in the given equation, we find the corresponding values of y. Thus, when $x = 0$ then $y = -5$, when $x = 7$ then $y = 9$, and so on. When we have located enough points, we sketch a smooth curve through them. This is called plotting the equation. Our graph is a straight line.



x	y
0	−5
7	9
−1	−7
4	3
2	−1
5	5

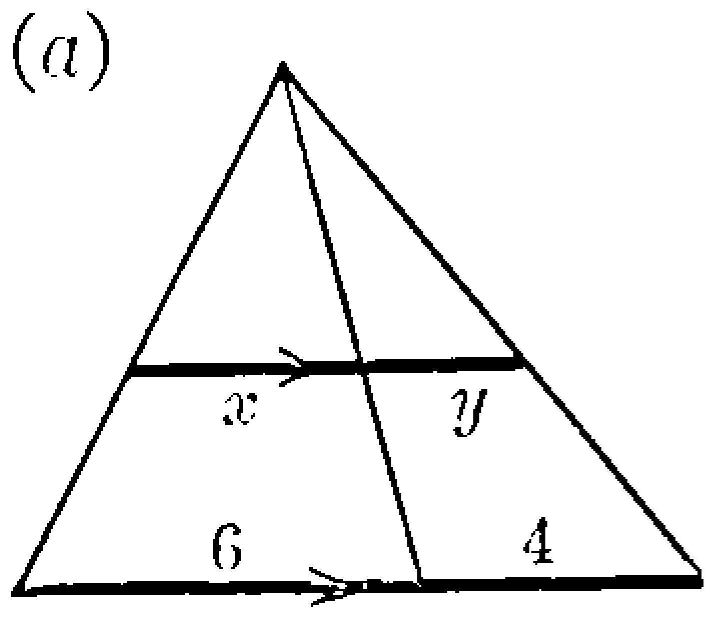
S1. Plot the locus $y = 3x$.

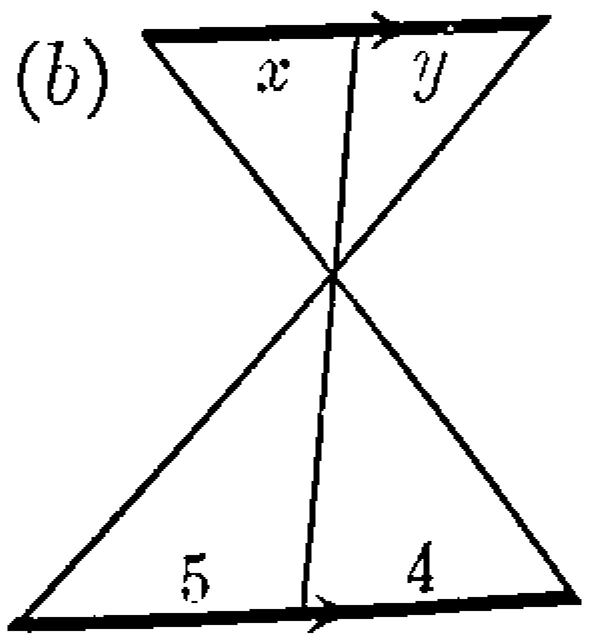
S2. Plot the locus $\frac{x}{8} + \frac{y}{5} = 1$.

S3. Plot the following two loci on the same axes:

$$3x + 2y = 12$$
$$2x - 5y = -5$$

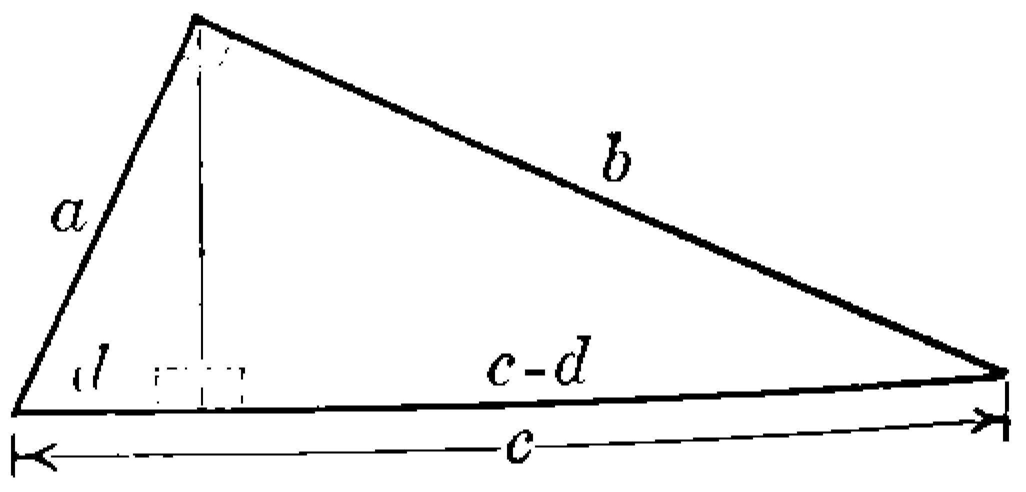
S4. Write an equation connecting x and y in each figure.





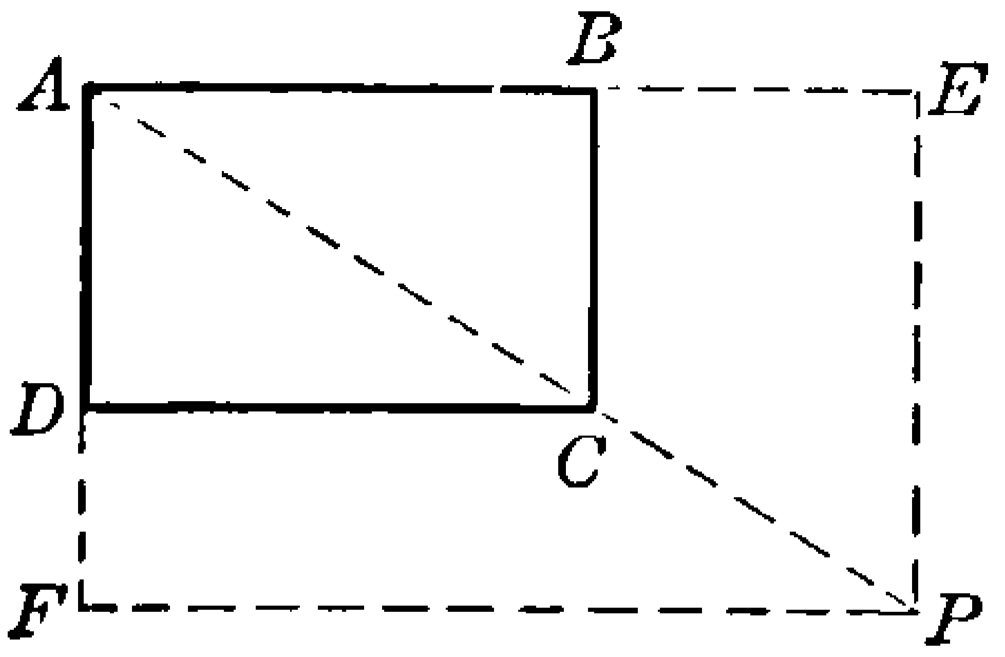
S5. By using similar triangles, prove that two altitudes of a triangle are inversely proportional to the sides to which they are perpendicular.

S6. By completing the equations $a^2 = $ —?— and $b^2 = $ —?—, as you did in S3, page 204, prove that $a^2 + b^2 = c^2$.



S7. The method indicated in the diagram at the top of the next page is often used to enlarge a given rectangle $ABCD$. Through the point P, on the prolongation of the diagonal AC, draw PE parallel to DA, and PF parallel to BA.

Prove that rectangle AP is similar to rectangle AC.



S8. On a given base, construct an isosceles triangle equal to a given square.

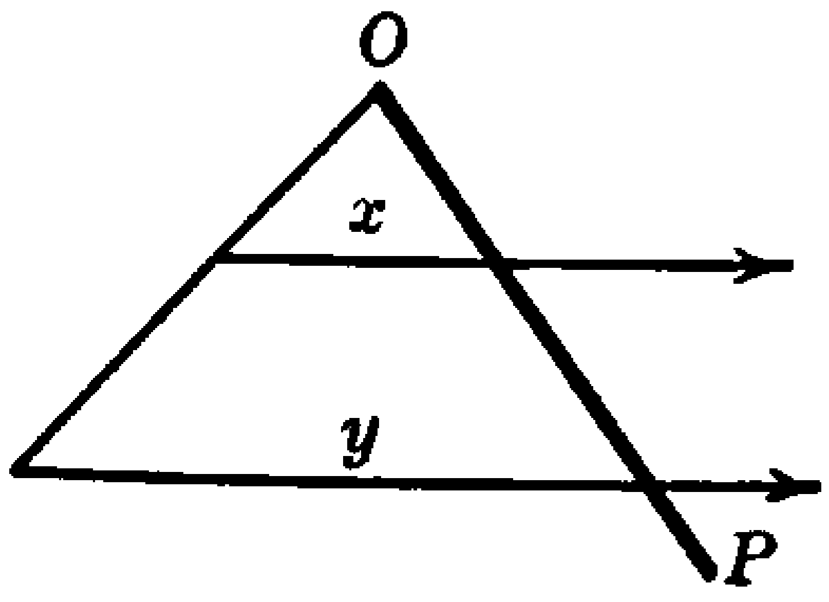
S9. With its perimeter given, construct a rectangle similar to a given rectangle.

S10. Through a given point on the base of a given trapezoid, construct two lines that divide the figure into three equal parts.

S11. A triangle grows 44% larger but retains its shape. By what percent have the sides increased?

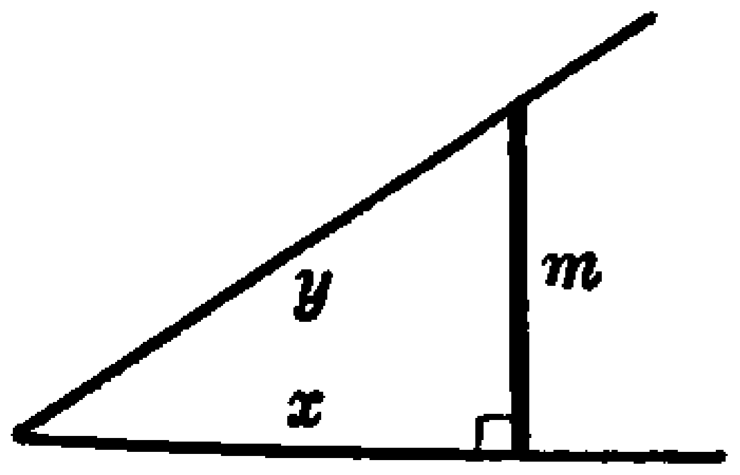
S12. In each of the following stationary figures, there is one moving part. Write an equation connecting x and y in each case:

(*a*)



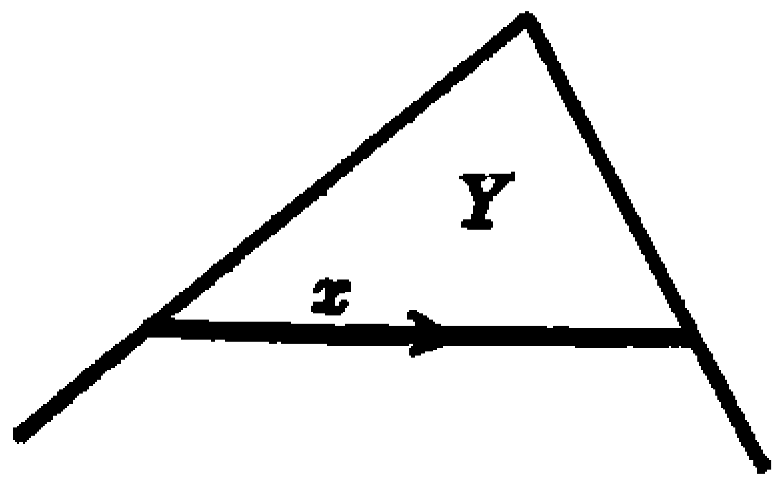
OP rotates about O $(x \parallel y)$

(*b*)



m moves

(*c*)



x is translated

S13. By referring to page 187, S4, explain how analogy is used in testing model planes in a wind tunnel.

S14. Analogous to the theorem about three parallel lines that are cut by two transversals is the following proposition in general space.

Theorem. If three parallel planes are cut by two lines, then the segments on these lines are proportional.

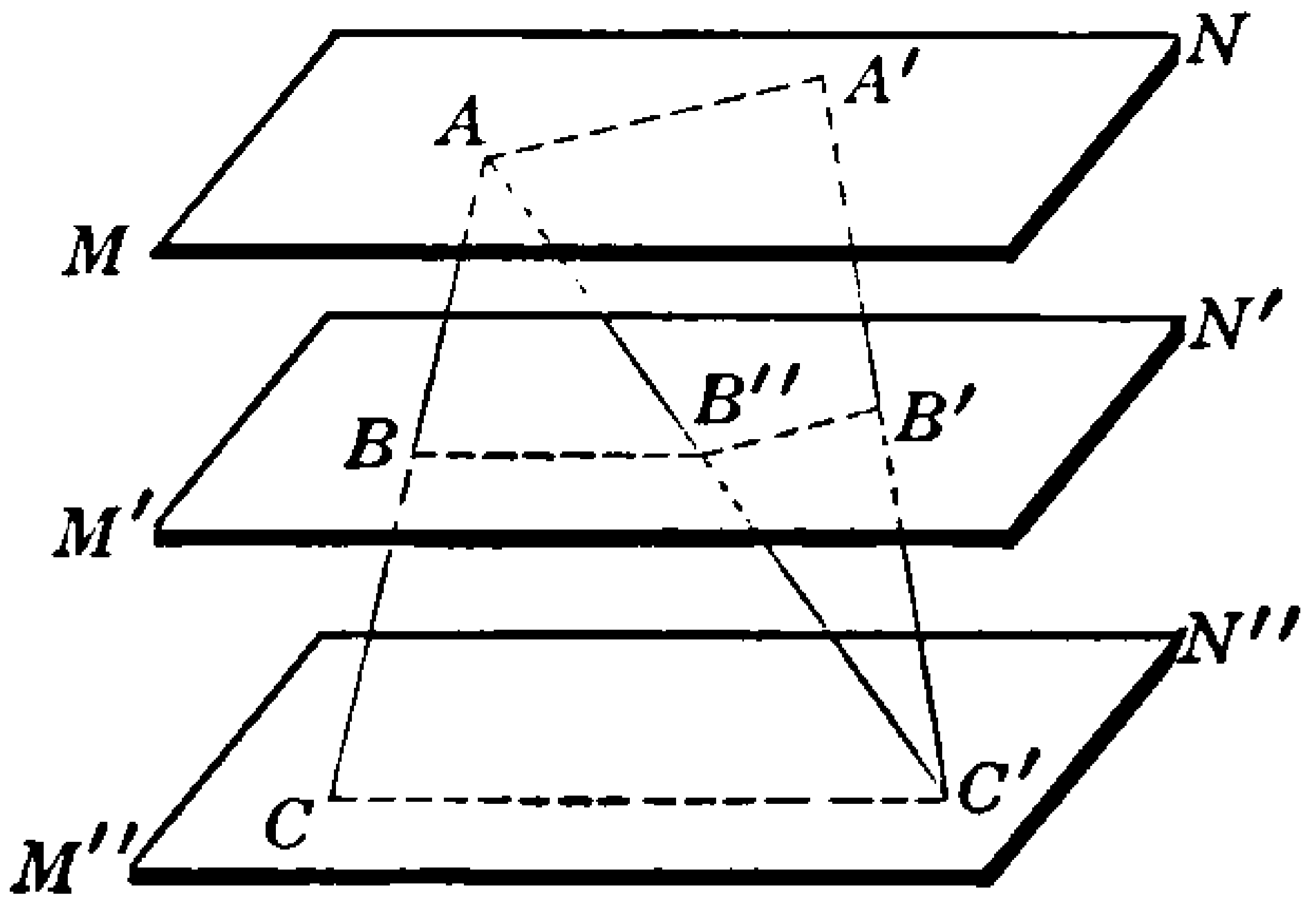
Given: Plane $MN \parallel$ plane $M'N' \parallel$ plane $M''N''$

Transversals AC and $A'C'$

Prove: $\dfrac{AB}{BC} = \dfrac{A'B'}{B'C'}$

Cons.: Pass a plane through AC and pt. C', cutting plane $M'N'$ in BB'' and plane $M''N''$ in CC'.

Also, pass a plane through $A'C'$ and pt. A, cutting plane $M'N'$ in $B''B'$ and plane MN in AA'.



PROOF

Statements	*Reasons*
1. $BB'' \parallel CC'$ $B''B' \parallel AA'$	1. Why?

Complete the proof.

S15. In general space, a prism (page 197, S6) is analogous to a parallelogram in a plane. The volume of a prism is also analogous to the area of a parallelogram; that is, the measure of each is the base times the altitude. Whereas the base of a parallelogram is a line segment, the base of a prism is a two-dimensional figure. When we multiply the two-dimensional base by the altitude, we get a three-dimensional magnitude for the volume.

(*a*) Each side of the base of a square prism is 5 inches. The altitude of the prism is 4 inches. Find the volume of the prism.

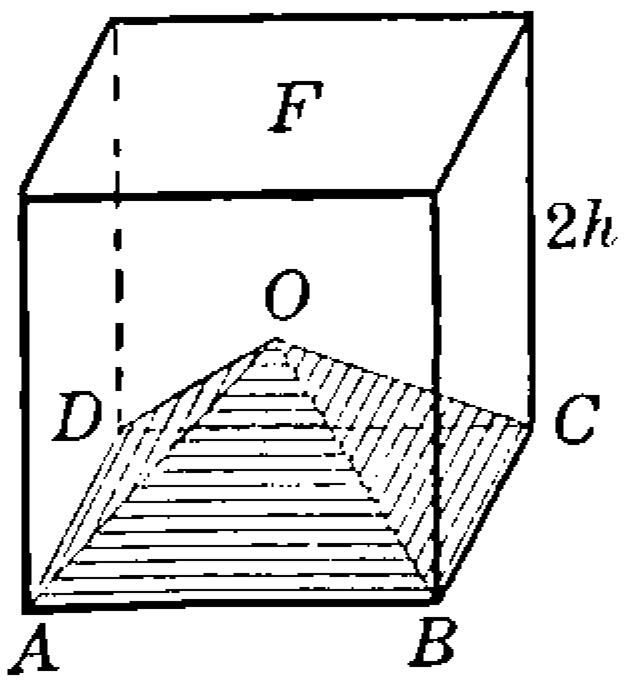
(*b*) The base of a prism is a right triangle whose legs are 6 feet and 9 feet. The volume of the prism is 135 cubic feet. Find the altitude of the prism.

S16. A swimming pool is 40 feet long and 18 feet wide. Its depth increases along the length from 3 feet to 12 feet. Find the capacity of the pool in gallons. (1 cubic foot = $7\frac{1}{2}$ gallons.)

S17. A railroad cut is to be 150 feet long and 9 feet deep. A cross section of the cut is to be a trapezoid 14 feet wide at the bottom and 22 feet wide at the top. Find the cost of removing the earth at \$6 a cubic yard.

S18. A pyramid (page 197, S10) is somewhat analogous to a triangle. Since the area of a triangle equals half the base times the altitude, one might guess that the volume of a pyramid is also half the base times the altitude. However, the analogy does not extend so far. This can easily be seen by cutting up a cube into pyramids.

We pass a plane through O, the center of the cube, and each of the edges. In this way, each face of the cube becomes the base of a new pyramid. The cube is now divided into six equal parts like the pyramid O–$ABCD$. Let F equal the area of the base of this pyramid (and also the base of the cube), and let h equal the altitude of the pyramid. The volume of the cube equals $F \cdot 2h$.



Therefore,

$$6 \times \text{pyramid } O\text{–}ABCD = F \cdot 2h$$

and $$\text{pyramid } O\text{–}ABCD = \tfrac{1}{3}Fh$$

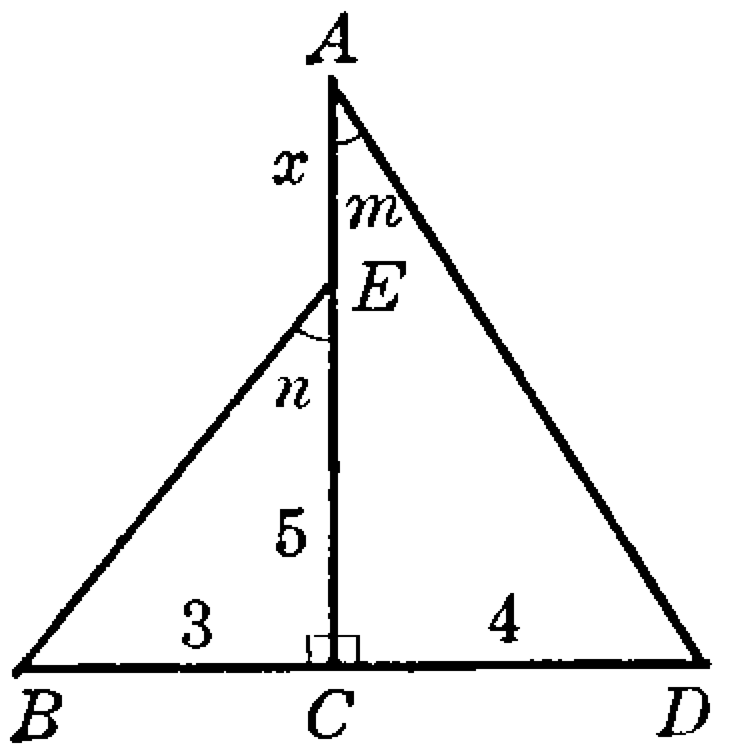
This suggests that the volume of a pyramid is one-third the base times the height. It can be proved that this is true for all pyramids (and also cones).

(*a*) The base of a pyramid is a rectangle 3 feet by 5 feet. The altitude of the pyramid is 4 feet 4 inches. Find the volume of the pyramid.

(*b*) The altitude of a square pyramid is 9 inches and the volume of the pyramid is $\frac{1}{6}$ of a cubic foot. Find each side of the base.

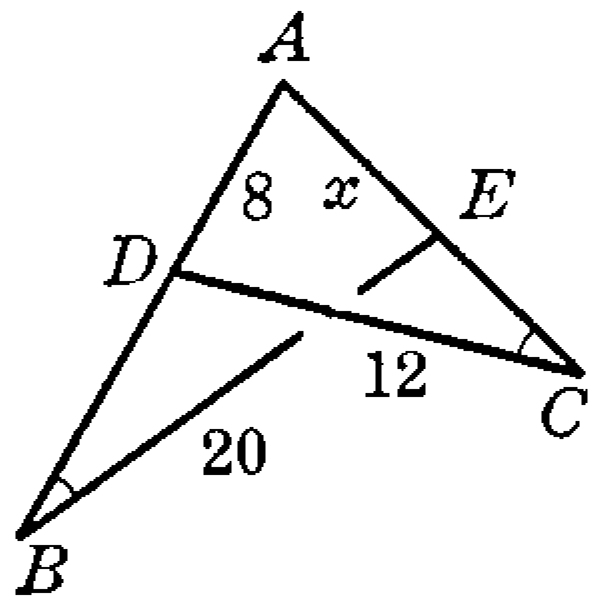
TEST

1.



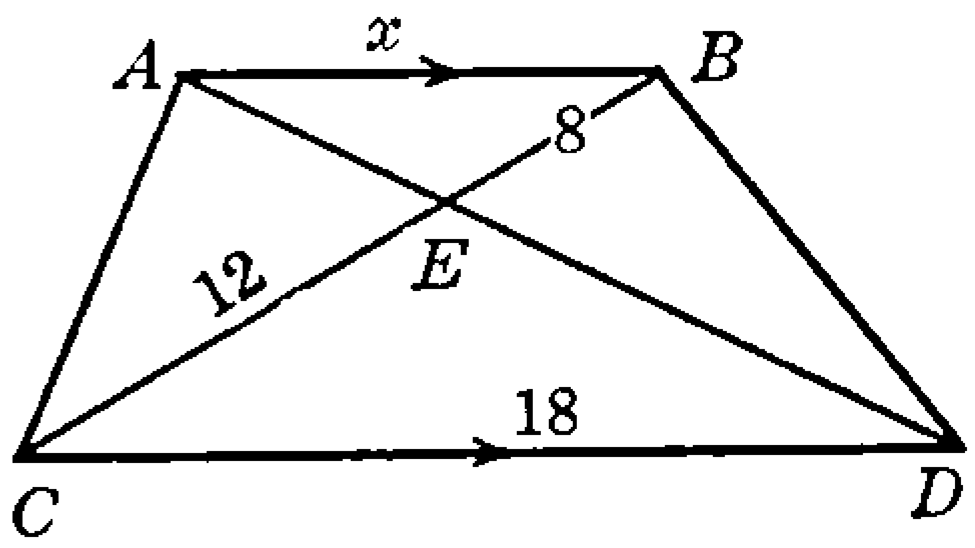
Given:
$\angle m = \angle n$
$AC \perp BD$
$BC = 3$
$CD = 4$
$CE = 5$
Find:
AE

2.



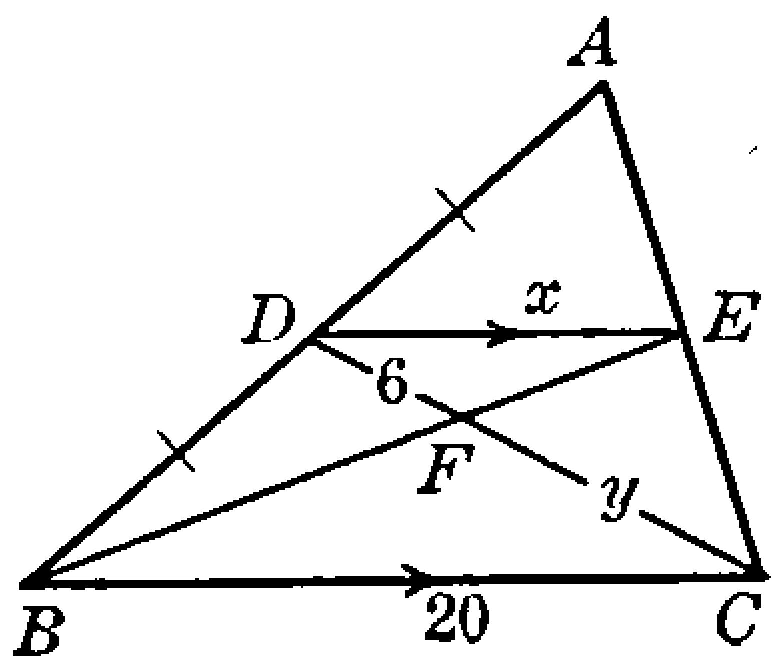
Given:
$\angle B = \angle C$
$AD = 8$
$BE = 20$
$CD = 12$
Find:
AE

3.



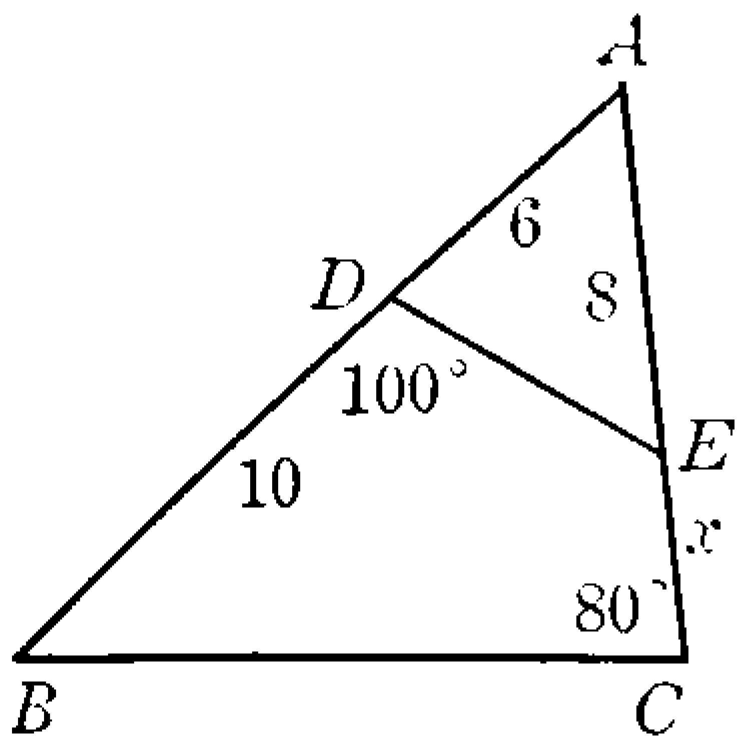
Given:
$AB \parallel CD$
$BE = 8$
$EC = 12$
$CD = 18$
Find:
AB

4.



Given:
$AD = DB$
$DE \parallel BC$
$BC = 20$
$DF = 6$
Find:
DE and FC

5.



Given:

$\angle BDE = 100^\circ$

$\angle C = 80^\circ$

$DB = 10$

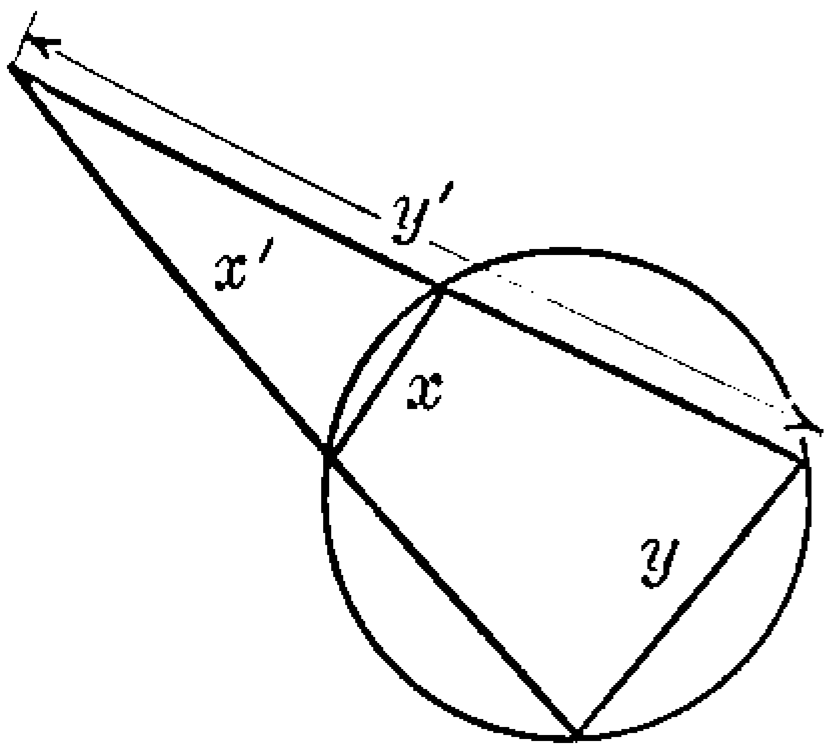
$AD = 6$

$AE = 8$

Find:

EC

6.



Prove:

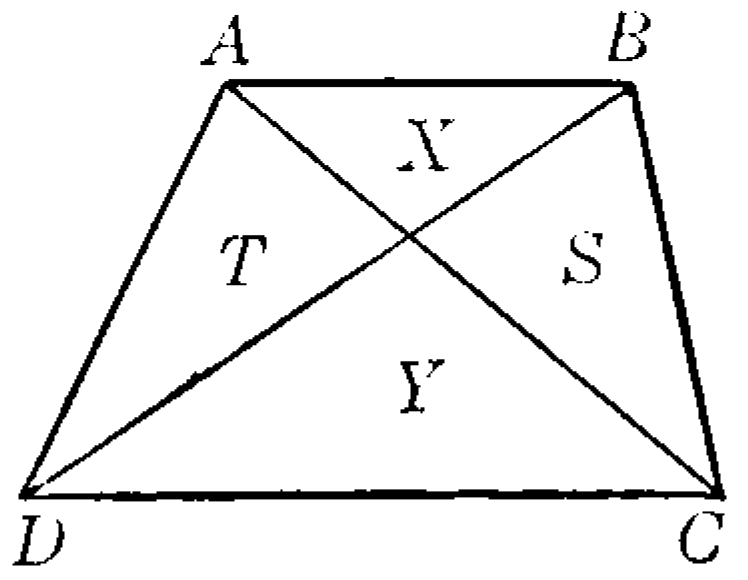
$$\frac{x}{y} = \frac{x'}{y'}$$

7. A building casts a shadow of 20 feet at the same time that a vertical yardstick casts a shadow of 16 inches. How high is the building?

8. Two corresponding sides of two similar triangles are 17 inches and 51 inches. If the area of the smaller triangle is 4 square feet, find the area of the larger triangle.

9. Divide a given line segment into two parts that have the same ratio as the leg and the hypotenuse of an isosceles right triangle.

10.



Given:

Quad. $ABCD$ with diags. AC and BD

$\triangle T = \triangle S$

Prove:

$\triangle X \sim \triangle Y$

9

Functional Thinking

Solution of Triangles

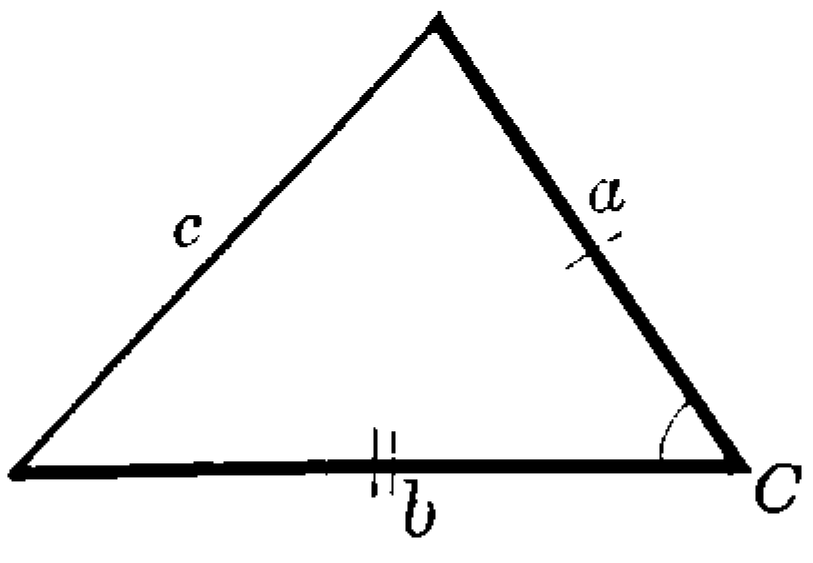
THE RIGHT TRIANGLE

Dependence

311. The following relationships in algebra illustrate the idea of *dependence:* $y = 2x$, $y = x^2$, $y = x - 5$.

In each case the value of y depends on the value of x. If the value of x is known, then y is completely determined. If $x = 5$, what is the value of y in each case? This idea of dependence is expressed by saying that y is a *function* of x.

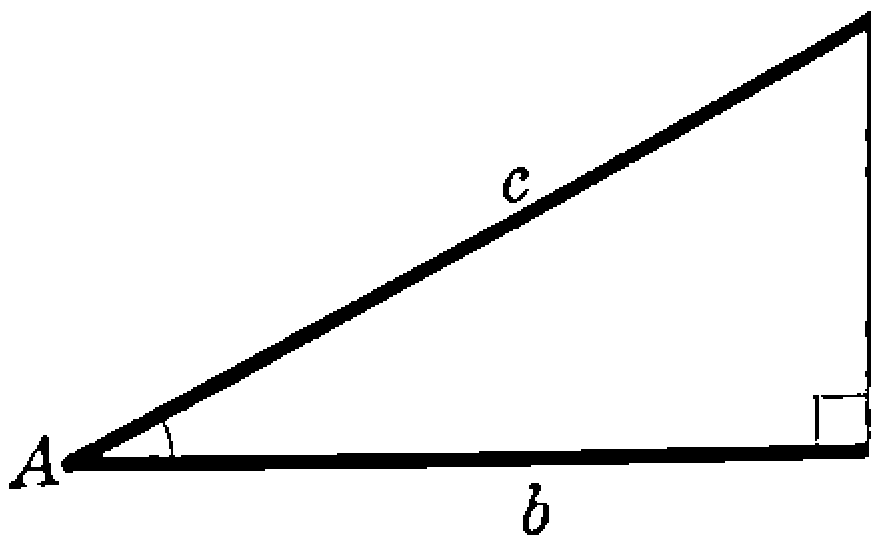
In the figure, the side c depends on the sides a and b and the angle C. The reason for this is that if a, b, and $\angle C$ are given, the whole triangle is completely determined (s.a.s.) and, in particular, the side c is determined. Hence, c is a function of a, b, and $\angle C$.



It is generally possible to calculate one part of a triangle when three other parts are known. To learn how to do this, we begin with the right triangle.

The Cosine Function

312. From our fundamental theorem about similar triangles (§ 298) we know that if two right triangles have an acute angle of one equal to an acute angle of the other, then the ratio of any two sides of one triangle equals the ratio of the corresponding sides of the other triangle. This means that no matter how large or how small the right triangle in the figure becomes, as long as the angle A remains the same, the ratio $\frac{b}{c}$ does not change.



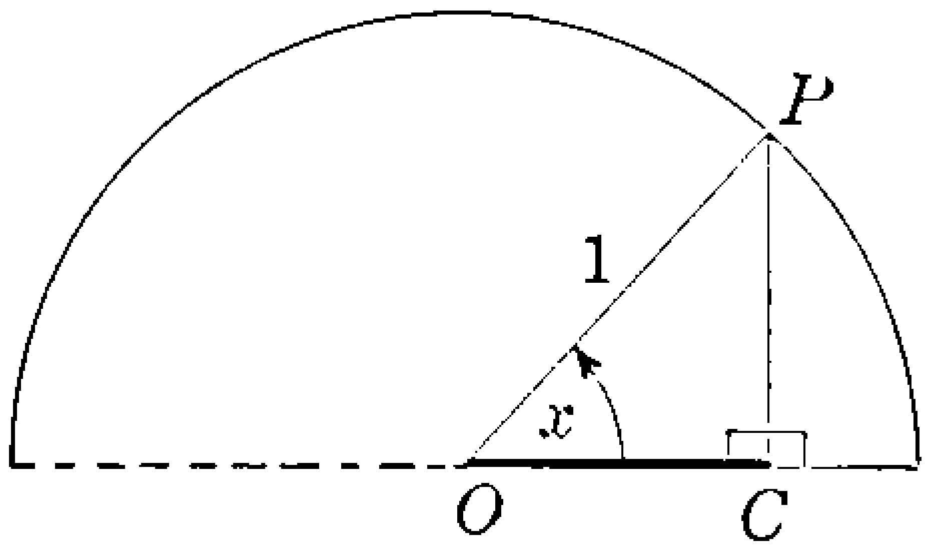
The ratio $\frac{b}{c}$ depends only on $\angle A$. It is therefore a function of $\angle A$. We call this function the *cosine* of the $\angle A$ (abbreviated cos A).

$$\cos A = \frac{\text{Adjacent Leg}}{\text{Hypotenuse}}$$

313. Definition. If from a point on one side of a given angle a perpendicular on the other side is drawn, then the leg of the resulting right triangle adjacent to the given angle divided by the hypotenuse is the ratio called the cosine of the given angle.

How the Cosine Changes

314. We shall now see how an increase in the size of an angle causes a change in the value of its cosine. Since the cosine of an angle x depends only on the angle, and not on the size of the triangle, we shall make one side OP of the angle x equal to 1 and draw PC perpendicular to OC. From the definition of the cosine, we have:



$$\cos x = \frac{OC}{OP} = \frac{OC}{1} = OC$$

Thus, OC alone measures the cos x.

As OP rotates to the left, the angle x grows larger, and the point C moves to the left. From this we observe that as $\angle x$ grows larger, OC, which is cos x, becomes smaller. The greatest value that OC can have is 1 (the radius of the circle) and then the angle x would be zero. As the angle x grows larger, the cosine becomes smaller until it reaches zero as the angle x becomes 90°.

As the angle x grows larger than 90°, the stationary side of the angle must be extended to the left in order to meet the perpendicular from P. The point C will now move to the *left* of O, and OC will become negative. Finally, when $\angle x$ becomes 180°, OC becomes -1. We see, then, that the cosine of an obtuse angle is negative.

The cosine is a decreasing function ranging from 1 to -1.

From the above diagram, we see that for any given angle there is only *one* cosine. The reason for this is that only one perpendicular can be drawn from the point P to the line OC. Conversely, for any given cosine there is only one angle between 0° and 180°, for the reason that from the point C only one perpendicular can be drawn to the line OC.

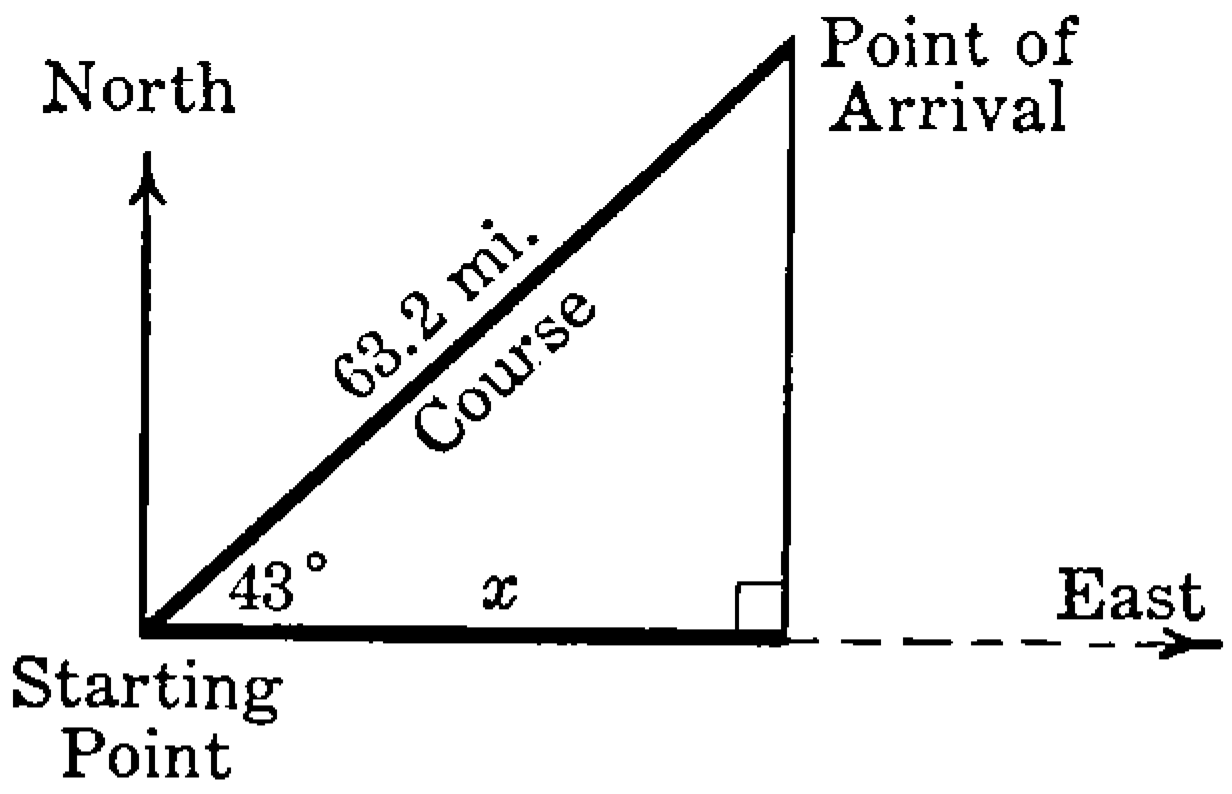
Numerical Value of the Cosine

315. If we wished to find the numerical value of the cosine of 40°, we could construct a figure like the last one we studied, making $\angle x = 40°$ with a protractor. The line segment OC would then measure the numerical value of cos 40°. By higher mathematics the cosine of any angle can be found to a degree of accuracy that does not depend on measurement. On page 279 you will find a table giving the cosines of angles accurately to four significant digits. The cosine of 40° = .7660.

EXERCISES

Illustrative Example. A ship sails 63.2 miles on a course E. 43° N. How far to the *east* does the ship sail?

The course of the vessel makes an angle of 43° with the east line.



From the definition of the cosine we have:

$$\frac{\text{Adjacent Leg}}{\text{Hypotenuse}} = \cos A$$

$$\frac{x}{63.2} = \cos 43°$$

From the table of cosines on page 279, cos 43° = .7314.

Substituting, $\frac{x}{63.2} = .7314,$

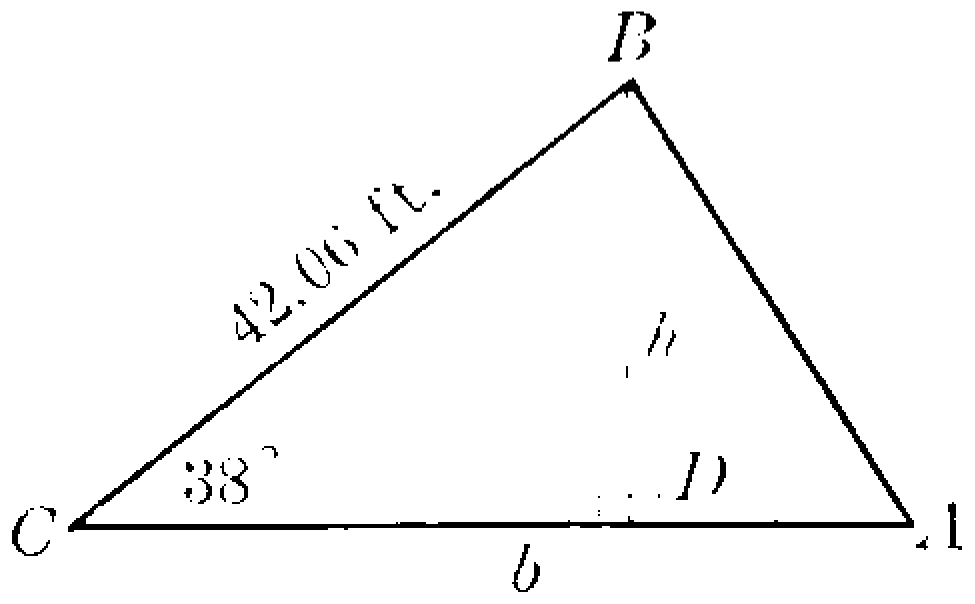
$$x = 63.2 \times .7314 = 46.2 \text{ miles}$$

$$\begin{array}{r} .7344 \\ 63.2 \\ \hline 14628 \\ 21942 \\ 13884 \\ \hline 46.22448 \end{array}$$

Note: The answer was rounded off to the same number of significant digits that were given in the problem (§ 277). In these problems we are assuming that the given angles have four significant digits; *i.e.*, 43° means 43.00°.

1. The hypotenuse of a right triangle is 24.3 feet and one angle is 38°. Find the leg adjacent to the given angle.

2. In △ABC, ∠C = 38° and a = 42.06 feet. Find the altitude to the side b. *Hint:* First find ∠CBD.

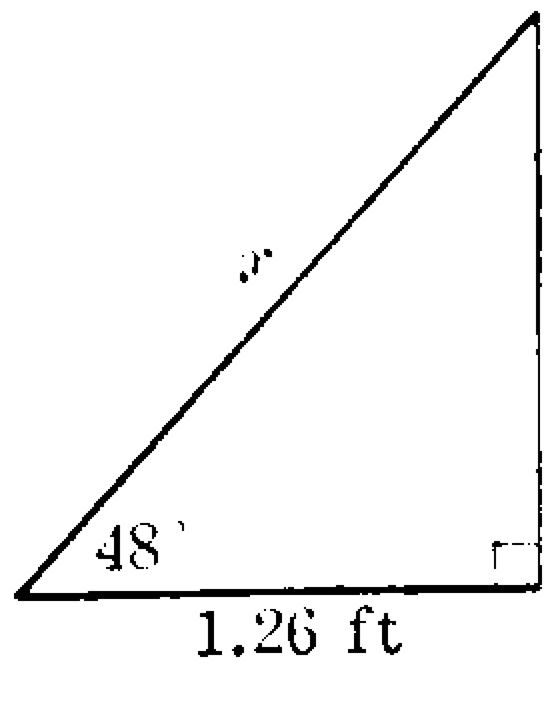


3. In a △ABC, ∠A = 106° and b = 29.0 inches. Find the altitude on the side c.

4. Find the area of a triangle in which two sides are 12.1 and 17.3, and the included angle is 103°.

5. Find the area of a parallelogram in which two sides are 18 and 13, and one angle is 62°.

Illustrative Example. One angle of a right triangle is 48° and the adjacent leg is 1.26 feet. Find the hypotenuse of the triangle.



From the definition of the cosine:

$$\cos 48° = \frac{\text{Adjacent Leg}}{\text{Hypotenuse}} = \frac{1.26}{x}$$

From the table on page 279, cos 48° = .6691. Substituting,

$$\frac{.6691}{1} = \frac{1.26}{x}$$

In this proportion, the product of the means equals the product of the extremes:

$$.6691x = 1.26$$

$$x = \frac{1.26}{.6691} = 1.88 \text{ feet}$$

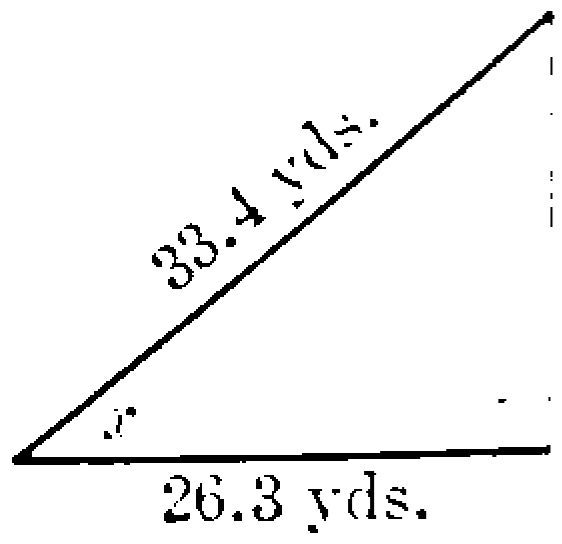
$$\begin{array}{r} 1.88 \\ .6691_{\wedge}\overline{)1.2600} \\ 6691 \\ \hline 59090 \\ 53528 \\ \hline 55620 \\ 53528 \\ \hline 2092 \end{array}$$

6. One angle of a right triangle is 62° and the opposite side is 48.4 inches. Find the hypotenuse.

7. The center pole for a tent is 8.6 feet high, and a rope from the top of the pole makes an angle of 50° with the pole. Find the length of the rope.

8. A straight road makes an angle of with the horizontal. If one end of the road is 6.8 feet higher than the other, what is the length of the road?

Illustrative Example. The hypotenuse of a right triangle is 33.4 yards and one leg is 26.3 yards. Find the included angle.



From the definition of the cosine:

$$\cos x = \frac{26.3}{33.4} = .787$$

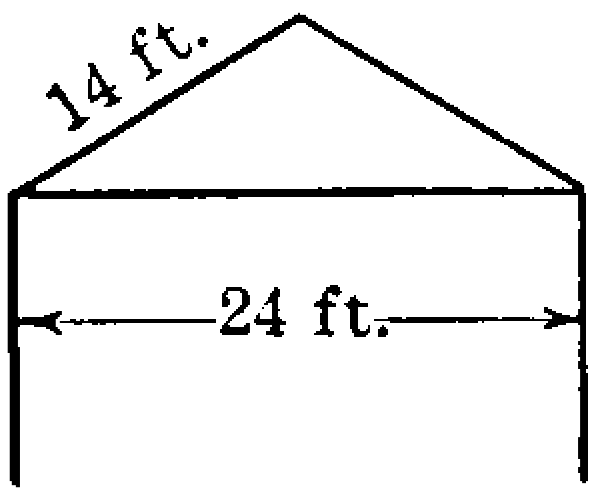
From the table we find the angle whose cosine is nearest to .787.

$$x = 38°$$

9. A ladder 16.5 feet long reaches a wall to a height of 12.8 feet. What angle does the ladder make with the ground?

10. After descending a straight mine tunnel a distance of 150 feet, an observer finds himself 85 feet below the surface. What angle does the tunnel make with the level ground?

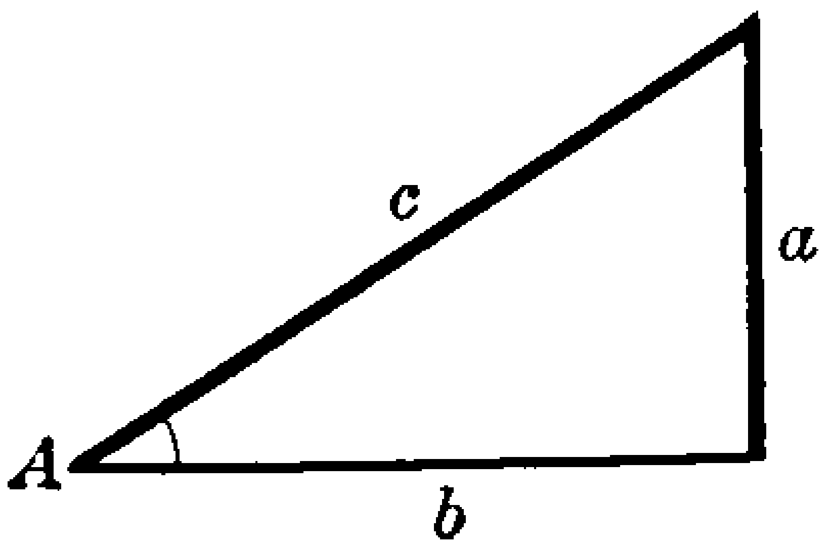
11. The rafters on a roof are 14 feet long and the width of the house is 24 feet. What angle do the rafters make with the ceiling? (Disregard the overhang.)



12. From a point 32.4 inches from the center of a circle, a tangent to the circle is 27.5 inches long. Find the angle between the tangent and the line joining its far end to the center of the circle.

The Sine Function (Optional)

In § 312 we saw that so long as ∠A remains the same, the ratio of *any* two sides remains the same. We defined the ratio $\frac{b}{c}$ as the cosine of the ∠A. We shall now define the ratio $\frac{a}{c}$ as the *sine* of the ∠A (abbreviated sin A).

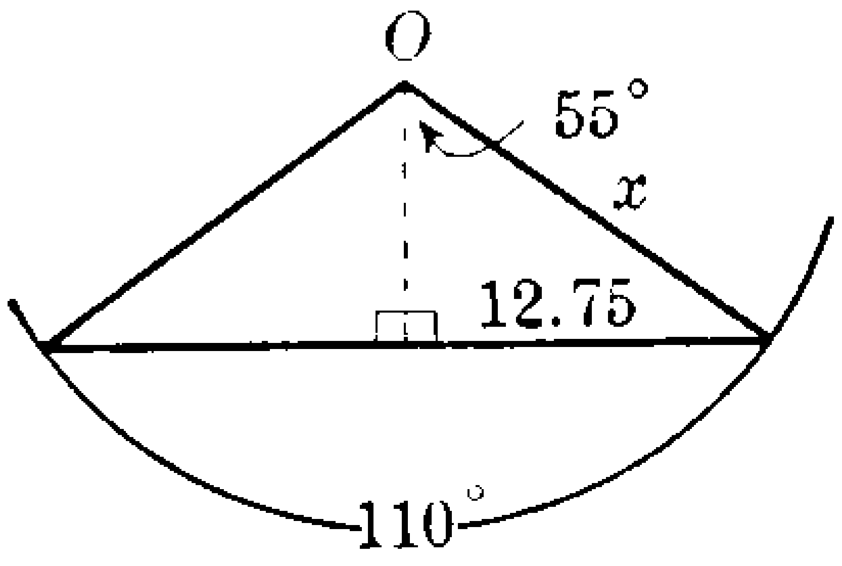


$$\sin A = \frac{\text{Opposite Leg}}{\text{Hypotenuse}}$$

316. Definition. If from a point on one side of a given angle a perpendicular on the other side is drawn, then the leg of the resulting right triangle opposite the given angle divided by the hypotenuse is the ratio called the sine of the given angle.

It is possible to work all the problems about right triangles with the cosine function alone. However, it is often more convenient to use other functions.

Illustrative Example. A chord 25.50 inches long has an arc of 110°. Find the radius of the circle.



The degree measure of an arc is its central angle. The perpendicular to the chord from the center will bisect the chord and the central angle.

From the definition of the sine:

$$\frac{\text{Opposite Leg}}{\text{Hypotenuse}} = \sin A, \qquad \frac{12.75}{x} = \sin 55°$$

From the table of sines on page 279, sin 55° = .8192.

Substituting,

$$\frac{12.75}{x} = \frac{.8192}{1},$$

$$.8192x = 12.75,$$

$$x = \frac{12.75}{.8192} = 15.56 \text{ inches}$$

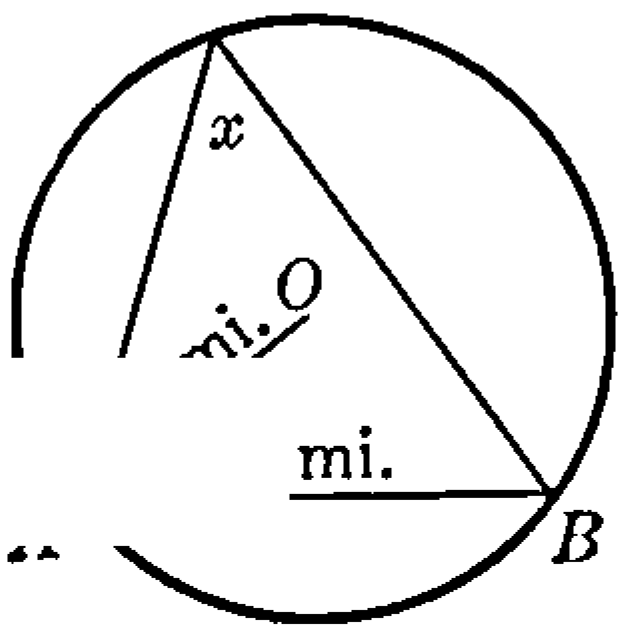
Solve these problems by using the sine function.

13. The hypotenuse of a right triangle is 16.3 feet and one angle is 32°. Find the shortest side.

14. Work Ex. 6 on page 222 by using the sine function.

15. A plane flies 180 miles in a direction of 328° (see page 41, S5). How far to the north of its starting point is the point of arrival?

16. A circular area of a 5-mile radius has been found dangerous for ships at sea. Two

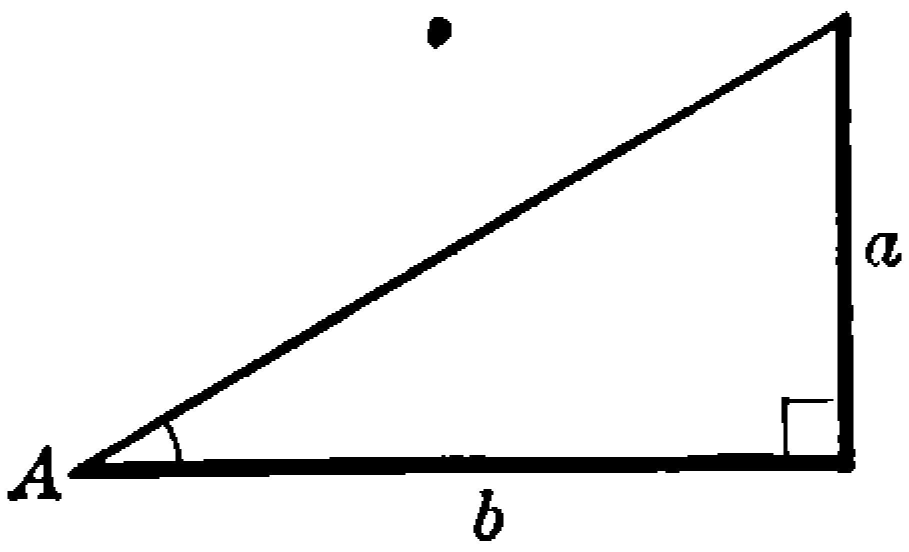


points A and B, 6 miles apart, are on the circle. Find the danger angle x that ships can use as a guide (page 160, S5; page 172, Ex. 6).

17. Two sides of a parallelogram are 24 units and 31 units, and its area is 580 square units. Find the angles.

The Tangent Function (Optional)

Another useful function is one that is formed by the ratio $\frac{a}{b}$. It is called the



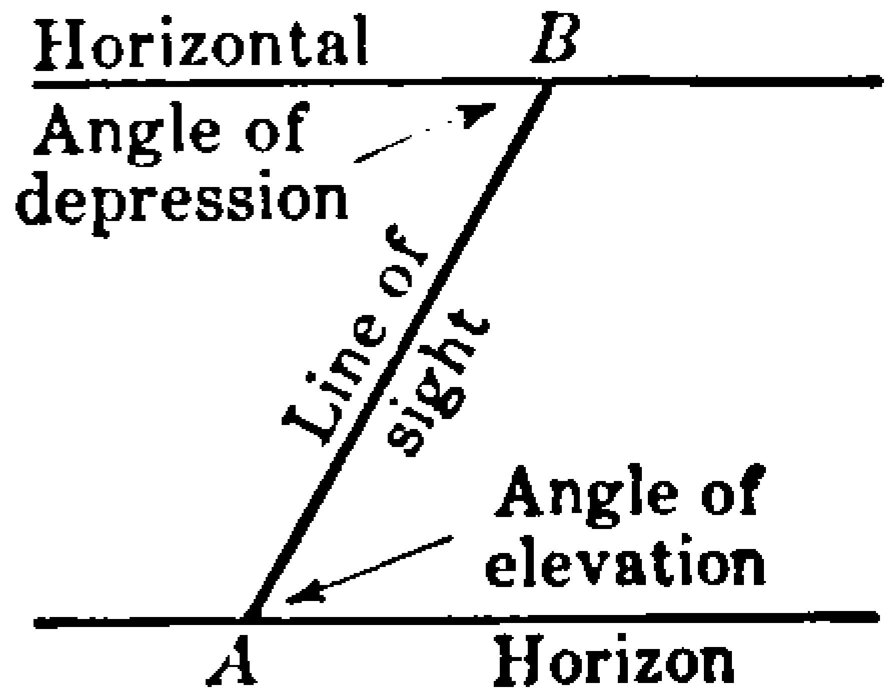
tangent of ∠A (abbreviated tan A).

$$\tan A = \frac{\text{Opposite Leg}}{\text{Adjacent Leg}}$$

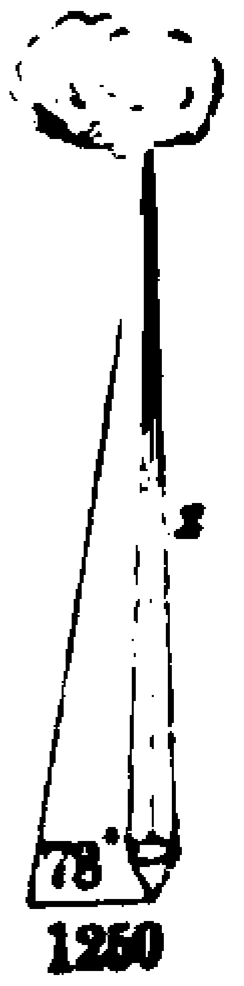
317. Definition. If from a point on one side of a given angle a perpendicular on the other side is drawn, then the leg of the resulting right triangle opposite the given angle divided by the adjacent leg is the ratio called the tangent of the given angle.

Illustrative Example. To find the height of a cloud over an airport, a searchlight is directed straight up. Then an observer 1250 feet from the searchlight measures the angle of elevation of the illuminated spot on the cloud and finds it to be 78°. How high is the cloud?

The *angle of elevation*, or the *angle of depression*, is the angle made by the line of sight and a horizontal line. If an observer is looking up from point A to point B, he calls it the angle of elevation; if looking down, the angle of depression.



By the definition of the tangent:



$$\frac{\text{Opposite Leg}}{\text{Adjacent Leg}} = \tan A,$$

$$\frac{x}{1250} = \tan 78°$$

From the table of tangents on page 279, tan 78° = 4.705.

Substituting,

$$\frac{x}{1250} = 4.705,$$

$$x = 1250 \times 4.705$$

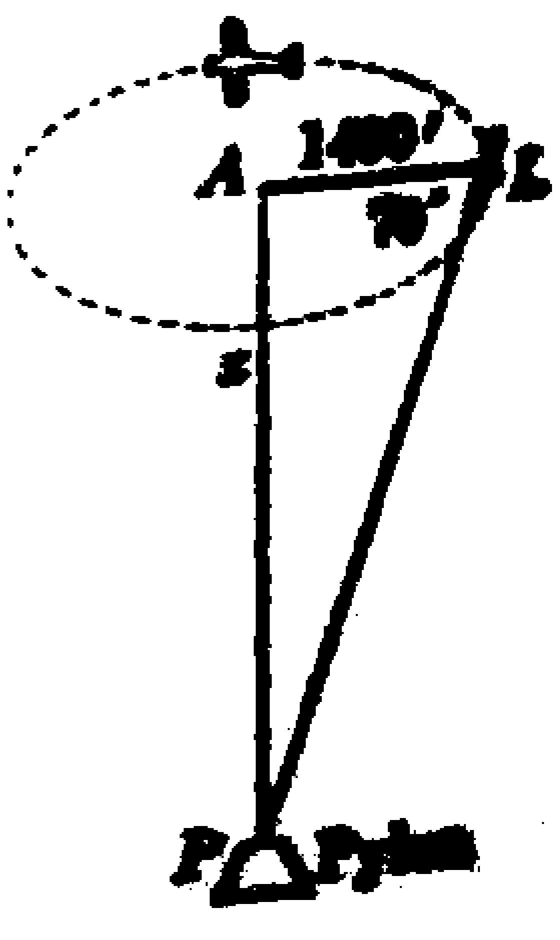
$$= 5881.25, \text{ or, rounded off, } 5880 \text{ feet}$$

Solve these problems by using the tangent function.

18. The legs of a right triangle are 24.5 centimeters and 19.6 centimeters. Find the smallest angle of the triangle.

19. An airplane pilot measured the angle of depression of a dock when he was 5100 feet directly over a boat. If the angle of depression was 8°, how far was the boat from the dock?

20. When flying a circle about a point P on the ground as a pivot, the wings of the plane are pointed to that point. If the maximum angle at which the plane is to be tilted from the horizon is 70° (called the *angle of bank*), at what height must the plane fly a circle of 1400 feet in radius?



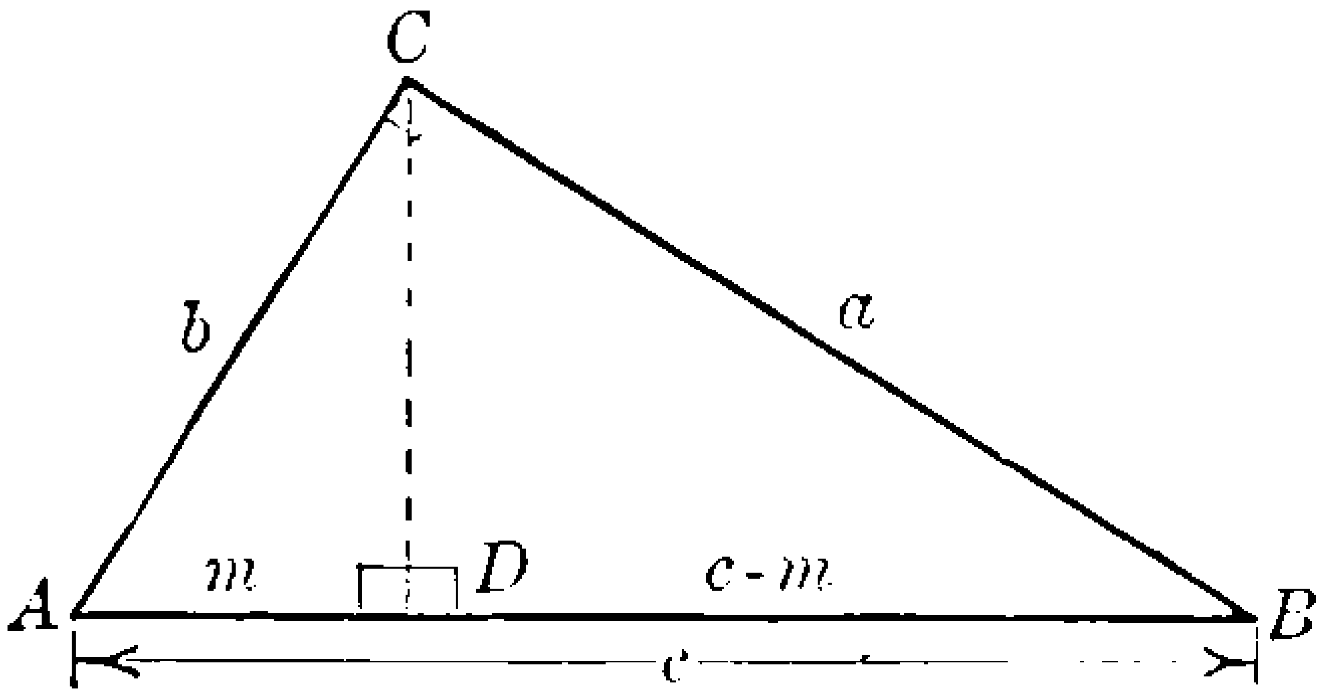
21. What is the angle of elevation of the sun when a building 40 feet high casts a shadow of 27 feet?

22. A 6% grade is one that rises 6 feet for every 100 feet horizontally. At what angle does a 6% grade rise?

The Pythagorean Theorem

Probably the most important theorem in geometry is the one that gives the relationship between the legs and the hypotenuse of a right triangle. It is named after an ancient Greek mathematician Pythagoras.

318. PYTHAGOREAN THEOREM. In a right triangle the square of the hypotenuse equals the sum of the squares of the legs.



Given: Rt. $\triangle ABC$
Prove: $c^2 = a^2 + b^2$
Cons.: Draw alt. CD.

PROOF

Statements	*Reasons*
1. In $\triangle ADC$, $\cos A = \frac{m}{b}$; in $\triangle ABC$, $\cos A = \frac{b}{c}$	1. Def. cos.
2. $\frac{m}{b} = \frac{b}{c}$, and $mc = b^2$ (1)	2. Subs. and Mult. Ax.
3. Similarly, $\cos B = \frac{c - m}{a}$ and $\cos B = \frac{a}{c}$	3. Same as 1.
4. $\frac{c - m}{a} = \frac{a}{c}$, and $c^2 - mc = a^2$ (2)	4. Same as 2.
5. Adding (1) and (2), $c^2 = a^2 + b^2$	5. Add. Ax.

You may have proved this theorem before in some other way (page 198, H1, H2), and you may prove it in still other ways later. This proof depends on similar triangles and is so pictured in the dependence chart on page 240. The cosine is a convenient way of denoting the constant ratio in the similar triangles. The plan of the proof given above is as follows:

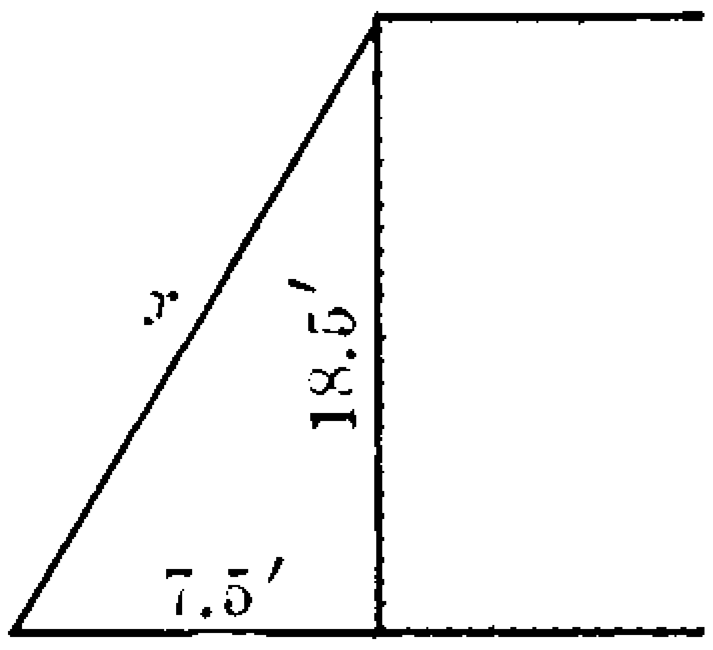
1. When the altitude to the hypotenuse is drawn, we find that $\angle A$ belongs in two right triangles. Consequently, we express its cosine by two different ratios. We set these ratios equal to each other and we write the product form (clear fractions).

2. We repeat this, using $\angle B$.

3. By adding the two equations in the product of the means and the product of the extremes, we get the Pythagorean Theorem.

EXERCISES

Illustrative Example. How long a ladder is needed to reach a roof 18.5 feet high, if the ladder is not to stand in a flower bed that extends 7.5 feet from the wall?



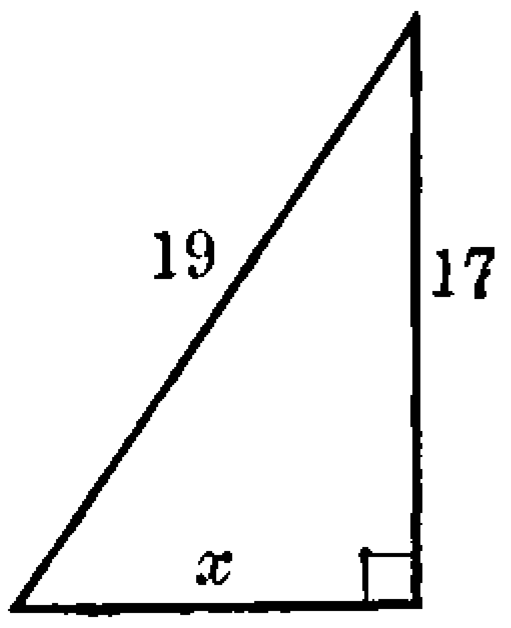
From the Pythagorean relationship,

$$\begin{aligned} x^2 &= 18.5^2 + 7.5^2 \\ &= 342.25 + 56.25 \\ &= 398.5 \\ x &= 20.0 \text{ feet (See § 277.)} \end{aligned}$$

Find the square root from the table on page 280. This table can also be used in reverse to square the given sides.

1. The legs of a right triangle are 4 units and 6 units. Find the hypotenuse correct to three significant digits.
2. Find the diagonal of a rectangle whose dimensions are 33 units and 56 units.
3. How large in diameter, to the nearest hundredth of an inch, must a hole be drilled to receive a square plug $2\frac{3}{4}$ inches on a side?
4. Two telephone poles 25 feet and 32 feet high respectively are 50 feet apart on level ground. How long, to the nearest tenth of a foot, is the wire connecting their tops?
5. The legs of a right triangle are 20 units and 21 units. Find the median to the hypotenuse.
6. A straight highway crosses a straight railroad track at right angles. A motorist, having stopped to let an express train pass at 70 miles per hour, proceeds at 40 miles per hour. How far, to the nearest tenth of a mile, is the motorist from the train 10 minutes later?
7. A chord 24 inches long is 5 inches from the center of the circle. Find the radius of the circle.
8. The diameter of a circle is 24 inches. A tangent measures 35 inches from the point of contact to its end. How far is this end from the center of the circle?
9. In a football game what is the longest run in a straight line that a ball carrier can make without scoring? A football field is 100 yards by 160 feet. The end zones are 10 yards deep.
10. A man who can row 4 miles per hour in still water has crossed a stream in 5 minutes by rowing at right angles to the course of the stream. The stream is flowing at 3 miles per hour. How far is the man from the starting point?

Illustrative Example. The hypotenuse of a right triangle is 19 units and one leg is 17 units. Find the other leg in terms of the simplest radical.



From the Pythagorean relationship,

$$\begin{aligned} x^2 &= 19^2 - 17^2 \\ &= 361 - 289 = 72 \\ x &= \sqrt{72} = \sqrt{36 \cdot 2} = 6\sqrt{2} \end{aligned}$$

By using the algebraic method of factoring the difference of two squares, the work can often be shortened as follows:

$$\begin{aligned} x^2 &= 19^2 - 17^2 = (19 + 17)(19 - 17) \\ &= 36 \cdot 2, \qquad x = 6\sqrt{2} \end{aligned}$$

11. The hypotenuse of a right triangle is 57 units and one leg is 43 units. Find the other leg in terms of the simplest radical.
12. The diagonal of a rectangle is 73 units and the width is 48 units. Find the length.
13. Find the area of a right triangle in which the hypotenuse is 37 feet and one leg is 12 feet.

14. The base of an isosceles triangle is 16 inches and each leg is 17 inches. Find the area of the triangle.

15. The side of an equilateral triangle is 10 units. Find the area of the triangle in terms of the simplest radical.

16. In a circle of 6.00-inch radius, how long is a chord 4.00 inches from the center? (See § 277.)

17. A pole 18 feet high is broken 6 feet from the ground, but the two parts remain attached. How far from the base of the pole does the top of the pole touch the ground?

18. The length of a flat box is 32 inches. How wide must it be to hold a yardstick?

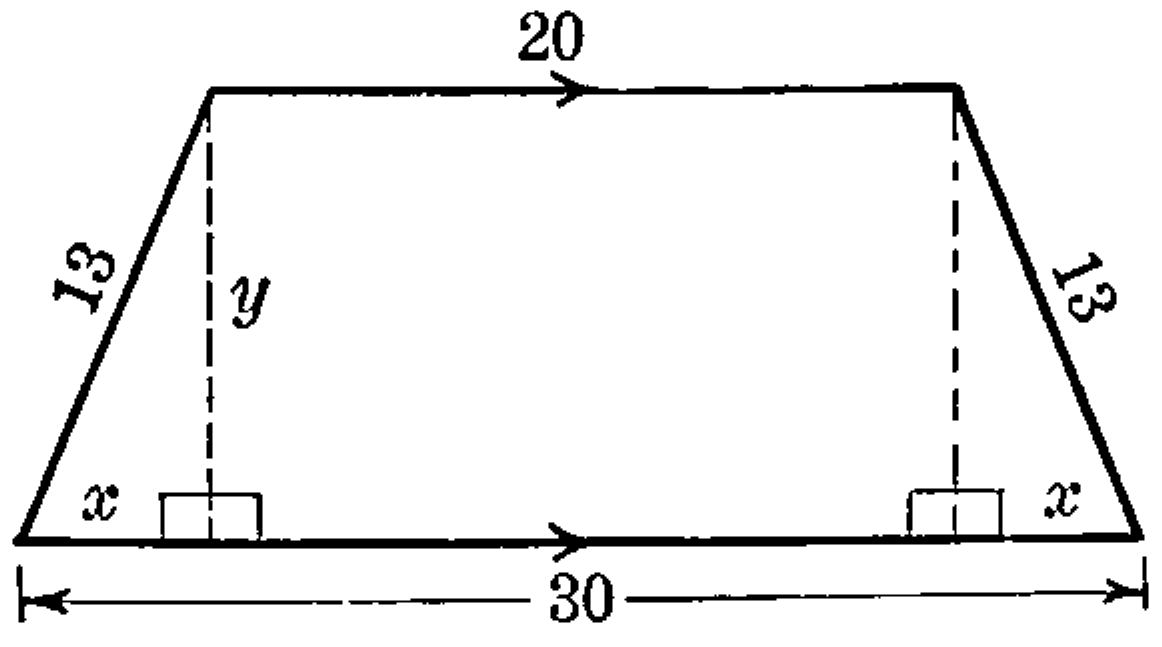
19. The diameters of two concentric circles are 12 inches and 20 inches. A chord of the larger circle is tangent to the smaller circle. How long is this chord?

20. When the wind is from the north, a pilot heads directly east with an airspeed of 115 miles per hour. In one hour he reaches a town 120 miles away. What is the speed of the wind?

21. The safety man on a football team received a punt and ran 60 yards in a straight line before being forced out of bounds. If the runback of the kick netted only 40 yards, how far from the nearest sideline was he when he received the ball? (The field is 160 feet wide.)

22. The bases of a trapezoid are 67 feet and 100 feet. One leg of the trapezoid is perpendicular to the bases, and the other leg is 65 feet long. Find the area of the trapezoid.

23. The bases of a trapezoid are 20 units and 30 units. Each leg is 13 units. Find the altitude of the trapezoid.



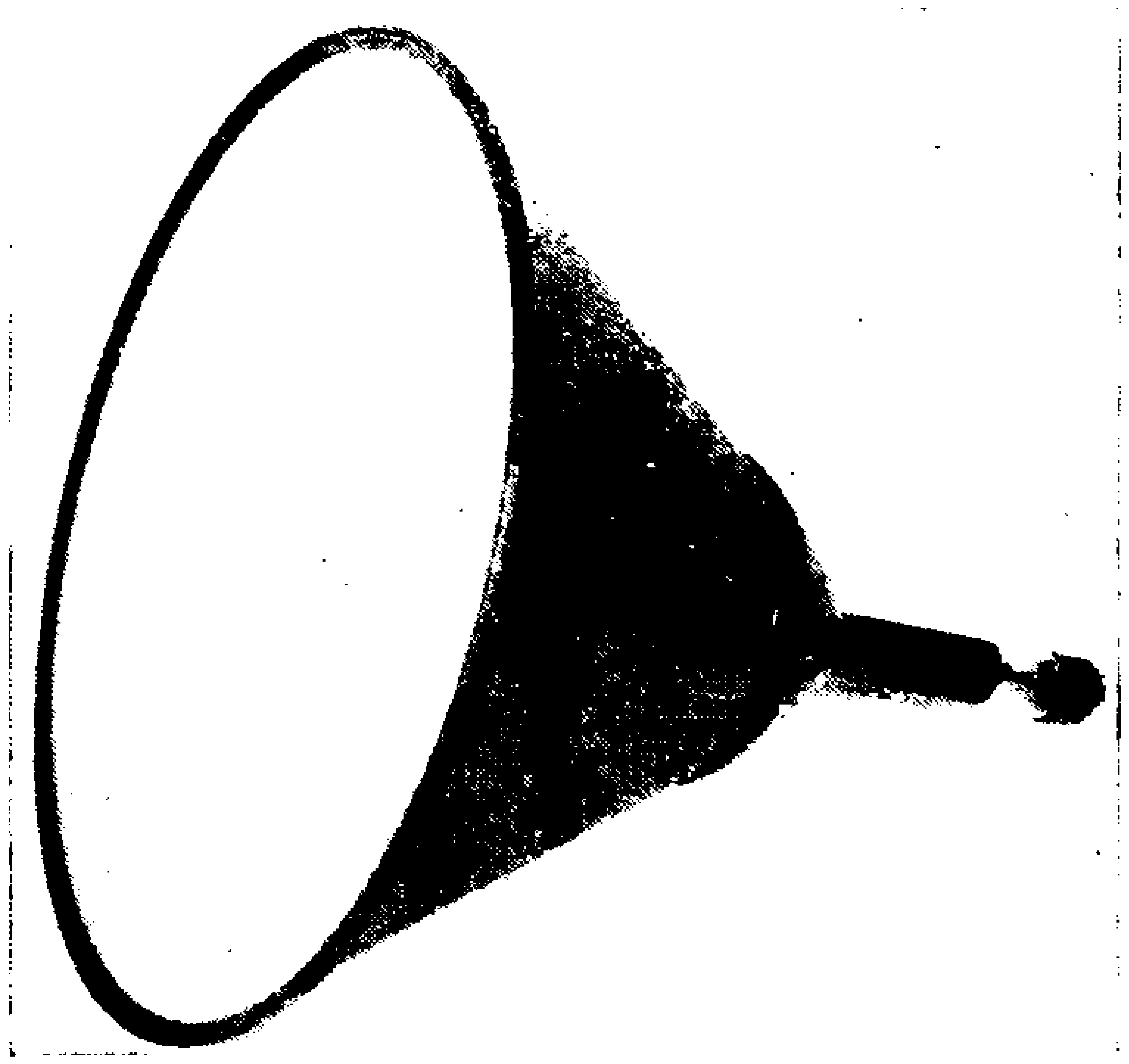
Hint: $x + 20 + x = 30$

24. The bases of a trapezoid are 17 inches and 25 inches. Each leg is 9 inches. Find the area of the trapezoid.

25. In triangle ABC, CD is an altitude. At A, AE is drawn perpendicular to AC and equal to BD. At B, BF is drawn perpendicular to BC and equal to AD. Prove that C is equidistant from E and F.

SUPERIOR WORK

S1. In television receivers the image is a rectangle with the ratio 3 : 4 for its dimensions when projected on the circular face of a cathode tube. The diameter of the cathode tube is usually 7 inches, 10 inches, $12\frac{1}{2}$ inches, 16 inches, or 19 inches. Find the number of square inches in the image of each of these tubes.



Courtesy Du Mont Laboratories

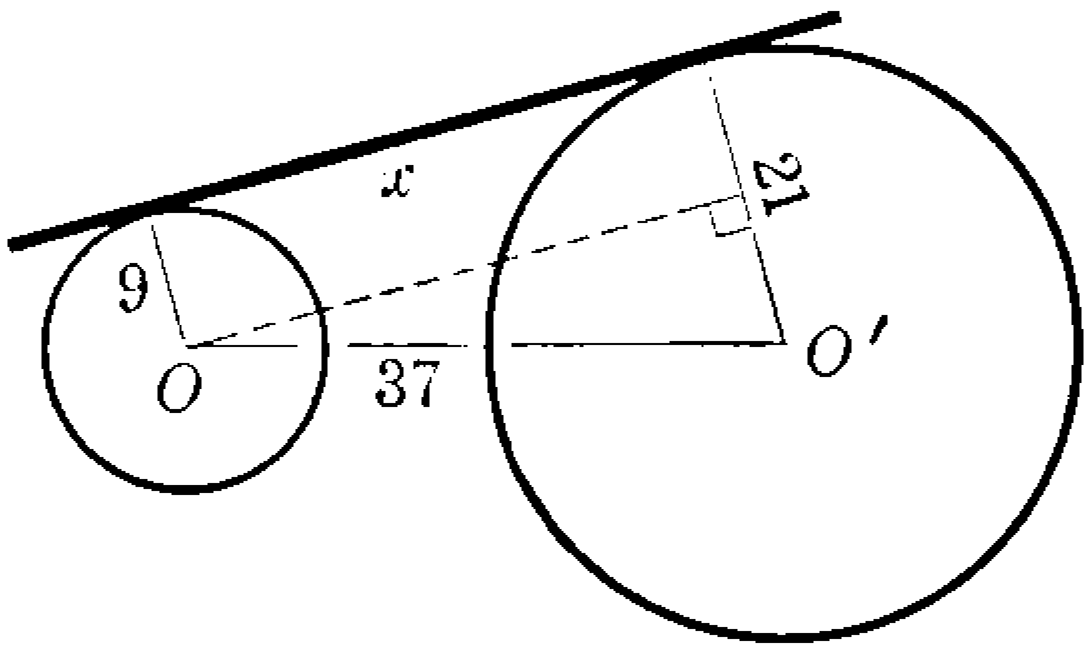
CATHODE-RAY TUBE

Note. You have found the area of the rectangle that can be inscribed in the circular face of a given tube. Actually, the image is a little larger because the corners are cut off. Thus, the image on a 7-inch tube is rated as 26 square inches, while a 10-inch tube is advertised from 52 to 55 square inches.

S2. The course of a ship is a straight line parallel to the shore and 10 miles from it. On the shore is a signal light that can be seen over a range of 12 miles. Over what distance of the ship's course can the sailors on board see the light?

S3. From a point 40 inches from the center of a circle whose radius is 14 inches, a secant is drawn. The nearest that the secant comes to the center of the circle is 10 inches. Find the over-all length of the secant.

S4. The radii of two circles whose centers are 37 inches apart are 9 inches and 21 inches.



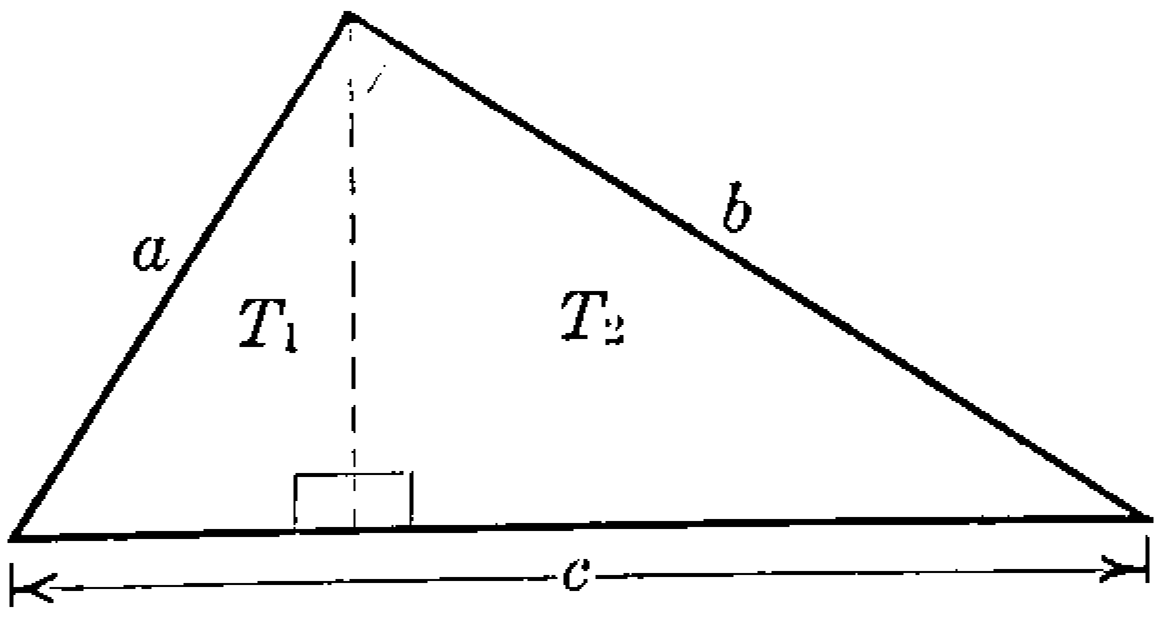
How long is the common external tangent to these circles? (See page 165, S5.)

S5. How long is the common internal tangent to the circles in S4? (See diagram on page 166, S6.)

S6. A corridor is 5 feet wide and has a right-angle turn. If a stretcher is 3 feet wide, what can be its maximum length to pass the turn?

S7. A barge 50 feet wide is being towed at 12 miles per hour. A deck hand drops an apple core from starboard and then walks directly across the barge at 4 miles per hour and drops another apple core. How far apart are the two apple cores?

S8. Stanley Jashinski, a schoolboy of Youngstown, Ohio, recently suggested the following proof of the Pythagorean Theorem:



$$\triangle T_1 \sim \triangle T_2 \sim \triangle(T_1 + T_2)$$

From the theorem that the areas of two similar triangles are proportional to the squares of the corresponding sides:

$$\frac{a^2}{T_1} = \frac{b^2}{T_2} = \frac{c^2}{(T_1 + T_2)}$$

Write out the complete proof, using the plan outlined.

S9. A plane flying at 200 miles per hour leaves Newark airport on a course of which the nearest point to Erie, Pa., is 192 miles from it. If Erie is 320 air miles from Newark, how long after leaving Newark will the plane be the same distance from the two cities?

S10. A room is 30 feet long, 20 feet wide, and 12 feet high. A housefly flies in a straight line between two far corners. How long is this line?

S11. Prove: (*a*) If two opposite angles of a quadrilateral are right angles, the sums of the opposite sides of the quadrilateral are inversely proportional to their differences.

(*b*) If two opposite angles of a circumscribed quadrilateral are right angles, then the quadrilateral is a kite. (Page 180, Ex. 9.)

S12. A circle O' passes through the center of another circle O. Through A, the intersection of the two circles, a tangent is drawn to circle O cutting circle O' again at B. From B a chord BC is drawn in circle O'. From C a line is drawn tangent to circle O at D. Prove: $AB^2 = BC^2 + CD^2$

SPECIAL RIGHT TRIANGLES

The Pythagorean relationship holds for all right triangles. There are two special cases of the right triangle in which the sides have additional relationships. These are the 45° right triangle and the 30°–60° right triangle.

The 45° Right Triangle

319. If one angle of a right triangle is 45°, then how many degrees are in the other acute angle? Do you see that the triangle is isosceles? The

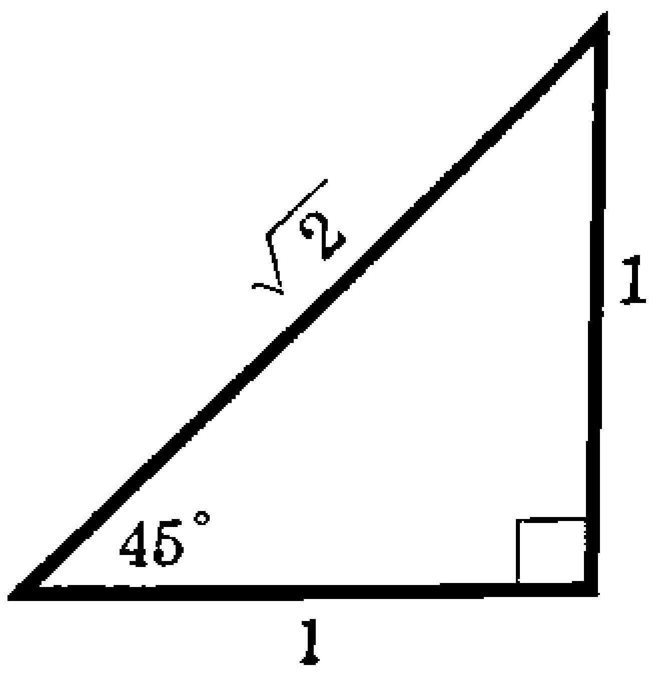
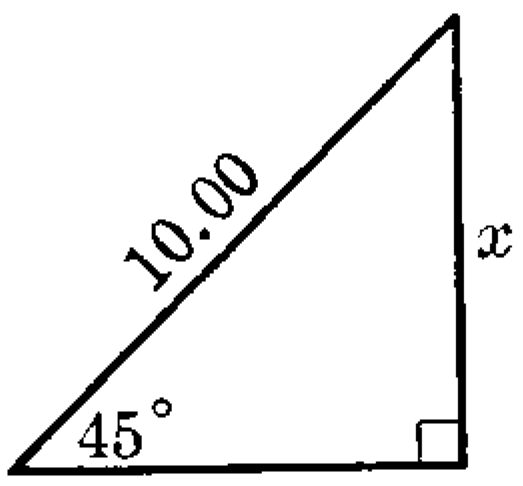


figure shows a 45° right triangle in which each leg is 1. From the Pythagorean relationship, the hypotenuse equals $\sqrt{1^2 + 1^2} = \sqrt{2}$. Every 45° right triangle is similar to this triangle. This triangle can therefore serve as a standard of comparison for solving any other 45° right triangle. The properties of this special right triangle may be stated as follows:

The hypotenuse equals either leg times $\sqrt{2}$.

Either leg equals half the hypotenuse times $\sqrt{2}$.

Illustrative Example. The hypotenuse of a 45° right triangle is 10.00 inches. Find each leg correct to four significant digits.



Solution A: Method of Similarity. The sides of the triangle in our problem are proportional to the sides 1, 1, $\sqrt{2}$ of the standard triangle.

$$\frac{x}{1} = \frac{10}{\sqrt{2}}, \qquad \frac{x}{1} = \frac{10.00}{1.414} = 7.072$$

or,

$$x = \frac{10}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{10\sqrt{2}}{2} = 5\sqrt{2}$$

$$= 5 \times 1.414 = 7.070$$

The answers to these problems are rational, or they can be expressed in terms of $\sqrt{2}$. Since such problems occur frequently, it is well to remember that $\sqrt{2} = 1.414$ and save time. In terms of the simplest radical, $x = 5\sqrt{2}$.

Solution B: Direct Method. In this method we do not use the standard triangle. Instead, we write directly that:

$$x^2 + x^2 = 10^2,\ 2x^2 = 100,\ x^2 = 50,\ x = 7.071$$

Which solution do you prefer? Why?

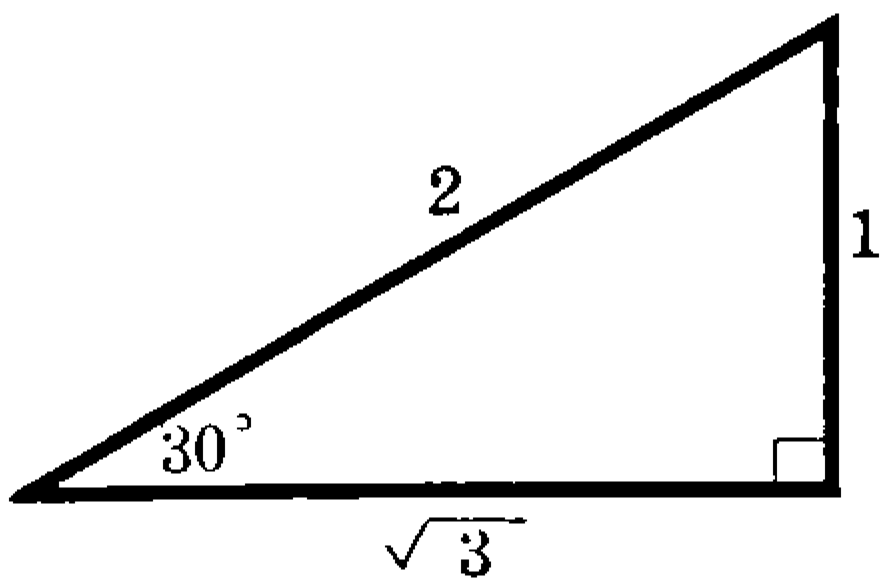
The reason for the first two answers being slightly different from the third answer is that we used only four significant digits for $\sqrt{2}$. Carry out the first two computations using the five significant digits 1.4142 for $\sqrt{2}$. Round off your results to four significant digits and compare the three answers. (See § 277.)

EXERCISES

1. The hypotenuse of an isosceles right triangle is 12 units. Find the length of each leg correct to four significant digits.
2. Find the side of a square to the nearest hundredth of an inch, if the diagonal is 2 feet.
3. Two sides of a triangle, 14 inches and 18 inches, include an angle of 45°. Find the area of the triangle correct to three significant digits.
4. A line segment to the center of a circle from an external point is 30 inches long and makes an angle of 45° with the tangent to the circle from that point. Find the diameter of the circle.
5. Find the cosine of 45° in terms of the simplest radical, and also correct to four significant digits.
6. A round shaft $4\frac{1}{2}$ inches in diameter is to be machined square. What is the maximum side of the square correct to four significant digits.
7. The area of a square is one acre. Find its diagonal to the nearest foot.

The 30°–60° Right Triangle

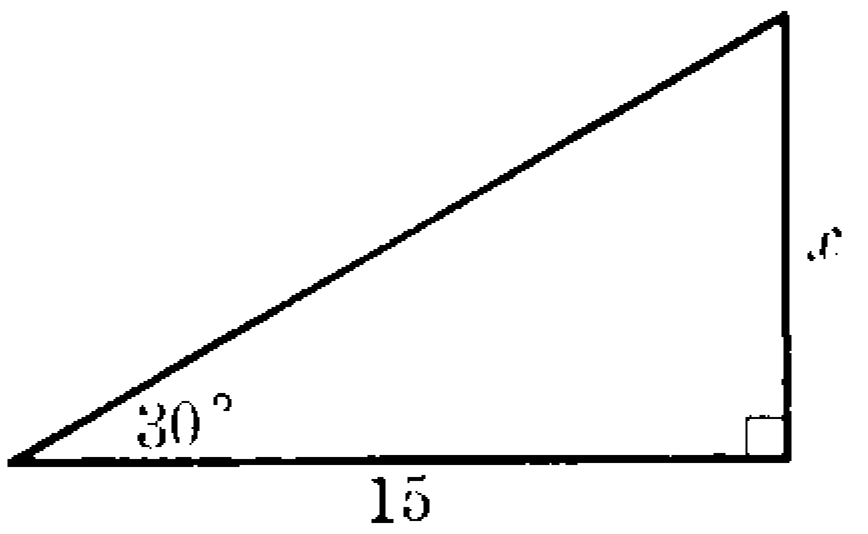
320. From the Theorem in § 192 you know that an important property of the 30°–60° right triangle is the fact that the *shorter leg is half the hypotenuse.* In the



figure, since the shortest side is 1 unit long, we know that the hypotenuse is 2 units long.

From the Pythagorean relationship, the other leg equals $\sqrt{2^2 - 1^2} = \sqrt{3}$. Another important property of this triangle, therefore, is: *The side opposite the 60° angle equals half the hypotenuse times* $\sqrt{3}$.

Illustrative Example. The side of a right triangle opposite a 60° angle is 15. Find the other leg.



Solution A: Method of Similarity. The sides of our triangle are proportional to the sides 1, 2, $\sqrt{3}$ of the standard triangle.

Therefore,

$$\frac{x}{1} = \frac{15}{\sqrt{3}}, \quad x = \frac{15}{1.732} = 8.661$$

or,

$$x = \frac{15}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{15\sqrt{3}}{3} = 5\sqrt{3}$$

$$= 5 \times 1.732 = 8.660.$$

Solution B: Direct Method. Let the hypotenuse $= 2x$.

$$\text{Then } (2x)^2 - x^2 = 15^2$$
$$4x^2 - x^2 = 225$$
$$3x^2 = 225$$
$$x^2 = 75$$
$$x = 8.660$$

The answers to these problems will either come out rational or in terms of $\sqrt{3}$. Again, it will save time to remember that $\sqrt{3} = 1.732$.

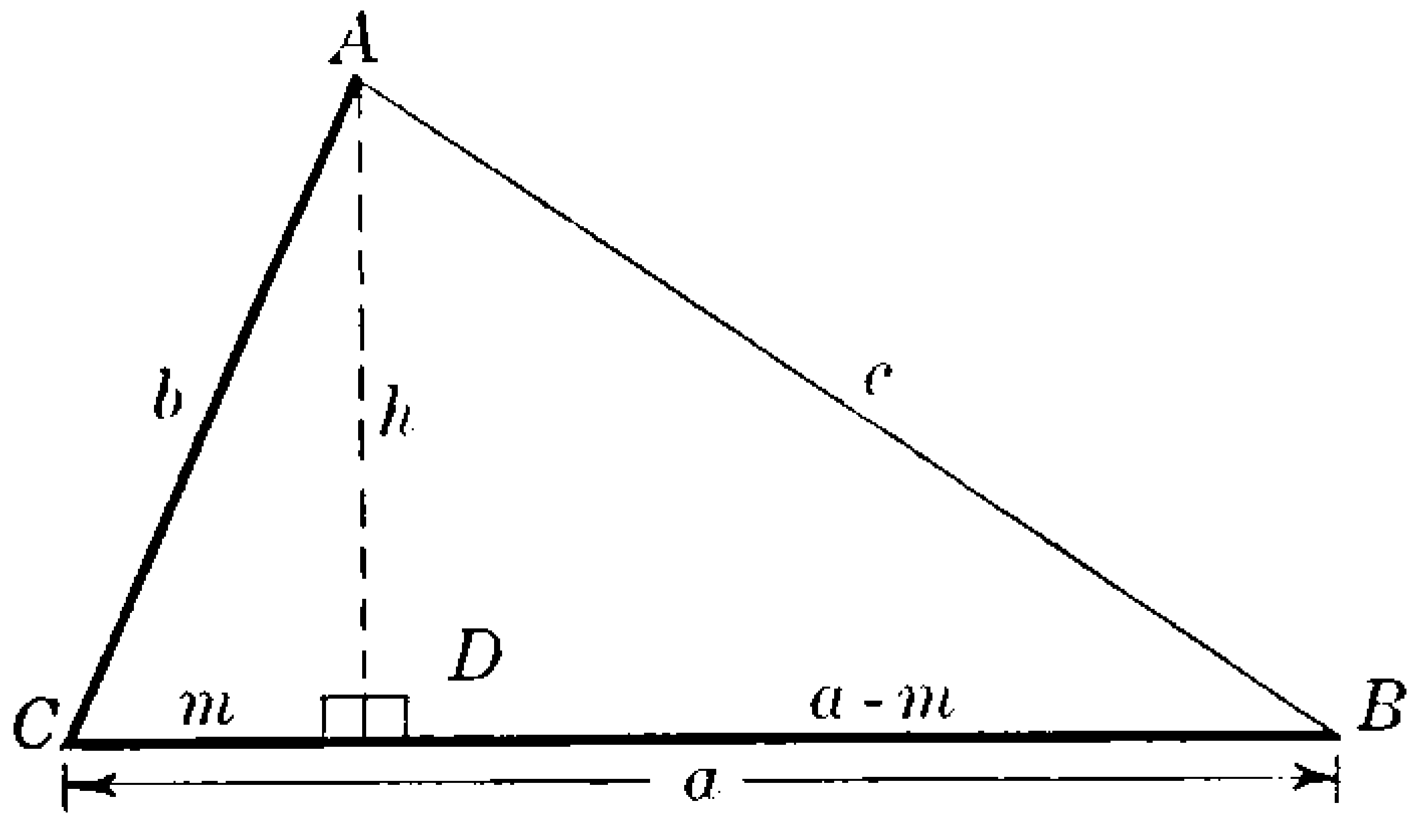
8. The leg of a right triangle adjacent to a 30° angle is 42 inches. Find the other leg of the triangle to four digit accuracy.
9. The shortest side of a 30° 60° right triangle is 20 units. Find the other leg in terms of the simplest radical.
10. The hypotenuse of a 30°-60° right triangle is 15 feet. Find the area of the triangle to three digit accuracy.
11. The side of an equilateral triangle is 30 units. Find the area of the triangle in terms of the simplest radical.
12. (*a*) Find the cosine of 60°.
    (*b*) Find the cosine of 30° in terms of the simplest radical and also correct to four significant digits.
13. Two sides of a parallelogram are 24 feet and 31 feet. One angle is 60°. Find the area of the parallelogram correct to three digit places.
14. The bases of a trapezoid are 28 feet and 46 feet. Two angles are each 30°. Find the area of the trapezoid.
15. The side of a right triangle opposite a 60° angle is 14 units. Find the area of the triangle correct to four significant digits.
16. The altitude of an equilateral triangle is 10. Find the area of the triangle in terms of the simplest radical.
17. A chord 16 inches long has an arc of 120°. Find the radius of the circle.
18. The hypotenuse of a 30°-60° right triangle equals the hypotenuse of a 45° right triangle. Find the ratio of the areas of the triangles.
19. The side of an equilateral triangle is a. Find the area of the triangle in terms of a and the simplest radical. Your answer is a useful formula.

THE OBLIQUE TRIANGLE

The Law of Cosines

We shall now find a relationship between the sides of an oblique triangle and its angles. We will do this by drawing an altitude of the oblique triangle, thus dividing it into two right triangles.

321. THEOREM. The square of one side of a triangle equals the sum of the squares of the other two sides, less twice the product of these sides and the cosine of the included angle (Law of cosines).



Given: $\triangle ABC$
Prove: $c^2 = a^2 + b^2 - 2ab \cos C$
Cons.: Draw altitude AD.

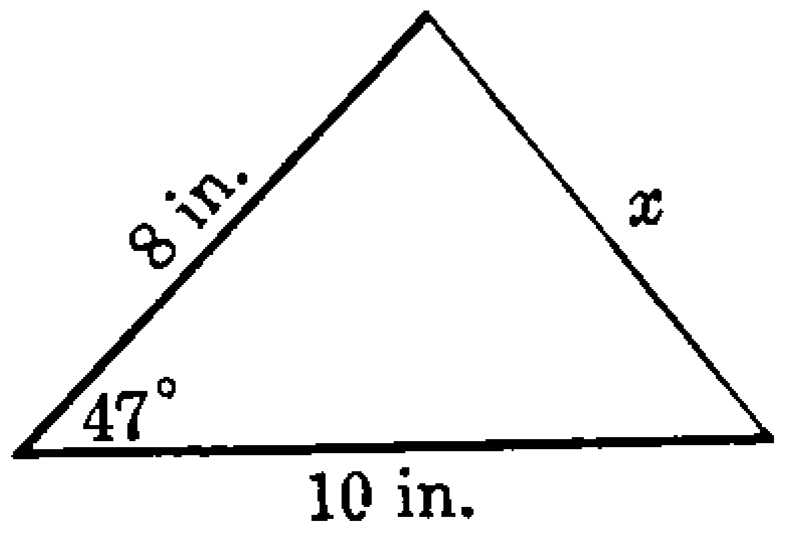
PROOF

Statements	*Reasons*
1. In rt. $\triangle ABD$, $c^2 = h^2 + (a - m)^2$ In rt. $\triangle ACD$, $h^2 = b^2 - m^2$	1. Pythagorean Theorem.
2. $c^2 = b^2 - m^2 + (a - m)^2$ or $c^2 = b^2 - m^2 + a^2 - 2am + m^2$ $c^2 = a^2 + b^2 - 2am$	2. Subs. Ax.
3. But $\frac{m}{b} = \cos C$, or $m = b \cos C$	3. Def. cos.
4. $c^2 = a^2 + b^2 - 2ab \cos C$	4. Subs. Ax.

If $\angle C = 90°$, then $\cos C = 0$ (§ 314) and the Law of Cosines reduces to the Pythagorean Theorem. If $c^2 = a^2 + b^2$, then the Law of Cosines at once reduces to $\cos C = 0$, making $\angle C = 90°$. This proves the converse of the Pythagorean Theorem.

EXERCISES

Illustrative Example. Two sides of a triangle are 10 inches and 8 inches. The included angle is 47°. Find the third side correct to three significant digits.



From the Law of Cosines

$$x^2 = 8^2 + 10^2 - 2 \cdot 8 \cdot 10 \cos 47°$$

From the table on page 279, we find that $\cos 47° = .6820$.

$x^2 = 8^2 + 10^2 - 2 \cdot 8 \cdot 10 \times .6820 = 54.9$

$x = 7.41$ inches

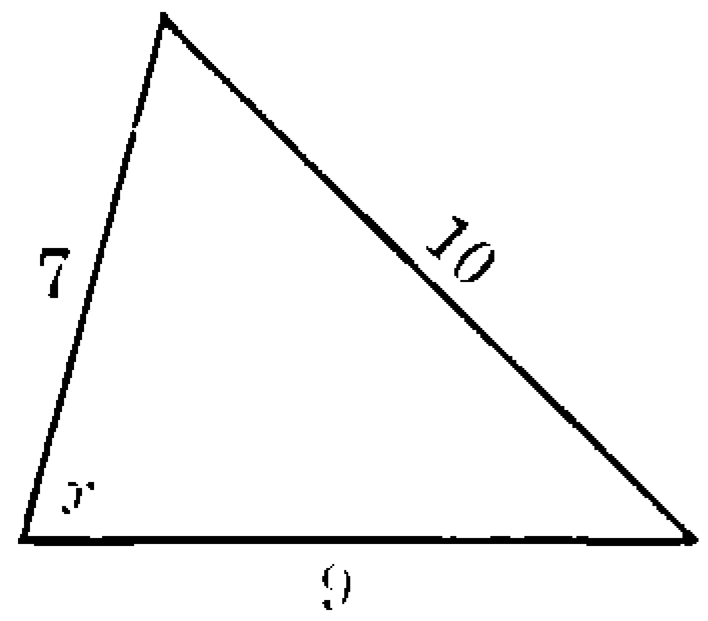
1. Two sides of a triangle are 6 and 5 units. The cosine of the included angle is $\frac{1}{3}$. Find the third side.
2. Two sides of a triangle, 8 and 4, form an angle of 60°. Find the third side.
3. The vertex angle of an isosceles triangle is 60° and each leg is 10. Find the base of the triangle by using the Law of Cosines.
4. Two sides of a triangle are 8 and 4 units. The cosine of the included angle is $\frac{1}{4}$. Find the third side of the triangle.
5. Two sides of a triangle, 12 and 9 units, include an angle whose cosine is $\frac{1}{4}$. Find the median to the 12-unit side.
6. The cosine of an angle of a triangle is $\frac{1}{4}$, and the sides forming the angle are 8 units and 12 units.

a. Find the third side of the triangle.

b. By the Law of Cosines, find the length of the line segment connecting the midpoints of the two given sides.

7. At 2 P.M. two pilots take off from an airport on courses that make an angle of 80° with each other. Their ground speeds are 90 and 120 miles per hour. How far apart are they at 3:12 P.M. when they communicate by radio?

Illustrative Example. Three sides of a triangle are 7, 9, and 10 units. Find the angle opposite the 10-unit side.



From the Law of Cosines

$$10^2 = 7^2 + 9^2 - 2 \cdot 7 \cdot 9 \cos x$$
$$100 = 49 + 81 - 126 \cos x$$
$$126 \cos x = 49 + 81 - 100 = 30$$
$$\cos x = \frac{30}{126} = \frac{5}{21} = .2381$$

From the table of cosines, we find that 76° has its cosine nearest to .2381. Therefore $x = 76°$ approximately.

8. The sides of a triangle are 6, 7, and 8 units. Find the three angles.

9. The sides of a triangle are 8, 15, and 17 units. Find the cosine of the angle opposite the 15-unit side as a common fraction in its lowest terms.

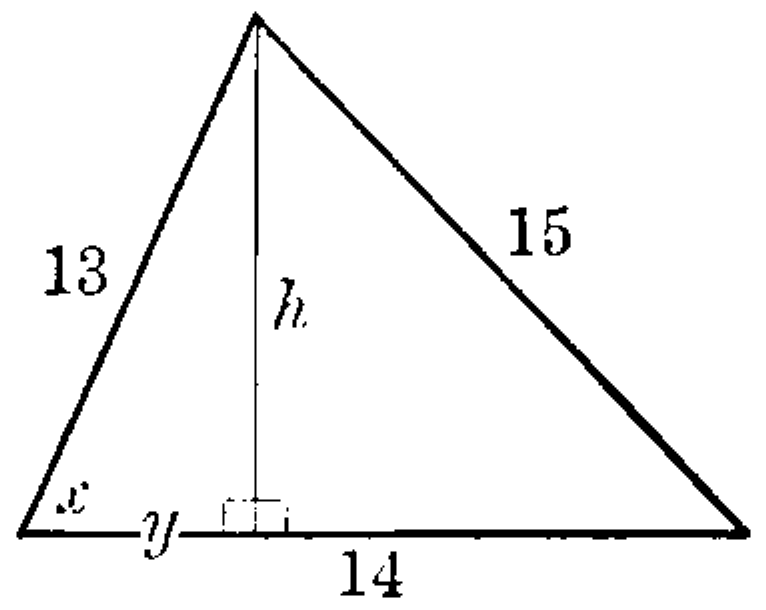
10. The sides of a triangle are 28, 45, and 53 units. Find the largest angle.

11. The sides of a triangle are twice as large as the sides of the triangle in Ex. 8. Find the angles.

12. How much nearer to first base is the pitcher on a baseball diamond than the batter? Find your answer to the nearest tenth of a foot. *Hint:* A baseball diamond is a square in which the side is 90 feet. The pitcher is 60 feet 6 inches from home plate.

SUPERIOR WORK

Illustrative Example. The sides of a triangle are 13, 14, and 15 units. Find the altitude to the 14-unit side.

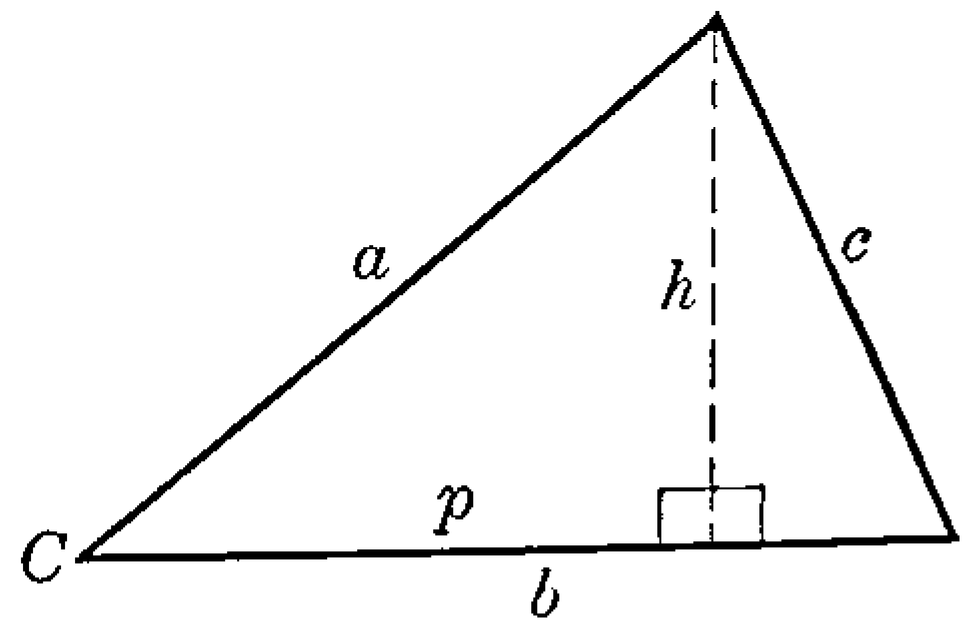


$$15^2 = 14^2 + 13^2 - 2 \cdot 14 \cdot 13 \cos x$$
$$225 = 196 + 169 - 364 \cos x$$
$$\cos x = \frac{196 + 169 - 225}{364} = \frac{140}{364} = \frac{5}{13}$$
$$\frac{y}{13} = \cos x, \quad y = 13 \cos x = 13 \cdot \frac{5}{13} = 5$$
$$h^2 = 13^2 - y^2 = 13^2 - 5^2 = 144, \quad h = 12$$

S1. The sides of a triangle are 10, 17, and 21 units. Find the area of the triangle.

Heron's Formula (Derivation)

S322. The sides of a triangle are a, b, and c units. Find the area.



The answer to this problem can be written as a formula known as Heron's (or Hero's) formula.

$$c^2 = a^2 + b^2 - 2ab \cos C;$$
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \tag{1}$$
$$\frac{p}{a} = \cos C; \qquad p = a \cos C \tag{2}$$

Substituting (1) in (2),

$$p = a \cdot \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + b^2 - c^2}{2b} \tag{3}$$

From the Pythagorean Theorem,

$$h^2 = a^2 - p^2 = (a + p)(a - p) \qquad (4)$$

Substituting (3) in (4),

$$h^2 = \left(a + \frac{a^2 + b^2 - c^2}{2b}\right)\left(a - \frac{a^2 + b^2 - c^2}{2b}\right)$$

$$h^2 = \frac{2ab + a^2 + b^2 - c^2}{2b} \cdot \frac{2ab - a^2 - b^2 + c^2}{2b} \qquad (5)$$

This formula can be greatly simplified by letting s stand for the semi-perimeter of the triangle.

Let $\quad a + b + c = 2s \qquad (6)$

Then $\quad a + b - c = 2s - 2c,$

$$a + b - c = 2(s - c) \qquad (7)$$

Similarly,

$$a - b + c = 2(s - b) \qquad (8)$$

and

$$-a + b + c = 2(s - a) \qquad (9)$$

(6) × (7),

$$a^2 + 2ab + b^2 - c^2 = 4s(s - c) \qquad (10)$$

(8) × (9),

$$2ab - a^2 - b^2 + c^2 = 4(s - a)(s - b) \qquad (11)$$

Substituting (10) and (11) in (5),

$$h^2 = \frac{4s(s - c)}{2b} \cdot \frac{4(s - a)(s - b)}{2b}$$

$$= \frac{4}{b^2} s(s - a)(s - b)(s - c)$$

$$h = \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$$

Area of the triangle,

$$T = \frac{b}{2} \cdot h$$

$$= \frac{b}{2} \cdot \frac{2}{b}\sqrt{s(s - a)(s - b)(s - c)}$$

$$T = \sqrt{s(s - a)(s - b)(s - c)} \qquad (12)$$

S2. The sides of a triangle are 15, 41, and 52 units. Find the area of the triangle, using Heron's formula.

S3. The sides of a triangle are 25, 29, and 36 units. Find the altitude upon the side 36. *Hint:* It is easier to find the area of the triangle by Heron's formula first.

S4. (*a*) Copy the figure in § 314 on page 221. In *your* figure, draw OP' so that $\angle COP'$ is the supplement of $\angle x$; *i.e.*, $\angle COP' = 180° - x$. Draw $P'C' \perp OC$. Prove that $OC' = OC$. It was explained in § 314 that OC' is negative, so that $OC' = -OC$. Since $\cos \angle COP' = OC'$ and $\cos x = OC$, $\cos \angle COP' = -\cos x$ or $\cos (180° - x) = -\cos x$.

The cosine of an angle equals the negative cosine of its supplement.

(*b*) Prove that the Law of Cosines is true for obtuse angles.

Help: Follow the proof in § 321, first drawing a figure in which $\angle C$ is obtuse.

Note the following changes: $BD = a + m$, $m = b \cos \angle ACD = b \cos (180° - C) = -b \cos C$.

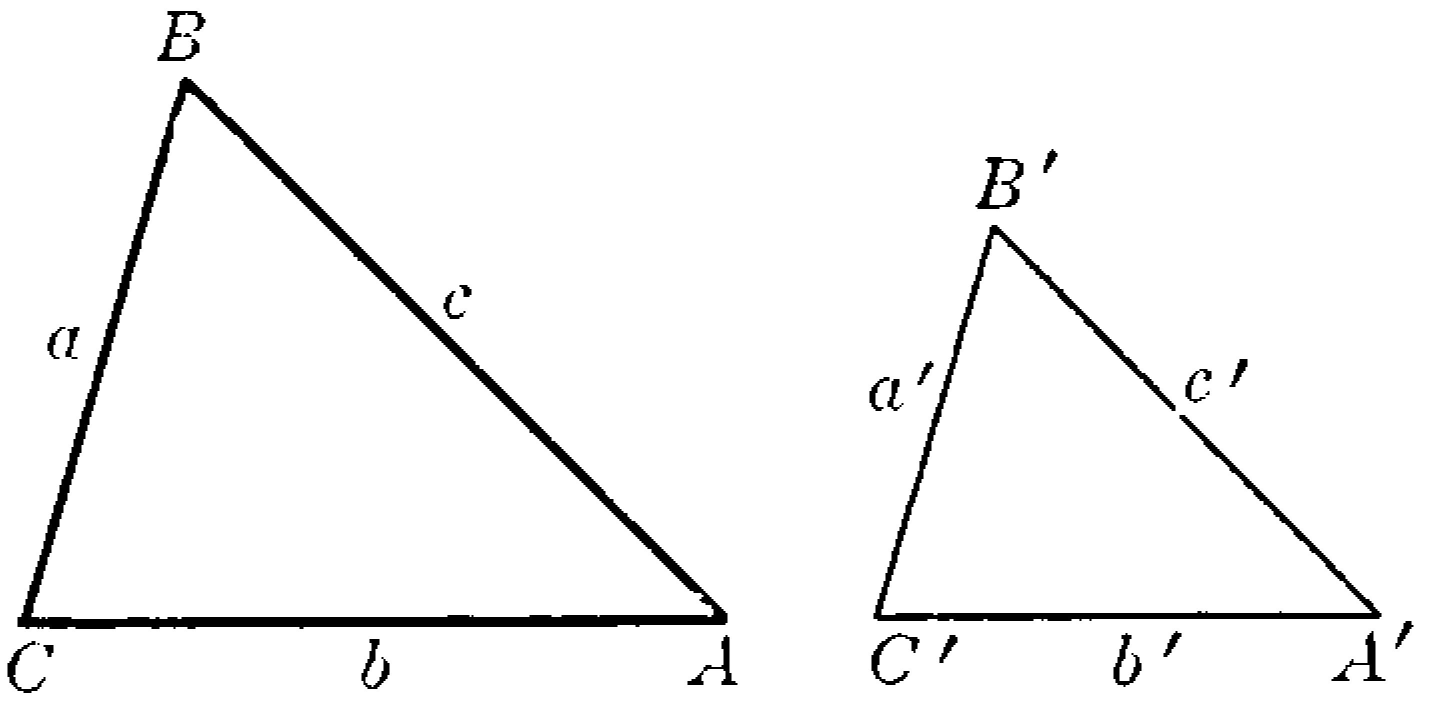
The Inverse Function

323. In an equation like $y = 2x$, we think of y as a function of x because y is completely determined by x. The converse is also true; that is, x is a function of y for the reason that when y is known we can find x. We think of x as the *inverse function.*

Similarly, in the Law of Cosines, $c^2 = a^2 + b^2 - 2ab \cos C$, we think of c as a function of a, b, and $\angle C$. Here $\angle C$ is an inverse function of a, b, and c. If the sides of a triangle are given, then the angles can be found.

The idea of the inverse function enables us to prove two important converses of the fundamental theorem (§ 298) about similar triangles.

324. THEOREM. If the sides of one triangle are proportional to the sides of another triangle, the triangles are similar.



Given: In $\triangle$ ABC and $A'B'C'$

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

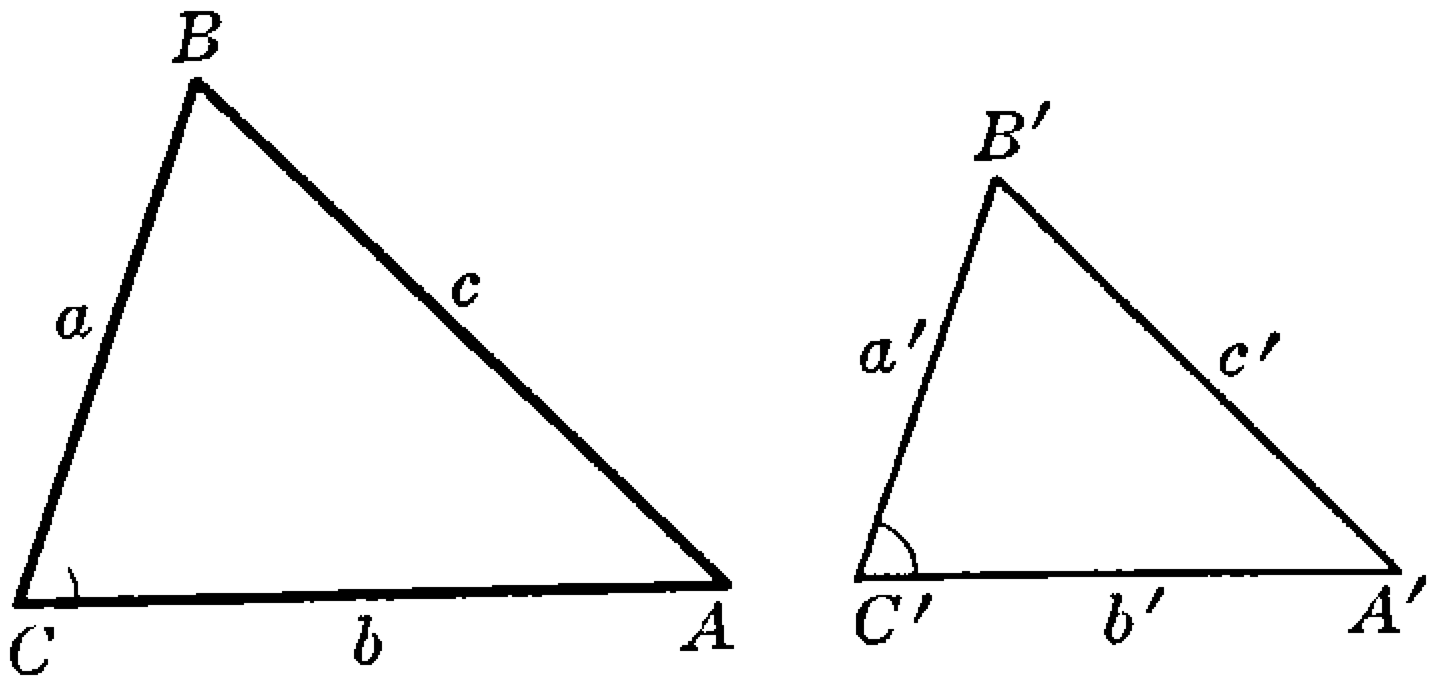
Prove: $\triangle ABC \sim \triangle A'B'C'$

PROOF

Statements	*Reasons*
1. $c^2 = a^2 + b^2 - 2ab\cos C$	1. Law of Cosines.
2. $a^2 = a^2$	2. Identity.
3. $\left(\frac{c}{a}\right)^2 = 1 + \left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)\cos C$ Similarly, $\left(\frac{c'}{a'}\right)^2 = 1 + \left(\frac{b'}{a'}\right)^2 - 2\left(\frac{b'}{a'}\right)\cos C'$	3. Div. Ax.
4. Since $\frac{c}{c'} = \frac{a}{a'}$ and $\frac{b}{b'} = \frac{a}{a'}$	4. Given.
5. $\frac{c}{a} = \frac{c'}{a'}$ and $\frac{b}{a} = \frac{b'}{a'}$	5. Alternation.
6. $\cos C = \cos C'$, or $\angle C = \angle C'$ In the same way, $\angle B = \angle B'$	6. Subs. Ax. (in equations in Statement 3).
7. $\triangle ABC \sim \triangle A'B'C'$	7. Def. $\sim$ polygons.

Two Sides Proportional and the Included Angles Equal

325. THEOREM. If two sides of one triangle are proportional to two sides of another triangle, and the included angles are equal, the triangles are similar.



Given: In $\triangle$ ABC and $A'B'C'$

$$\frac{a}{a'} = \frac{b}{b'} \text{ and } \angle C = \angle C'$$

Prove: $\triangle ABC \sim \triangle A'B'C'$

PROOF

Statements	*Reasons*
1. $c^2 = a^2 + b^2 - 2ab\cos C$	1. Law of Cosines.
2. $a^2 = a^2$	2. Identity.
3. $\left(\frac{c}{a}\right)^2 = 1 + \left(\frac{b}{a}\right)^2 - 2\left(\frac{b}{a}\right)\cos C$ Similarly, $\left(\frac{c'}{a'}\right)^2 = 1 + \left(\frac{b'}{a'}\right)^2 - 2\left(\frac{b'}{a'}\right)\cos C'$	3. Div. Ax.
4. $\angle C = \angle C'$ and $\frac{a}{a'} = \frac{b}{b'}$ or $\frac{b}{a} = \frac{b'}{a'}$	4. Given.
5. $\frac{c}{a} = \frac{c'}{a'}$	5. Subs. Ax. (in Statement 3).
6. $\triangle ABC \sim \triangle A'B'C'$	6. If their sides are proportional, the $\triangle$s are $\sim$ (§ 324).

A Line Divides Two Sides of a Triangle Proportionally (Converse of § 302)

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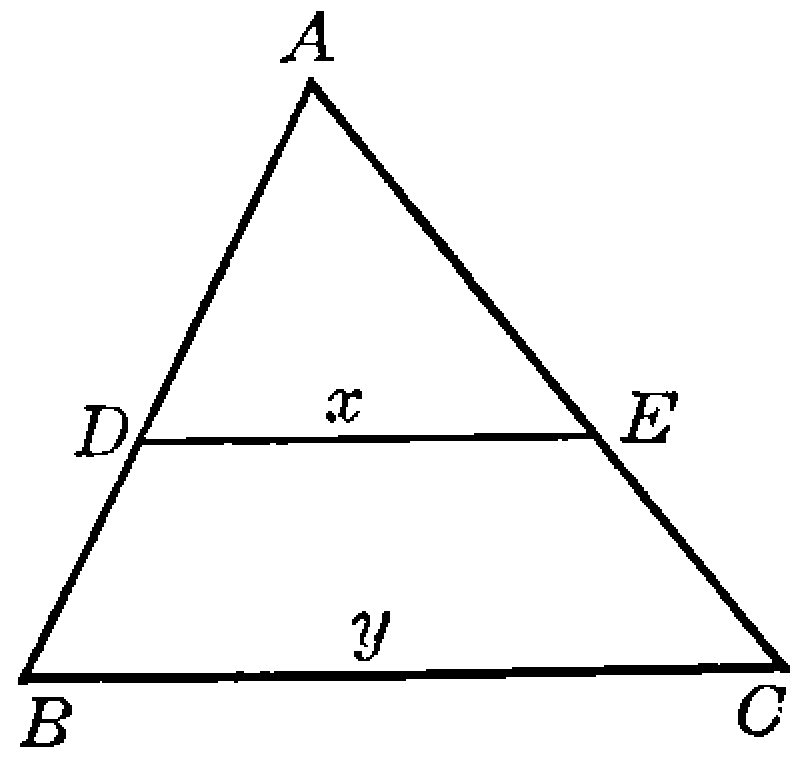
326. THEOREM. If a line divides two sides of a triangle proportionally, it is parallel to the third side.



Given: $\frac{AD}{DB} = \frac{AE}{EC}$

Prove: $DE \parallel BC$

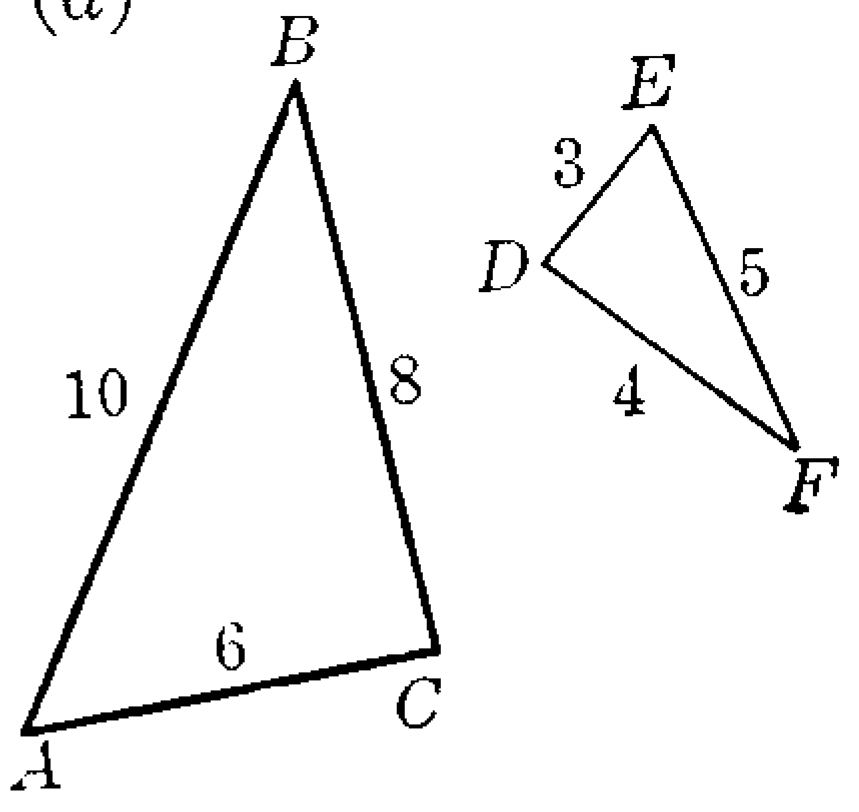
PROOF

Statements	*Reasons*
1. $\frac{AD}{DB} = \frac{AE}{EC}$	1. Given.
2. $\frac{AD + DB}{AD} = \frac{AE + EC}{AE}$ or $\frac{AB}{AD} = \frac{AC}{AE}$	2. Addition Transformation.
3. $\angle A = \angle A$	3. Identity.
4. $\triangle ABC \sim \triangle ADE$	4. If 2 sides are proportional and the included $\angle$s are =, the $\triangle$s are $\sim$ (§ 325).
5. $\angle B = \angle ADE$	5. Def. $\sim$ polygons.
6. $DE \parallel BC$	6. Corr. $\angle$s are =, $\parallel$s.

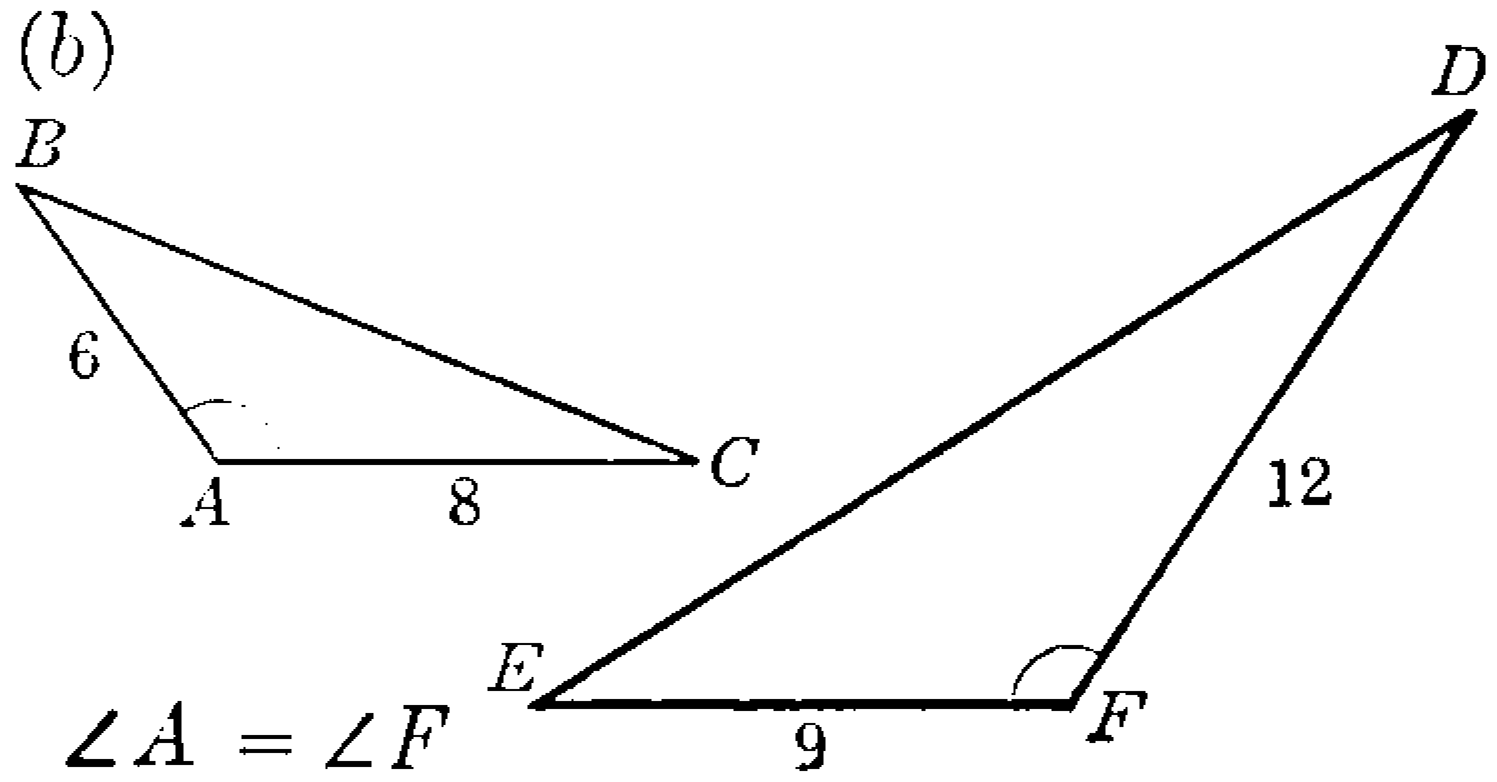
EXERCISES

1. From the measurements given in each figure, show that two triangles are similar. Name the equal angles.

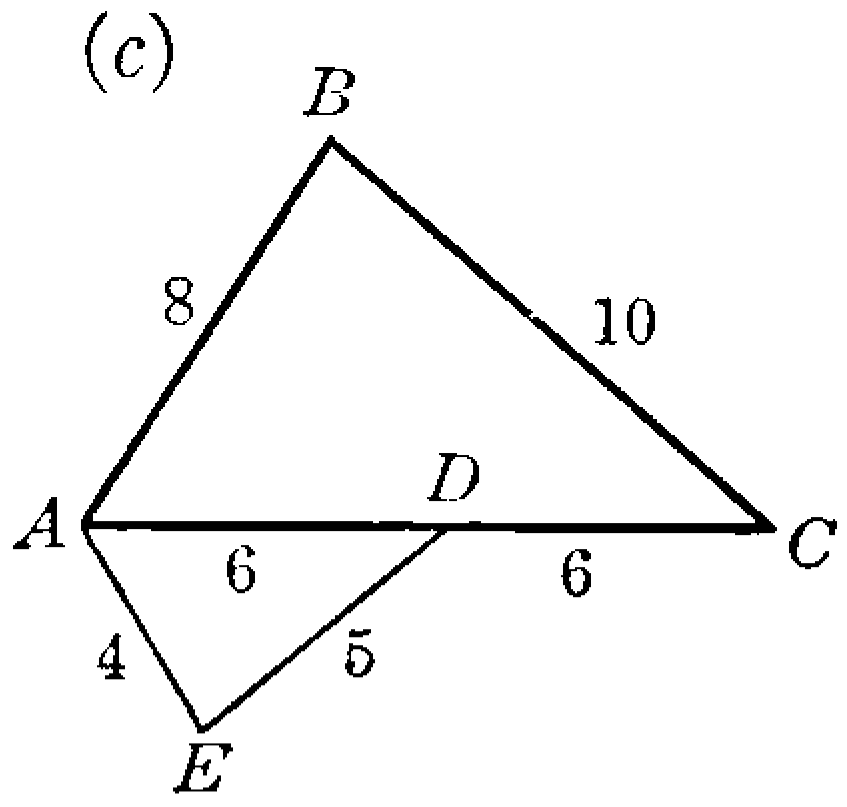
(*a*)



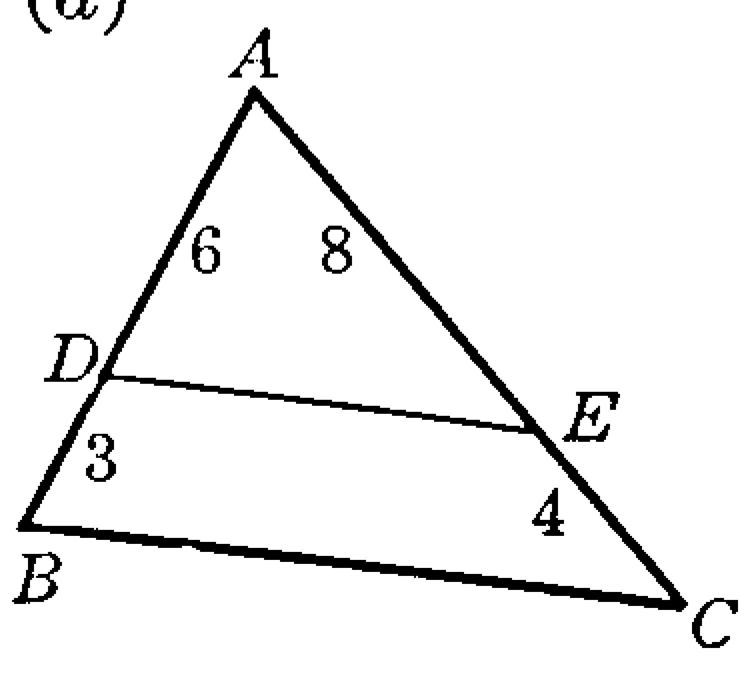
(*b*)



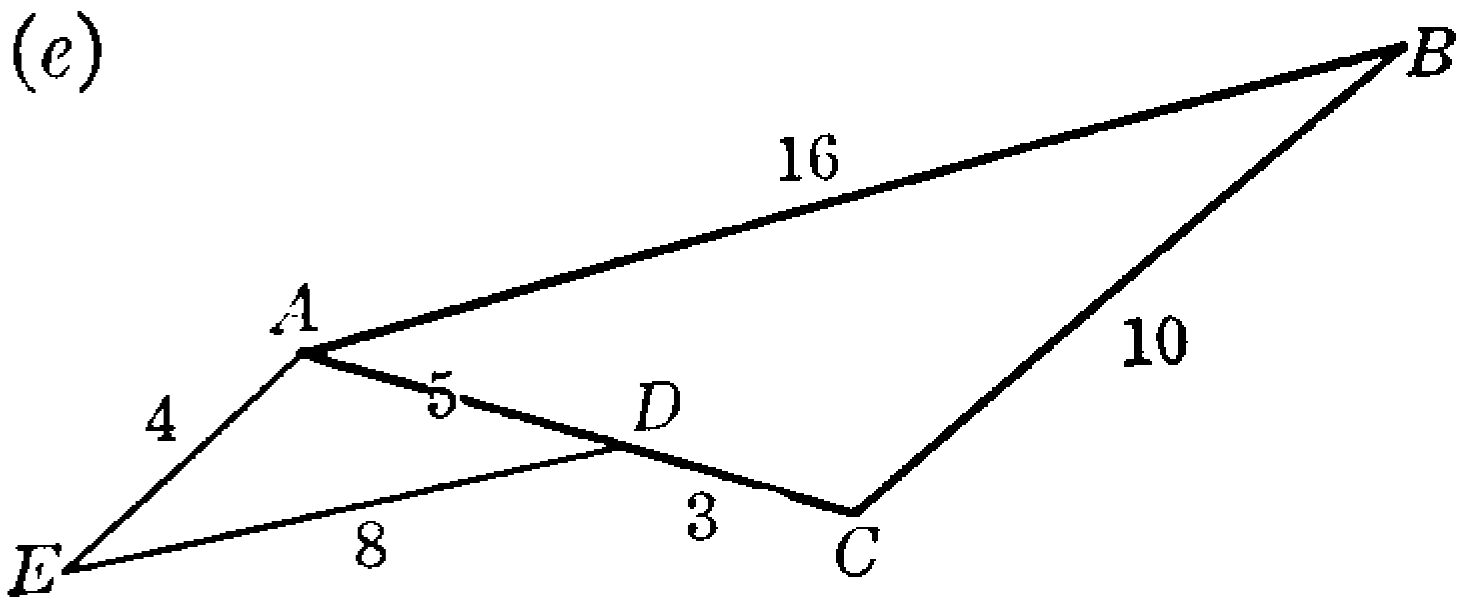
(*c*)



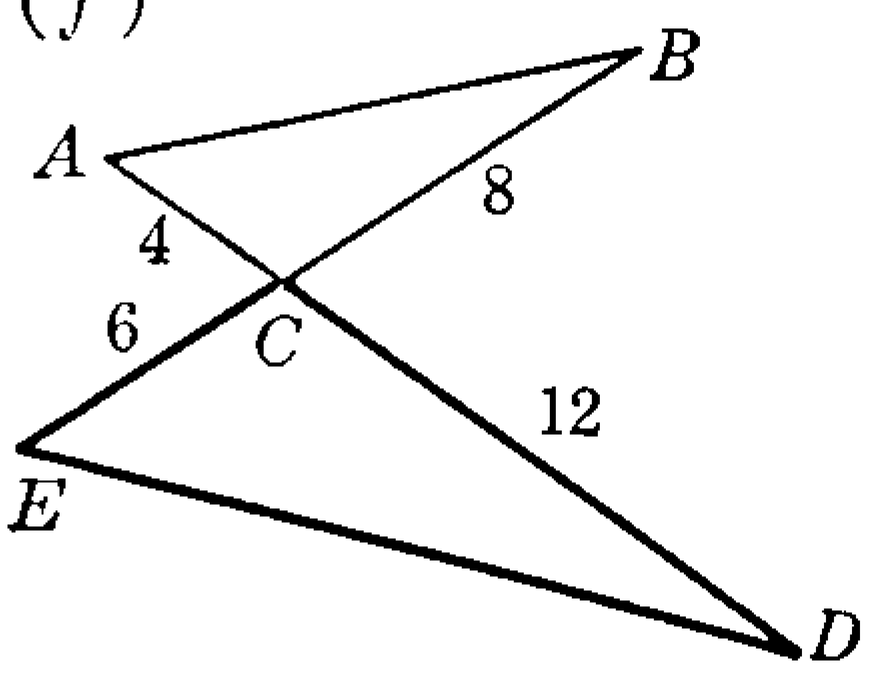
(*d*)



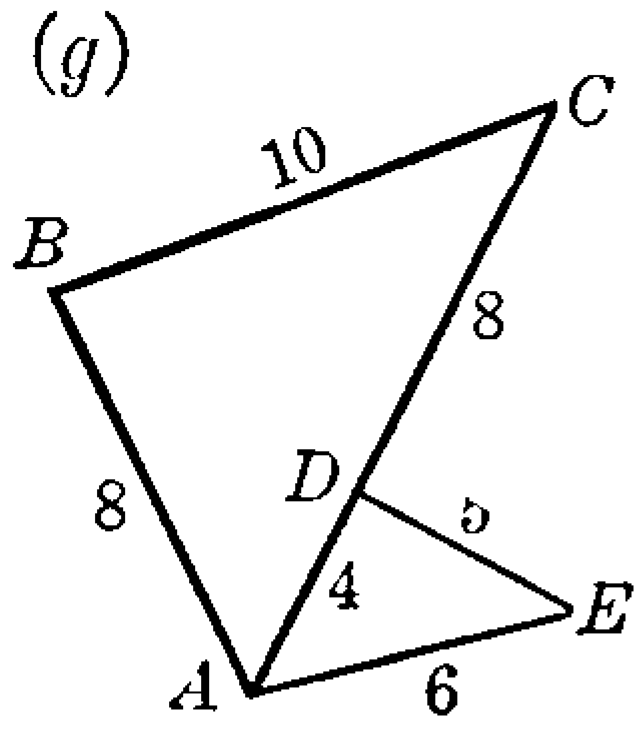
(*e*)



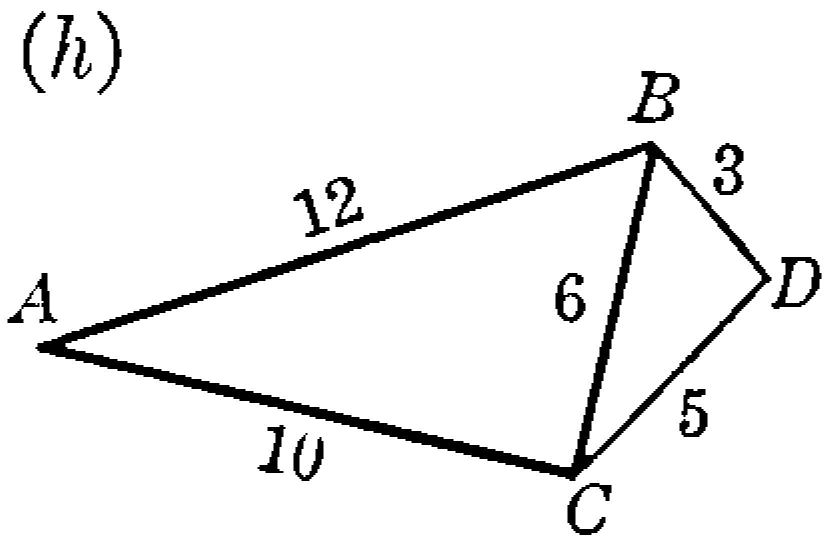
(*f*)



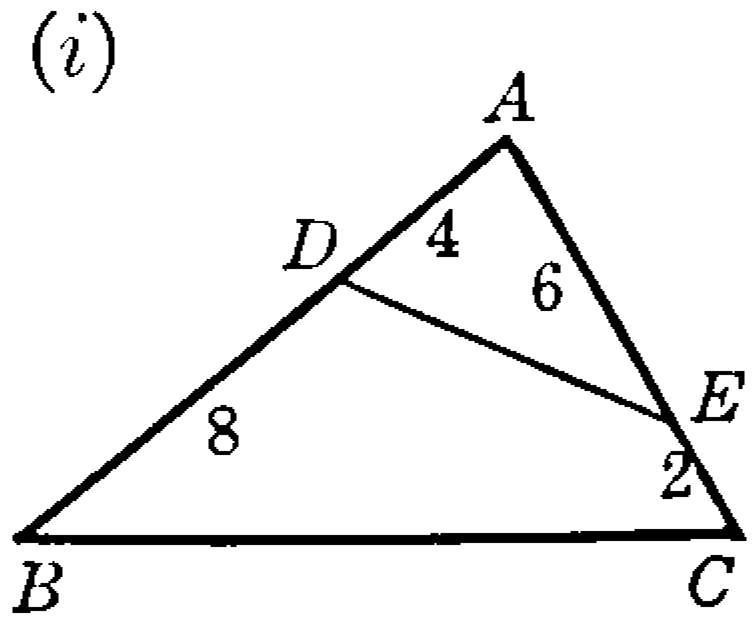
(*g*)



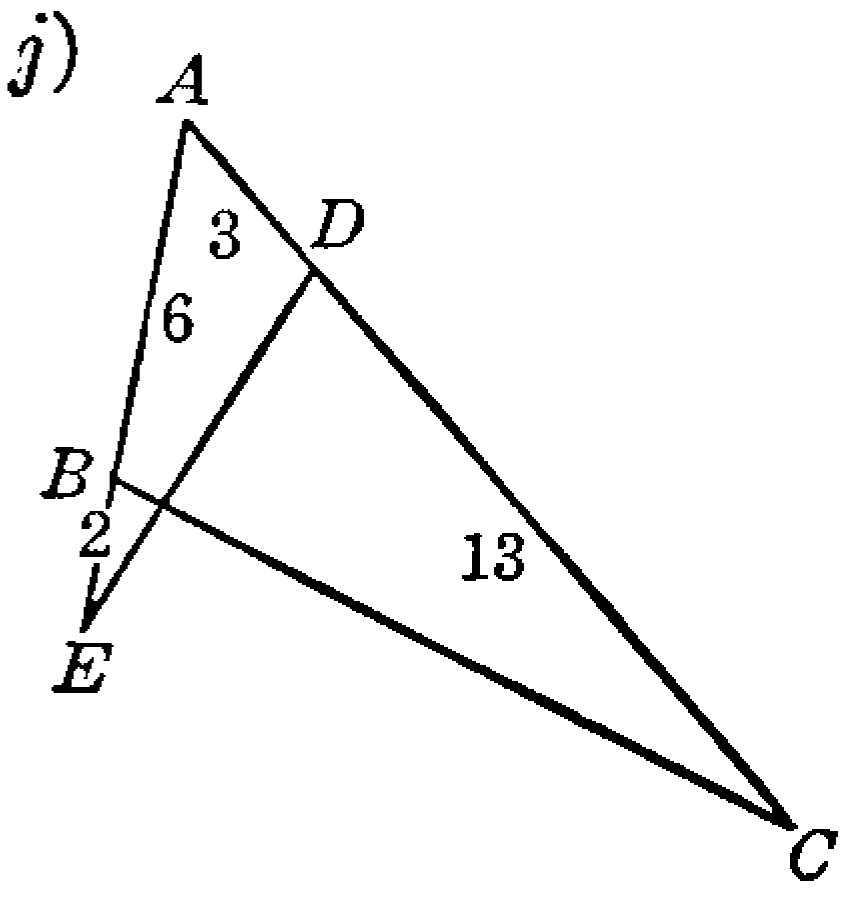
(*h*)



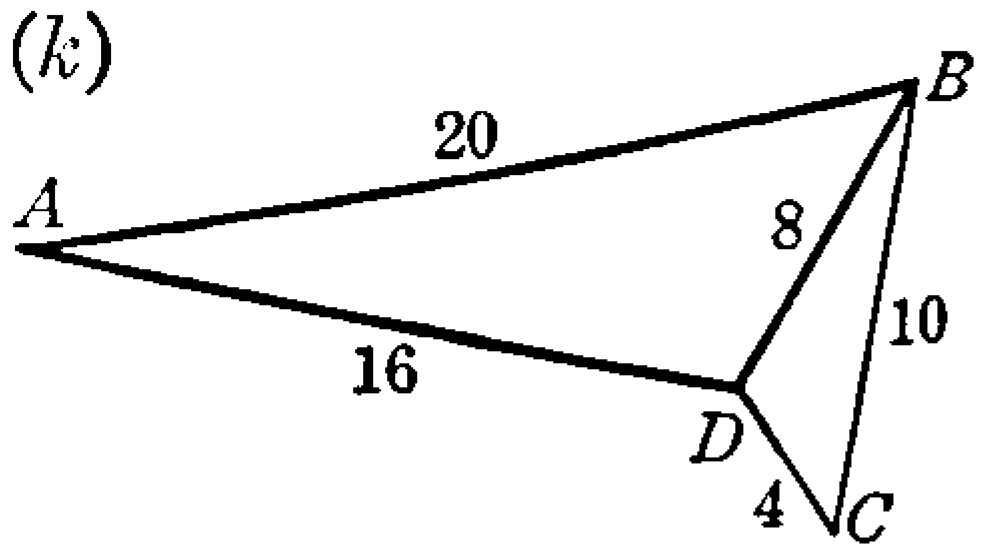
(*i*)



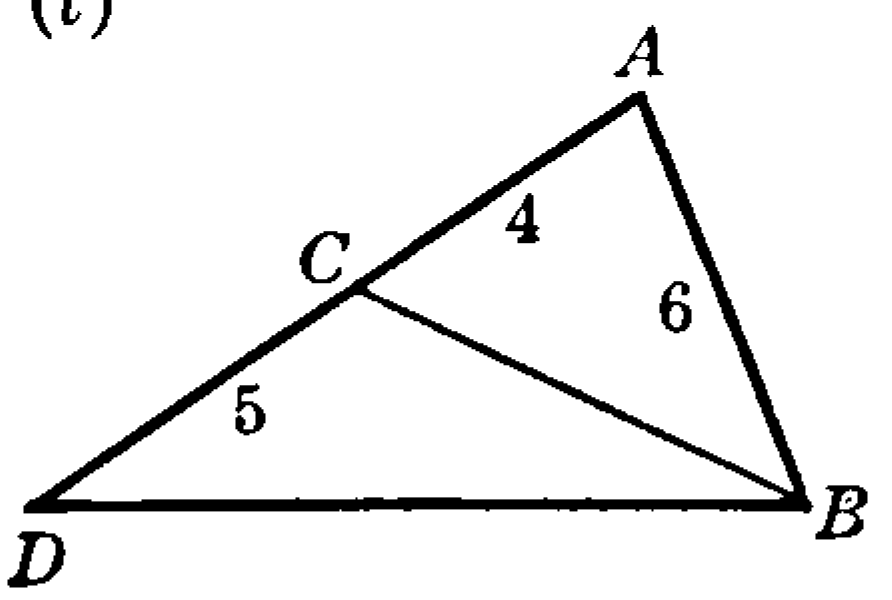
(*j*)



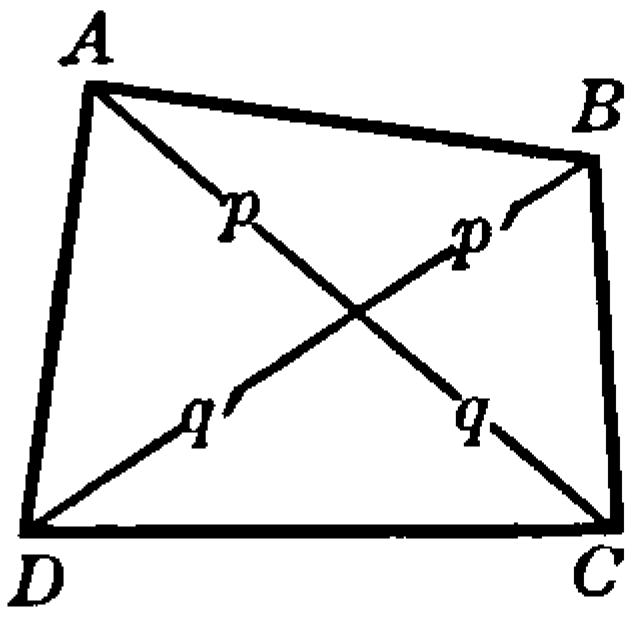
(*k*)



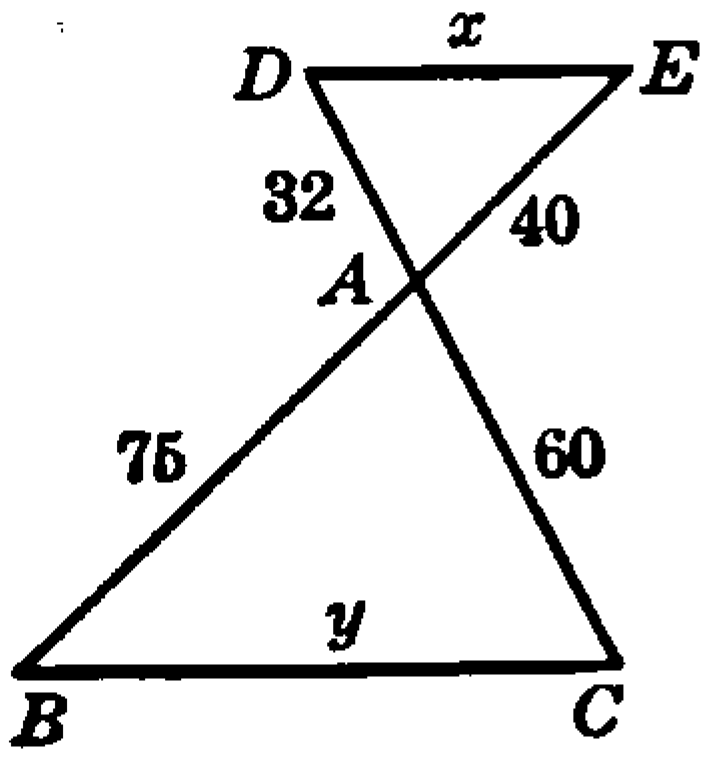
(*l*)



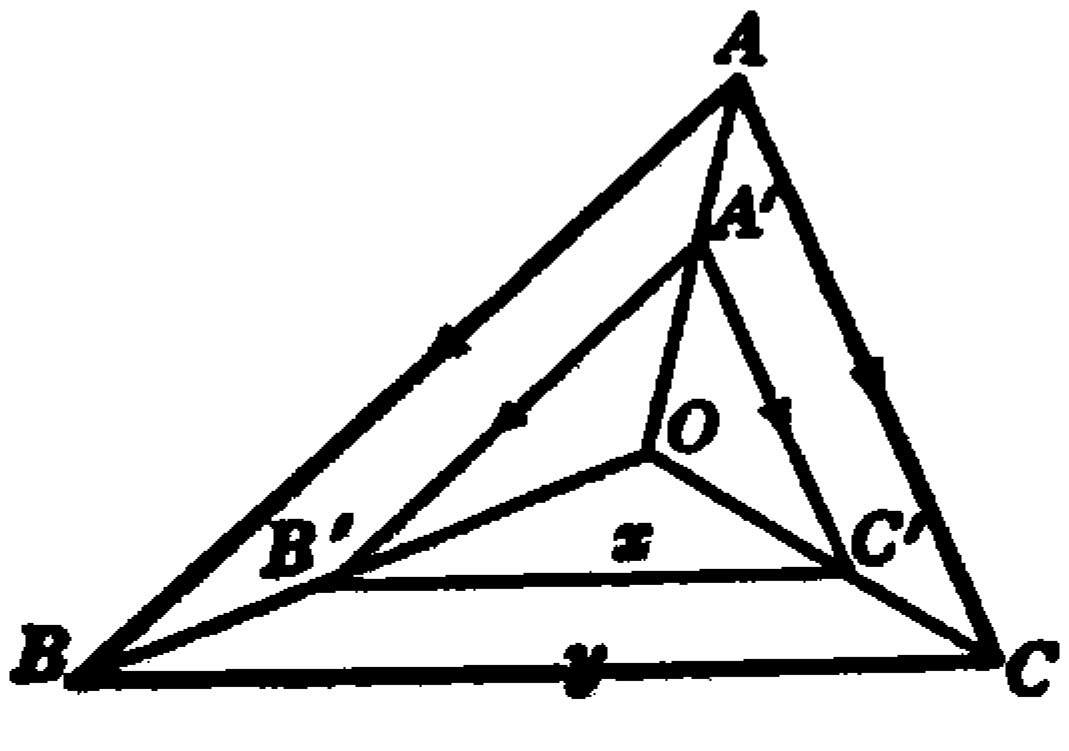
2. In the figure, $pq = p'q'$. Prove two sets of triangles similar. Which are the equal angles?



3. The sides of a triangle are 6, 10, and 14 units. The sides of another triangle are 125, 75, and 175 units. Find the ratio of their areas.
4. Prove that the corresponding medians of two similar triangles are proportional to a pair of corresponding sides.
5. Using the measurements given in the figure, prove that DE is parallel to BC.



6. Prove that if the diagonals of a quadrilateral divide each other proportionally, the figure is a trapezoid.
7. The figure shows a method that may be used for reducing (or enlarging) a triangle without altering its shape. A point O is joined to the vertices of the triangle ABC. From point A' on OA, draw $A'B'$ parallel to AB, and $A'C''$ parallel to AC. Prove that $B'C''$ will be parallel to BC.



Heron's Formula

327. The following is known as Heron's formula for the area of a triangle when the sides are given:

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$

where a, b, and c are the sides of the triangle and s is the semi-perimeter; that is, $s = \frac{1}{2}(a + b + c)$. The formula was derived in § S322.

Illustrative Example. Find the area of a triangle whose sides are 18, 42, and 48 units.

Let $a = 18$

$b = 42$

$c = 48$

Then $a + b + c = 108$

$$s = \tfrac{1}{2}(a + b + c) = 54$$
$$s - a = 54 - 18 = 36$$
$$s - b = 54 - 42 = 12$$
$$s - c = 54 - 48 = 6$$

Substituting in

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$
$$T = \sqrt{54 \cdot 36 \cdot 12 \cdot 6}$$

Instead of multiplying these four factors, it is sometimes shorter to factor them *further* as follows:

$$T = \sqrt{9 \cdot 6 \cdot 36 \cdot 4 \cdot 3 \cdot 6}$$
$$= \sqrt{9(6 \cdot 6) \cdot 36 \cdot 4 \cdot 3}$$
$$T = 3 \cdot 6 \cdot 6 \cdot 2\sqrt{3} = 216\sqrt{3}$$
$$T = 216 \times 1.732 = 374$$

(to three-digit accuracy).

8. Find the area of a triangle whose sides are 13, 14, and 15 units.
9. Find the area of a triangle whose sides are 29, 35, and 28 units.
10. The sides of a triangle are 8 inches, 11 inches, and 13 inches. Find the area of the triangle correct to three significant digits.

11. The sides of a triangular park are 42 yards, 68 yards, and 87 yards. What is the shortest walk that can be constructed from the longest side to the opposite corner? *Hint:* First find the area of the triangle.

12. Each side of a triangle is 10. Using Heron's formula, find the area in terms of the simplest radical.

13. Each side of an equilateral triangle is a. Using Heron's formula, find the area in terms of a and the simplest radical.

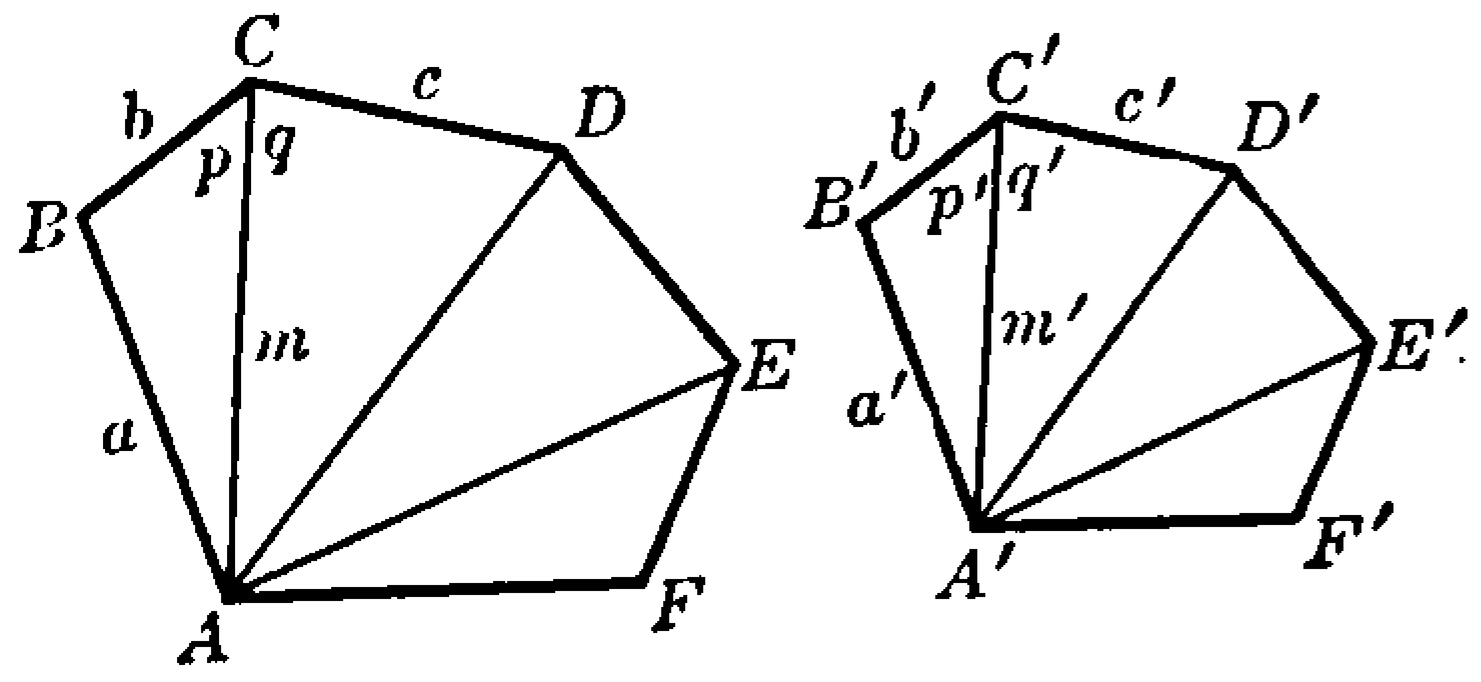
14. Two sides of a parallelogram are 25 inches and 26 inches. One diagonal is 17 inches. Find the area of the parallelogram.

SUPERIOR WORK

S1. Copy the proof for the following theorem, supplying all the reasons.

Theorem. Two similar polygons can be divided into triangles that are similar each to each and are similarly placed.

Given: Polygon $ABCDEF \sim$ Polygon $A'B'C'D'E'F'$
Prove: $\triangle ABC \sim \triangle A'B'C'$
$\triangle ACD \sim \triangle A'C'D'$
$\triangle ADE \sim \triangle A'D'E'$, etc.



PROOF

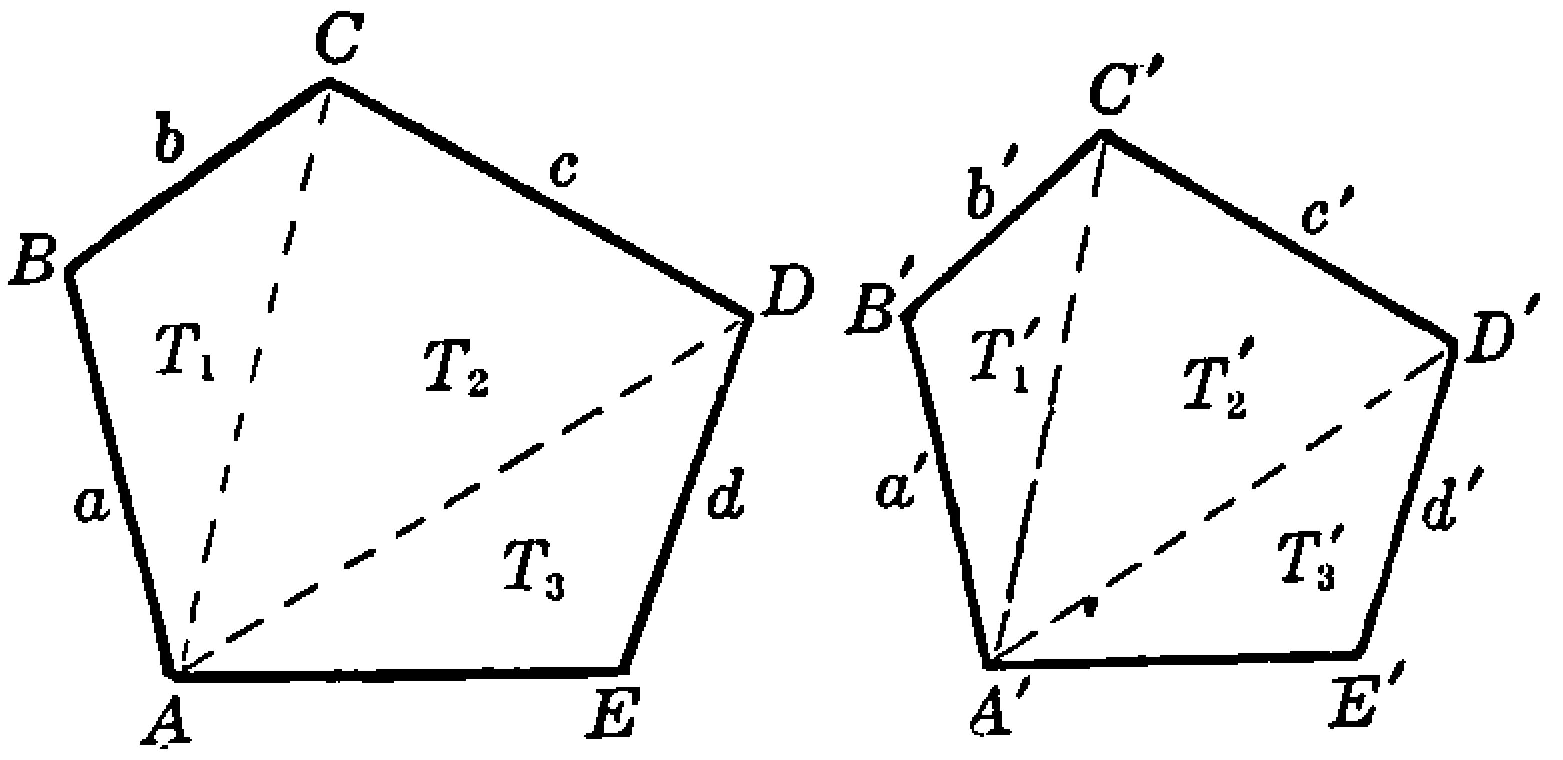
Statements	*Reasons*
1. $\angle B = \angle B'$	1. —?—.
2. $\frac{a}{a'} = \frac{b}{b'}$	2. —?—.
3. $\triangle ABC \sim \triangle A'B'C'$	3. —?—.
4. $\angle C = \angle C'$	4. —?—.
5. $\angle p = \angle p'$	5. —?—.
6. $\angle q = \angle q'$	6. —?—.
7. $\frac{b}{b'} = \frac{m}{m'}$	7. —?—.
8. $\frac{b}{b'} = \frac{c}{c'}$	8. —?—.
9. $\frac{m}{m'} = \frac{c}{c'}$	9. —?—.
10. $\triangle ACD \sim \triangle A'C'D'$	10. —?—.

Similarly, we can prove the remaining $\triangle s \sim$.

S2. *Theorem.* The areas of two similar polygons are proportional to the squares of the corresponding sides.

Given: Polygon $ABCDE \sim$ Polygon $A'B'C'D'E'$
S = Area polygon $ABCDE$
S' = Area polygon $A'B'C'D'E'$

Prove: $\frac{S}{S'} = \frac{a^2}{a'^2}$



Outline of Proof:

1. $\frac{T_1}{T_1'} = \frac{a^2}{(a')^2}, \quad \frac{T_2}{T_2'} = \frac{c^2}{(c')^2}, \quad \frac{T_3}{T_3'} = \frac{d^2}{(d')^2}$

2. $\frac{T_1}{T_1'} = \frac{T_2}{T_2'} = \frac{T_3}{T_3'} = \frac{a^2}{(a')^2}$

3. $\frac{T_1 + T_2 + T_3}{T_1' + T_2' + T_3'} = \frac{a^2}{(a')^2}$

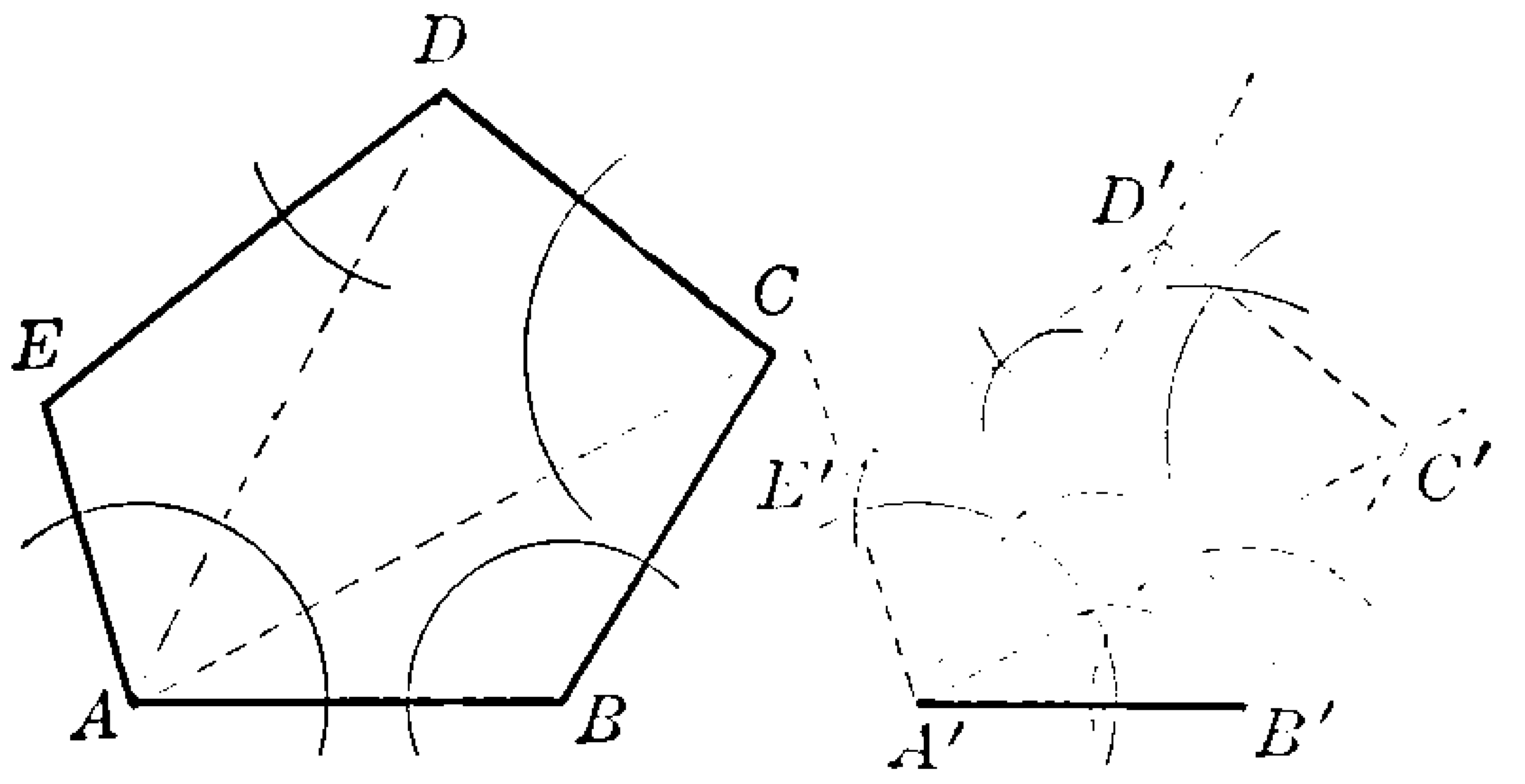
Write out the complete proof.

S3. Following the plan in S1, prove the following converse.

Theorem. If two polygons can be divided into triangles that are similar each to each and similarly placed, they are similar.

S4. *Problem.* On a given base, construct a polygon similar to a given polygon.

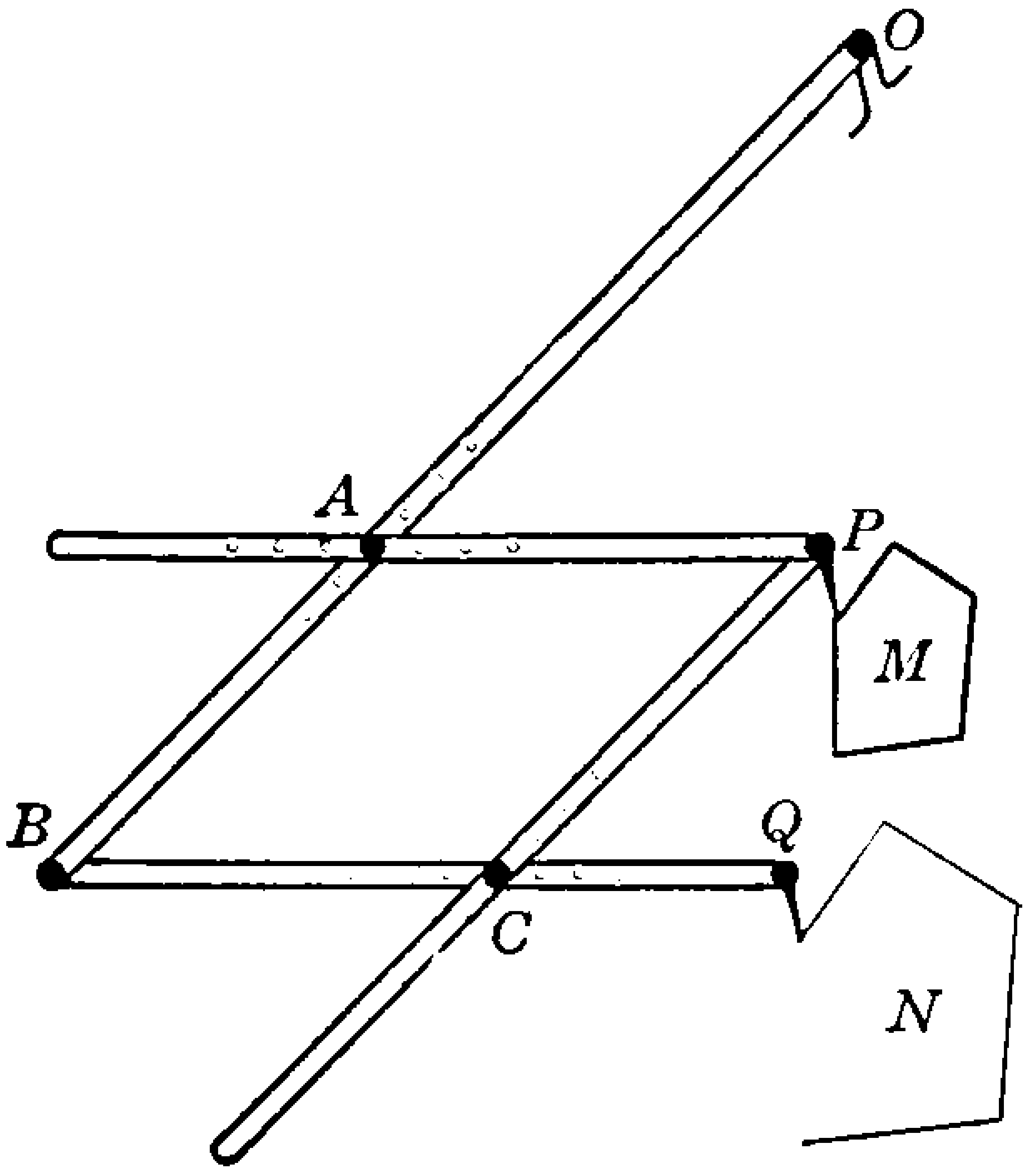
Given: Polygon $ABCDE$, and base $A'B'$
Req.: To construct polygon $A'B'C'D'E'$ similar to polygon $ABCDE$, so that $A'B'$ will correspond to AB



Using the theorem in S3, construct $\triangle A'B'C'$ similar to $\triangle ABC$ by copying two angles. Then construct $\triangle A'C'D'$ similar to $\triangle ACD$, and so on. Perform the construction, using as few arcs as possible.

HONOR WORK

H1. The *pantograph* is a simple instrument to enlarge or reduce a figure, while retaining its shape. Four rods are so adjusted that $ABCP$ is a parallelogram and $\frac{OA}{OB} = \frac{BC}{BQ}$. The point O is made stationary; pencils are attached to the points P and Q. As the point P traces the figure M, the point Q draws the figure N similar to M, the ratio being $\frac{OA}{OB}$.



To prove the theory of the pantograph, draw two positions of it and prove that O, P, and Q are and remain collinear; also that the displacements of P and Q are in a constant ratio— $\frac{PP'}{QQ'} = \frac{OA}{OB}$.

SUMMARY

328. From the theorems about congruent triangles it is evident that, in general, three parts of a triangle determine the remaining parts. For example, an angle of a triangle depends on the three sides of the triangle and is determined by them. This is expressed by saying that an angle of a triangle is a function of the sides of the triangle. In order to express one part of a triangle in terms of three other parts as a function that can be computed numerically, we begin with the right triangle.

Two right triangles that have the same acute angle are similar, and therefore agree in the ratio of any two corresponding sides. One of these ratios, the adjacent leg divided by the hypotenuse, is called the cosine of the acute angle. This ratio depends only on the size of the acute angle and is completely determined by it. The cosine, therefore, is a function of the acute angle. With the aid of the cosine, all the parts of a right triangle can be found when two independent parts (in addition to the right angle) are given. This is called solving the right triangle.

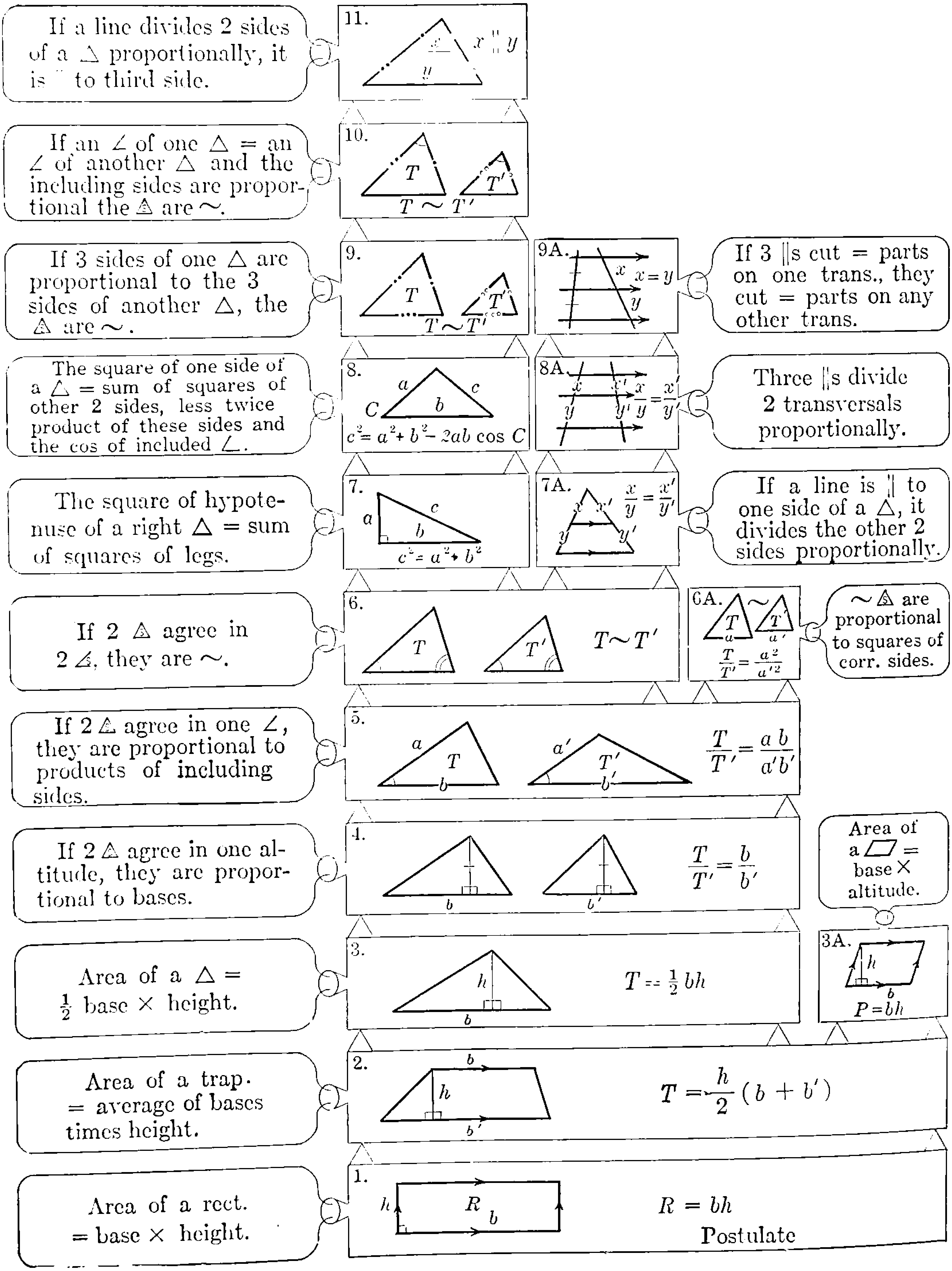
A special case of the solution of the right triangle is the Pythagorean relationship among the sides: The square of the hypotenuse equals the sum of the squares of the legs.

As special cases of the right triangle, the 45° right triangle and the 30°–60° right triangle require no tables for their solution. The cosines of these angles can be found from the simple relationships among the sides of these special triangles.

The oblique triangle is solved by drawing an altitude, thereby breaking it up into two right triangles. The Pythagorean relationship is applied to each right triangle and the Law of Cosines is found to be: $c^2 = a^2 + b^2 - 2ab \cos C$.

Using the Law of Cosines, two oblique triangles whose sides are proportional are compared. The corresponding angles are found to be equal. This proves that if two triangles have their sides proportional, they are similar.

Then two triangles that agree in one angle and have the including sides proportional are compared. The third sides of the two triangles are found to be proportional to the



11. If a line divides 2 sides of a △ proportionally, it is ∥ to third side. — $x \parallel y$

10. If an ∠ of one △ = an ∠ of another △ and the including sides are proportional the △ are ∼. — $T \sim T'$

9. If 3 sides of one △ are proportional to the 3 sides of another △, the △ are ∼. — $T \sim T'$

9A. If 3 ∥s cut = parts on one trans., they cut = parts on any other trans. — $x = y$

8. The square of one side of a △ = sum of squares of other 2 sides, less twice product of these sides and the cos of included ∠. — $c^2 = a^2 + b^2 - 2ab \cos C$

8A. Three ∥s divide 2 transversals proportionally. — $\frac{x}{y} = \frac{x'}{y'}$

7. The square of hypotenuse of a right △ = sum of squares of legs. — $c^2 = a^2 + b^2$

7A. If a line is ∥ to one side of a △, it divides the other 2 sides proportionally. — $\frac{x}{y} = \frac{x'}{y'}$

6. If 2 △ agree in 2 ∠, they are ∼. — $T \sim T'$

6A. ∼ △ are proportional to squares of corr. sides. — $\frac{T}{T'} = \frac{a^2}{a'^2}$

5. If 2 △ agree in one ∠, they are proportional to products of including sides. — $\frac{T}{T'} = \frac{a\,b}{a'b'}$

4. If 2 △ agree in one altitude, they are proportional to bases. — $\frac{T}{T'} = \frac{b}{b'}$

3. Area of a △ = ½ base × height. — $T = \frac{1}{2} bh$

3A. Area of a ▱ = base × altitude. — $P = bh$

2. Area of a trap. = average of bases times height. — $T = \frac{h}{2}(b + b')$

1. Area of a rect. = base × height. — $R = bh$ Postulate

other sides. From the previous theorem, then, the triangles are similar.

Finally, a line is drawn dividing two sides of a triangle proportionally. The figure then contains two triangles with a common angle and the including sides proportional. From the similarity of these triangles, it is proved that the line is parallel to the third side of the original triangle.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

function	§ 311
cosine	§ 313
Pythagorean relationship	§ 318
Law of Cosines	§ 321

329. *On page 240 you will find a chart that shows the dependence of the fundamental theorems, beginning with dimensional thinking. By studying the chart, complete the following statements. Refer to the propositions by their numbers.*

1. Prop. 2 depends upon Prop. 1 because a trapezoid can be the average of two —?—.

2. Prop. 3 depends upon Prop. 2 because a trapezoid becomes a triangle when . . . .

3. Prop. 3A depends upon Prop. 2 because a trapezoid becomes a parallelogram when . . . .

4. Prop. 4 is derived from Prop. 3 by using the —?— axiom.

5. In the proof of Prop. 5, we could compare each of the given triangles with a third triangle because a certain —?— was common.

6. The —?— (hypothesis, conclusion) in Prop. 6 is a special case of the —?— (hypothesis, conclusion) in Prop. 5.

7. The conclusion in Prop. 6A is a special case of the conclusion in Prop. —?—.

8. To prove Prop. 8, we divide the oblique triangle into two —?— triangles.

9. The most useful proposition is Prop. —?—.

10. All the theorems in the chart depend directly or indirectly on Prop. —?—.

11. Props. —?— and —?— are both special cases of Prop. 8A.

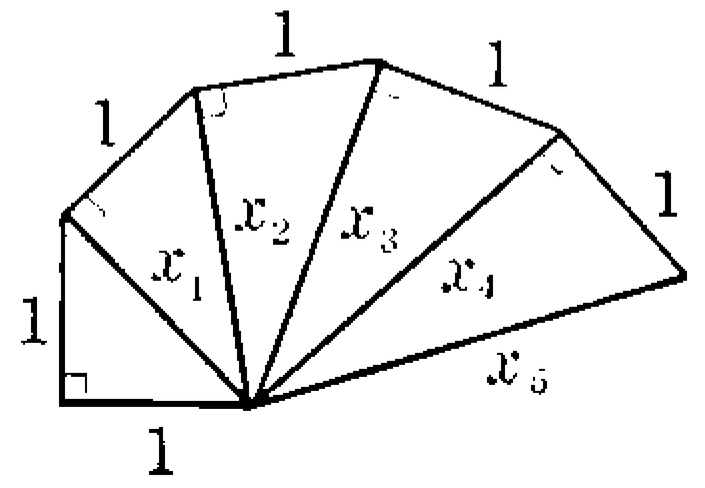
12. Prop. —?— is a generalization or extension of Prop. 7.

REVIEW EXERCISES

1. A road is said to have a 30% grade if it rises 30 feet vertically for every 100 feet horizontally. Find the distance between two points along such a road, if one of the points is 22 feet higher than the other.

2. A ship has sailed from a port as follows: 86 miles east, 10 miles south, 14 miles west, and 75 miles north. How far from port is the ship?

3. A football is on the 20-yard line, 15 yards from the sideline at the end of the first quarter. How many yards is it from the position of the ball at the end of the first quarter to the position of the ball at the beginning of the second quarter? (The width of the field is 160 feet.)



4. In the figure, find x_1, x_2, x_3, etc., in terms of the simplest radicals.

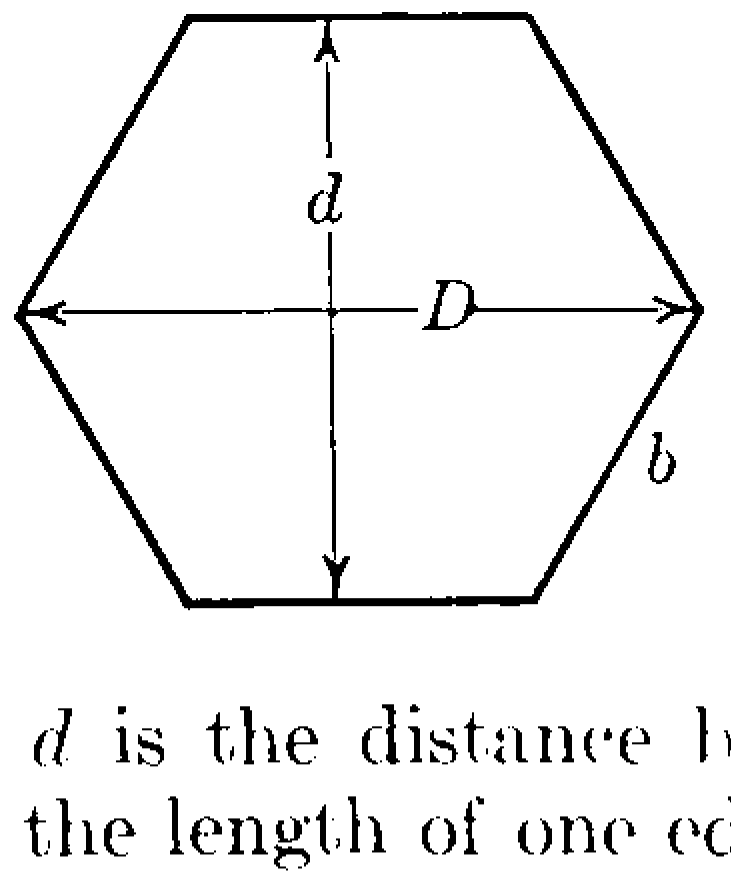
5. A ladder resting against a wall reaches to a height of 16 feet when the foot of the ladder is 12 feet from the wall.

(*a*) How much higher will the ladder reach if it is moved 6.4 feet closer to the wall?

(*b*) In each case, how far is the middle rung from the base of the wall?

6. The hypotenuse of a right triangle is 53 and one leg is 45. Using the table of cosines, but not the Pythagorean relationship, find the other leg.

7. The figure shows a hexagonal head of a bolt. *D* is called the distance between corners, *d* is the distance between flats, and *b* is the length of one edge.



Prove: $D = 2b$ and $d = 1.732b$

8. Each diagonal of a rectangle is 22 feet, and they make an angle of 30° with each other. Find the area of the rectangle.

9. An outfielder caught a fly ball that was driven 60 feet past second base and directly over it. If he completed a double play by throwing to first base, how long was his throw? A baseball diamond is a square 90 feet on each side. Disregard the first baseman's stretch.

10. Two straight roads intersect at an angle of 45°. Two cars, one on each road, are 100 and 160 feet respectively from the intersection of the roads. How far apart are the cars? (Two answers are possible.)

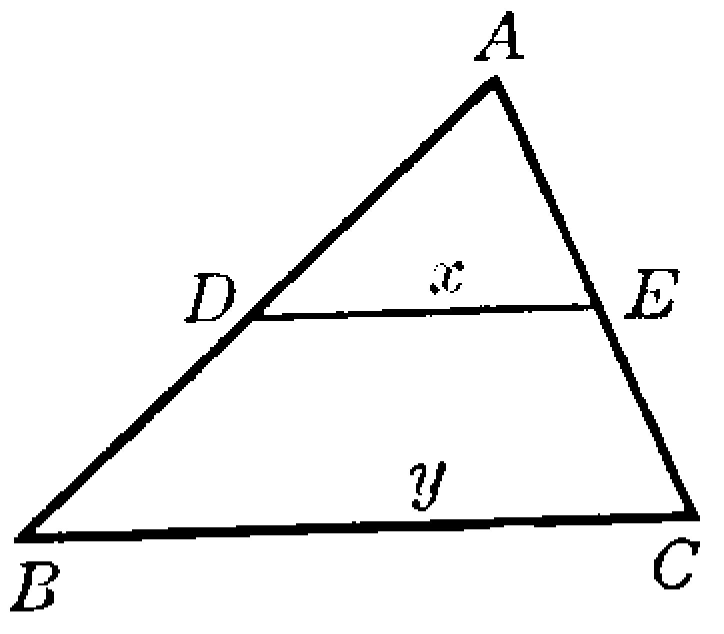
11. A railroad connects two towns 112 miles apart. Rockville is 50 miles from one town and 78 miles from the other. It is proposed to build a station on the railroad line as near to Rockville as possible. How near is that?

12. (*a*) Substitute 3, 4, and 9 units as the sides of a triangle in Heron's formula. Work out the answer for the area in radical form. Comment on your answer.
    (*b*) Using 3, 4, and 9 units as the sides of a triangle in the Law of Cosines, find the cosine of the angle opposite the 3-unit side. Comment on your answer.

13. Is the triangle whose sides are 3, 5, and 6 obtuse? Use §§ 321, 314.

14. Prove that if the areas of two right triangles are proportional to the squares of their legs, the triangles are similar.

15.



Given:
$$\frac{\triangle ADE}{\triangle ABC} = \left(\frac{AD}{AB}\right)^2$$
Prove:
$DE \parallel BC$

SUPERIOR WORK

S1. Prove that if the diagonals of a quadrilateral make one pair of opposite triangles similar, then the other pair of opposite triangles are either similar or equal.

S2. The diagonals of a quadrilateral are 10 units and 13 units and are perpendicular to each other. Find the area of the quadrilateral. *Help:* Let the segments of one diagonal be x and $13 - x$.

S3. Prove that the area of a quadrilateral equals one-half the product of the diagonals times the sine of the angle between them.

S4. (*a*) If the sides of a right triangle are also the corresponding sides of three similar triangles, prove that the largest of the three similar triangles equals the sum of the other two.

(*b*) Prove that $\sin^2 A + \cos^2 A = 1$. ($\sin^2 A$ means $\sin A \times \sin A$.)

S5. A pilot flying at 100 miles per hour at a height of 1 mile, observes the angle of depression of a speedboat as 30°. Six minutes later, after passing directly over the boat, the pilot again measured the angle of depression of the boat and found that it was 45°. Assuming that the plane and the boat were going in the same direction, how fast was the boat traveling? The angle of depression is the angle between the line of sight and a horizontal line. (See page 224, Illustrative Example.)

S6. A submarine is heading in the direction of N. 30° E. at a submerged speed of 12 knots (12 nautical miles per hour). At 1200 (12 o'clock noon) a destroyer is sighted 9 (nautical) miles to the east steaming directly north at 24 knots.

(*a*) How far apart are the two craft when the submarine crosses the course of the destroyer?

(*b*) How far apart are the two craft at 1220?

S7. (*a*) From the Pythagorean Theorem, we know that the diagonal d of a rectangle whose dimensions are l and w is given by the formula $d^2 = l^2 + w^2$. Prove that the diagonal d of a rectangular box whose dimensions are l, w, and h is given by the analogous formula $d^2 = l^2 + w^2 + h^2$. (See page 228, S10.)

(*b*) The base of a rectangular box is 12 units by 20 units and the diagonal of the box is 25 units. Find the height of the box.

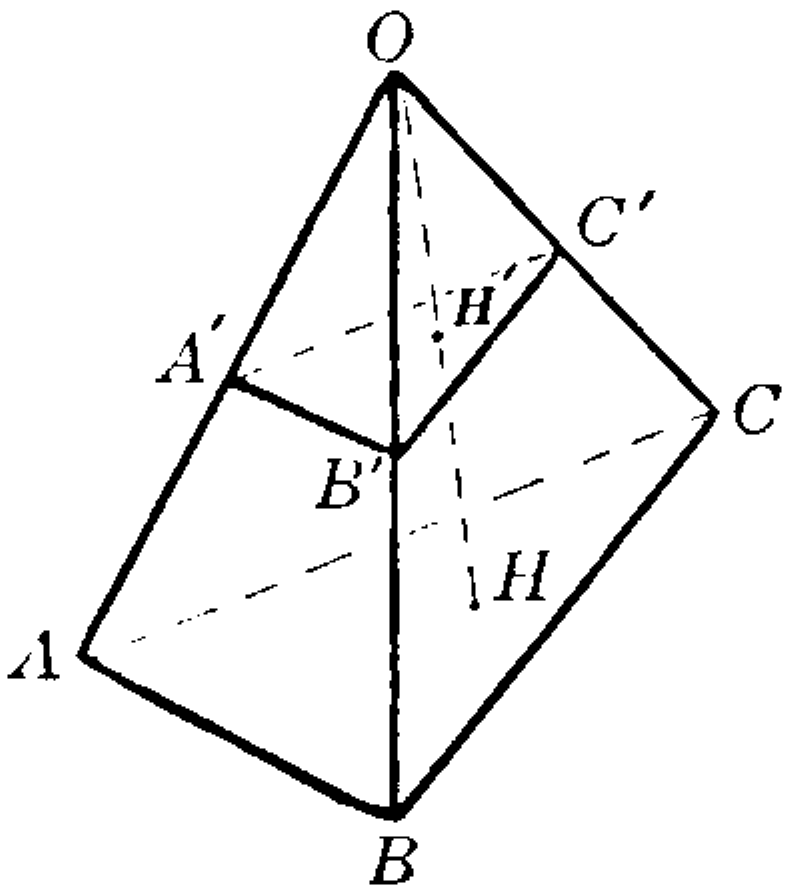
S8. The base of a pyramid is a triangle with sides of 25, 39, and 40 units. The altitude of the pyramid is 24 units. Find the volume of the pyramid. (See page 218, S18.)

S9. A plane cuts three edges of a rectangular box 9 inches, 12 inches, and 16 inches, from their common vertex. This cuts off a tetrahedron (page 197, S5). Find the volume, all the sides, and the total area of the tetrahedron.

S10. *Theorem.* The area of a plane section parallel to the base of a triangular pyramid varies as the square of its distance from the vertex.

Given: Pyramid O-ABC
Plane $A'B'C' \parallel$ plane ABC
Alt. OH cuts plane $A'B'C'$ in H'

Prove: $\dfrac{\triangle A'B'C'}{\triangle ABC} = \left(\dfrac{OH'}{OH}\right)^2$



Give the reasons for the statements below.

PROOF

1. $\dfrac{OA'}{OA} = \dfrac{OB'}{OB} = \dfrac{OC'}{OC} = \dfrac{OH'}{OH}$ (See page 217, S14.)

2. $\triangle OA'B' \sim \triangle OAB$
$\triangle OB'C' \sim \triangle OBC$

3. $\dfrac{A'B'}{AB} = \dfrac{OB'}{OB}$
$\dfrac{B'C'}{BC} = \dfrac{OB'}{OB}$

4. $\dfrac{A'B'}{AB} = \dfrac{B'C'}{BC}$
Similarly, $\dfrac{A'B'}{AB} = \dfrac{A'C'}{AC}$

5. $\triangle A'B'C' \sim \triangle ABC$

6. $\dfrac{\triangle A'B'C'}{\triangle ABC} = \left(\dfrac{A'B'}{AB}\right)^2$

7. $\dfrac{\triangle A'B'C'}{\triangle ABC} = \left(\dfrac{OH'}{OH}\right)^2$

Note: Since any pyramid can be divided into triangular pyramids, we can prove that this theorem holds for all pyramids.

S11. The base of a certain pyramid is a triangle with sides 3 inches, 4 inches, and 5 inches. The altitude is 10 inches. Find the area of the cross section parallel to the base and 7 inches from it.

S12. (*a*) The triangular base of a pyramid has an area of 80 square inches. The altitude of the pyramid is 12 inches. A plane cross section of the pyramid parallel to the base has an area of 20 square inches. How far from the vertex is the cross section?

(*b*) Find the volume of the part of the pyramid enclosed between the parallel sections in (*a*).

S13. The idea of a *function* is important in everyday thinking. It is used outside of mathematics almost as much as in mathematics, and with the same general meaning.

(*a*) What is the function of a sentence?

(*b*) Explain how a sentence is a function of the words in it. Give an example of a sentence in which a slight change in one word would change the function of the sentence.

(*c*) Write two sentences using the same word with different meanings. Explain how a word is sometimes the inverse function of a sentence.

S14. In mathematics we sometimes express a function by a formula, like $\cos x$. Here we have concentrated a lot of meaning into a symbol.

(*a*) Give some examples of words or phrases in everyday thinking that have a lot of meaning condensed in them and may therefore be compared to formulas.

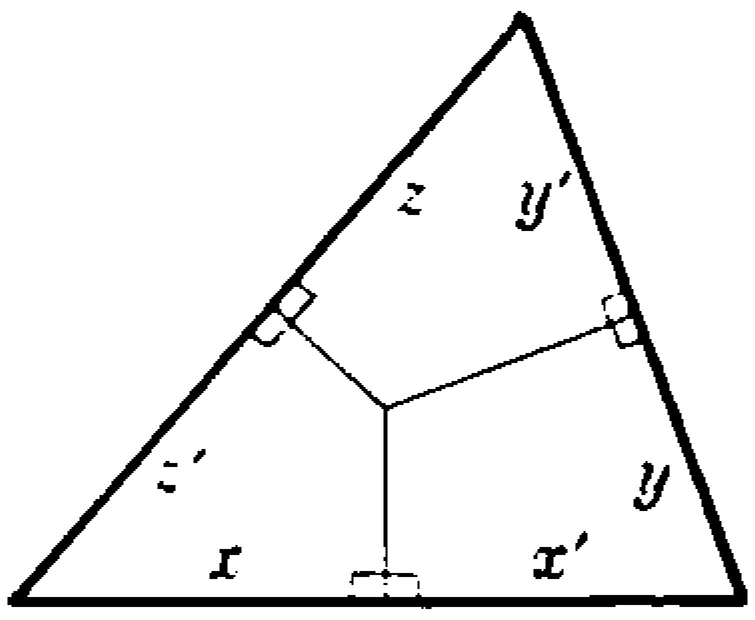
(*b*) Comparing your examples with the mathematical function $\cos x$, is the meaning equally clear?

(*c*) Is there a danger that a formula in nonmathematical thinking that is set aside its one meaning, or set of meanings, may take on different meanings? Can you give some examples?

HONOR WORK

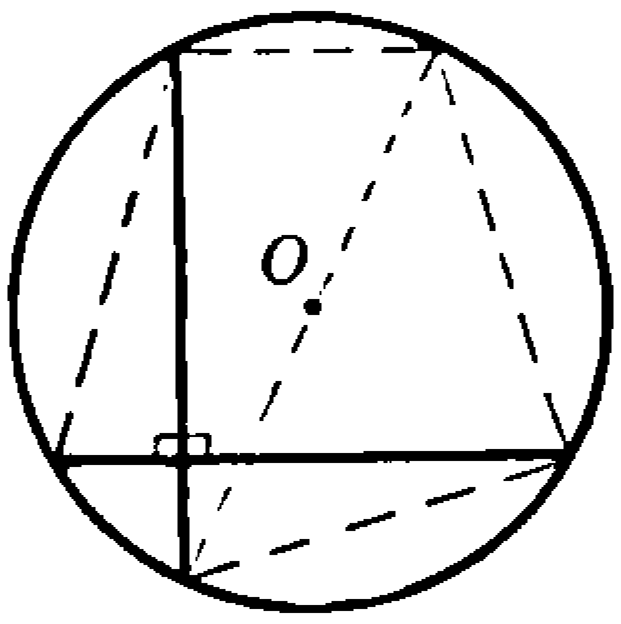
H1. (*a*) In the figure, prove that:

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2$$



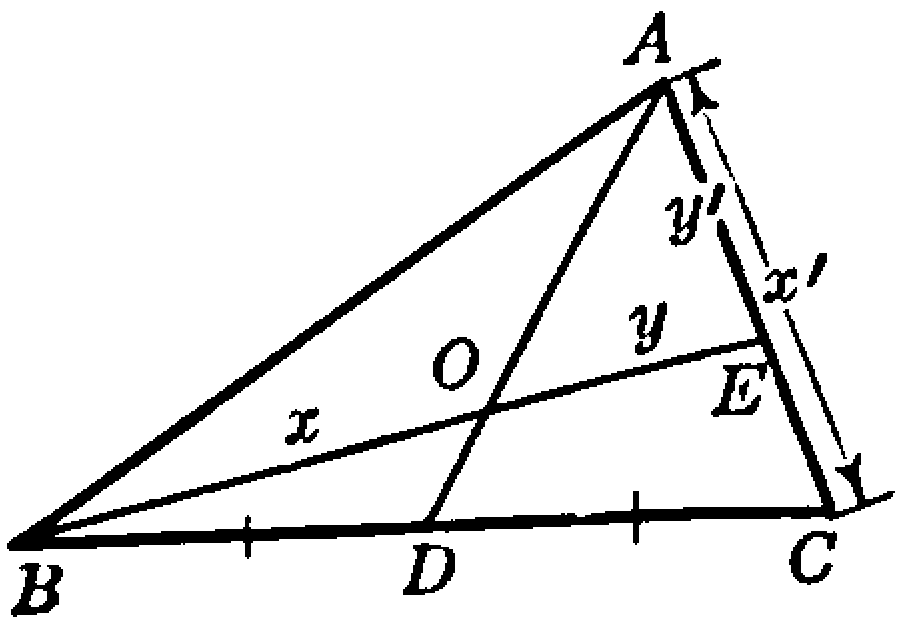
(*b*) State this proposition in words so that it will be analogous to Ceva's Theorem (page 193, S6).

H2. (*a*) Prove: If two intersecting chords of a circle are perpendicular to each other, then the sum of the squares of the four segments of the chords equals the square of the diameter.



(*b*) Explain the three special cases: when one segment is zero; when one chord is bisected; when both chords are bisected.

H3.



Given: AD a median

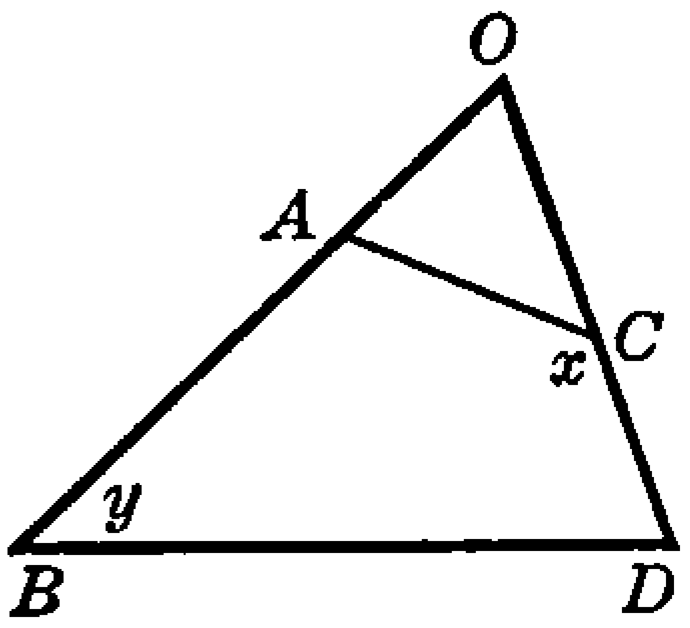
Prove: $\dfrac{x}{y} = \dfrac{x'}{y'}$

Help: Draw CO cutting AB at F, and draw FE. Use Ceva's Theorem (page 193, S6).

TEST

1. (*a*) Find the area of a square whose diagonal is $\sqrt{3}$.
   (*b*) Find the perimeter of a rhombus whose diagonals are 6 and 8 units.
2. A pilot climbs at an angle of 20° with a speed of 80 miles per hour. How high is he after 3 minutes?

   $\cos 20° = .9397, \qquad \cos 70° = .3420$
3. Two parallel roads 6 miles apart are crossed by a third road at an angle of 60°. How far apart are the intersections?
4. The hypotenuse of a 30°–60° right triangle is 12 units. Find the altitude on the hypotenuse.
5. The diagonal of a quadrilateral is 20 inches long. On one side of it, the diagonal forms angles of 30° and 60°; on the other side, it forms two angles 45° each. Find the area of the quadrilateral.
6. Two sides of a triangle that include a 45° angle are 8 and 15 units. Find the area of the triangle.
7. A 25-foot ladder rests against a wall with its foot 9 feet from the wall. If the foot of the ladder is moved two feet closer to the wall, how much higher will the middle rung be?
8. The sides of a triangle are 7, 24, and 25 units. Find the cosine of the angle opposite the side 25.
9.



Given: $OA \times OB = OC \times OD$

Prove: $\angle x + \angle y = 180°$

10. The vertex angle of an isosceles triangle is 30°, and each leg is 10 feet. A line cuts the legs 4 feet and 7 feet respectively from the vertex. Find the area of the quadrilateral that is formed.

10

Group Measures

Means, Mean Proportionals, and Circle Measure

SMALL GROUPS

How a Group Is Measured

330. It is often convenient to have a single measure of a group containing two or more things. For example, if 25 students in your geometry class have taken a test, your teacher may wish to have a single measure of the results. To get this measure he may *add* all the marks and divide the sum by 25. He would then have the *average* of the group. This measure is based on addition, and the technical name for it is the *arithmetic mean.* Most people think that this is the only way of measuring a group. We shall find out, however, that there are other types of measures.

You can obtain another measure of the group of test marks in the following way. Arrange the papers in the order of the marks, from the lowest to the highest, and call the middle paper (the thirteenth) the measure of the group. This measure is called the *median.* If the mark on the thirteenth paper is 78, then 78 would be the median of the group. (If the group contained an even number of papers, the median would be the average of the two middle papers.) The median is a measure that can be found more quickly than the arithmetic mean and, for many purposes, is just as useful or even more useful.

The measurement of groups is a very important branch of mathematics known as *statistics.* It includes such things about a group as income, education, opinion, life expectancy, and many others.

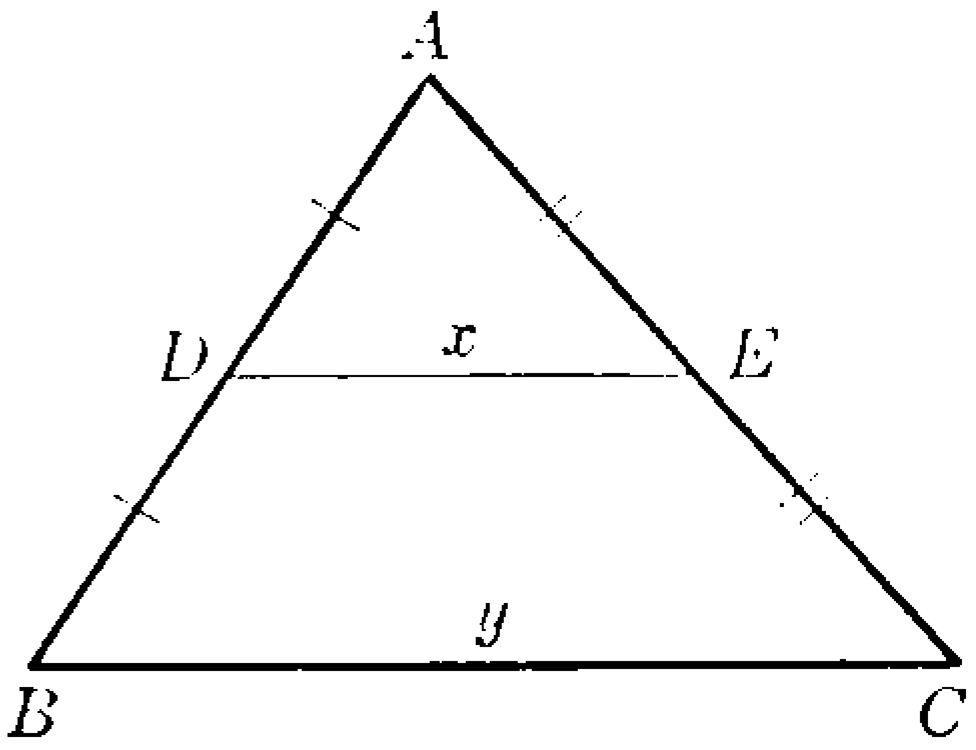
When you read that the median of adult education in the United States is 8.4 years in school, what does it mean? Does it mean that American adults average 8.4 years in school?

The Average

331. First we shall consider groups that contain just two things. The simplest measure of such groups is the average; that is, the arithmetic mean. This measure has already been used to find the number of degrees in an angle whose sides cut certain arcs on a circle. By using the average of the bases of a trapezoid we found the area of the trapezoid. We shall now learn that the average of the bases of a trapezoid is the line segment that connects the midpoints of the legs of the trapezoid.

As you have learned previously (§ 267), a trapezoid becomes a triangle if one base shrinks to a point. Thus, we begin by proving the following theorem.

332. THEOREM. The line segment between the midpoints of two sides of a triangle is parallel to the third side and equal to half of it.

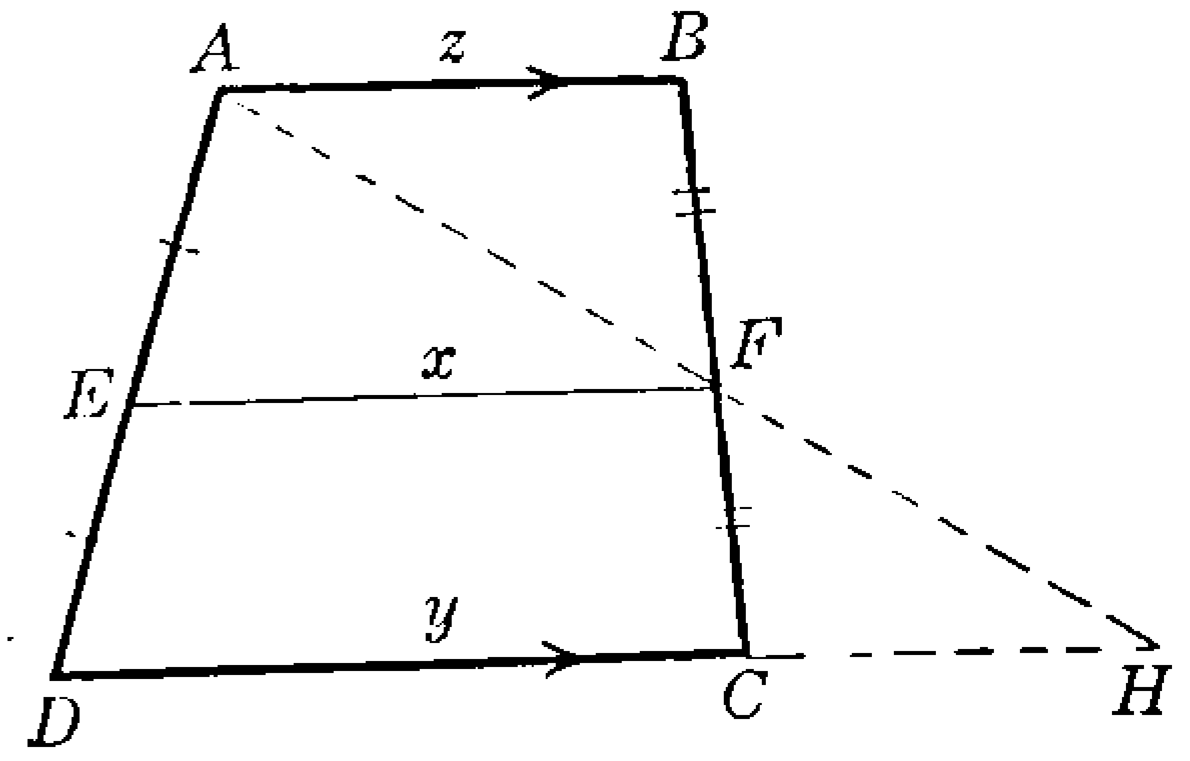


Given: In $\triangle ABC$
$AD = DB$
$AE = EC$
Prove: $DE \parallel BC$
$DE = \frac{1}{2}BC$

PROOF

Statements	*Reasons*
1. $DE \parallel BC$	1. If a line divides 2 sides of a △ proportionally, it is ∥ to third side (§ 326).
2. $\angle ADE = \angle B$ $\angle AED = \angle C$	2. ∥s, corr. ∠s are =.
3. $\triangle ADE \sim \triangle ABC$	3. If 2 △s agree in 2 ∠s, they are ∼.
4. $\frac{DE}{BC} = \frac{AD}{AB}$	4. Def. ∼ polygons.
5. Since $\frac{AD}{AB} = \frac{1}{2}$	5. Given.
6. $\frac{DE}{BC} = \frac{1}{2}$, or $DE = \frac{1}{2}BC$	6. Subs. Ax.

333. THEOREM. The line segment between the midpoints of the legs of a trapezoid is parallel to the bases and equal to their average.



Given: In trap. $ABCD$
$AE = ED$
$BF = FC$
Prove: $EF \parallel DC$
$EF = \frac{1}{2}(AB + DC)$
Cons.: Draw AF meeting DC prolonged at H.

Write out the proof. First prove $\triangle ABF \cong \triangle HCF$.

EXERCISES

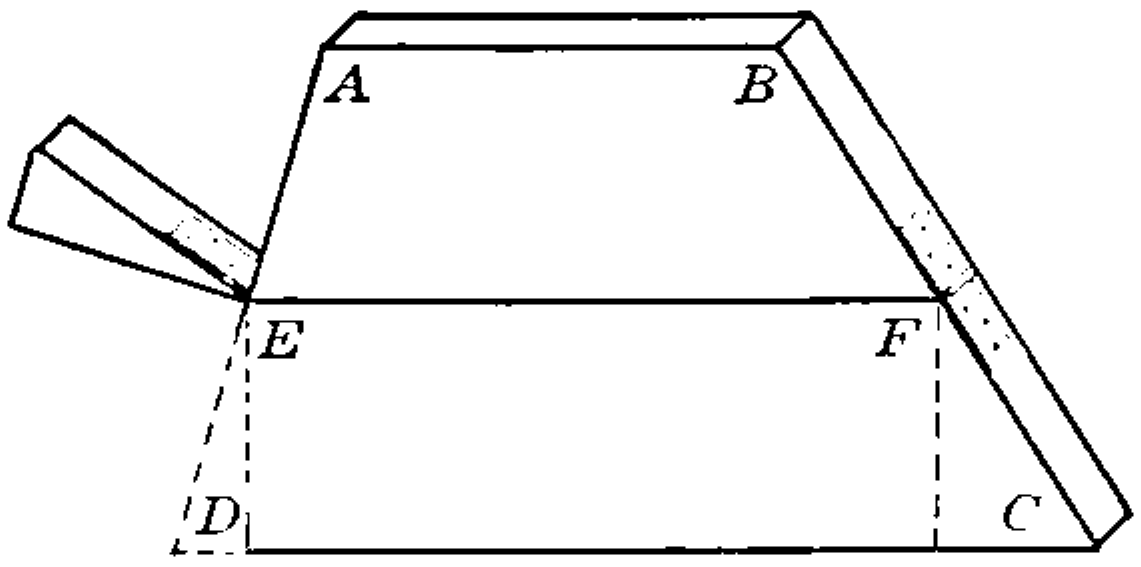
1. Prove: If the midpoints of the sides of a quadrilateral are joined in succession by line segments, then a parallelogram is formed. (*Hint:* Draw a diagonal of the quadrilateral.)
2. Prove: The line segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

3. Prove: If the midpoints of the sides of a rhombus are joined in succession by line segments, a rectangle is formed.

4. Prove: If from the midpoints of two sides of a triangle, line segments are drawn perpendicular to the third side, they are equal.

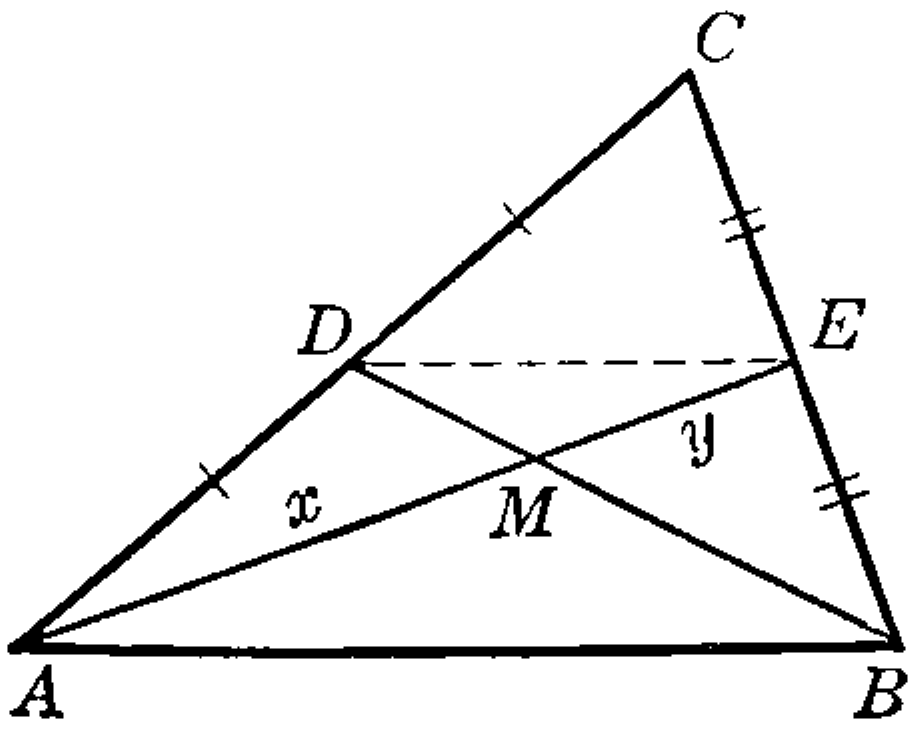
5. Prove: The triangle formed by connecting the midpoints of the sides of a given triangle equals one-fourth the given triangle.

6. (*a*) State in words the proposition that gives the area of a trapezoid in terms of the altitude and the line segment between the midpoints of the legs.
   (*b*) State the corresponding proposition for a triangle.
   (*c*) Construct the model shown in the figure and use it to demonstrate the proposition you formed in (*a*).



Cut out the trapezoid $ABCD$ from a board (or cardboard) and draw the line segment EF between the midpoints of the legs. Then cut off the two right triangles along the dotted lines. Finally, attach two small hinges (or gummed stickers) at E and F.

7. *Theorem.* The part of a median of a triangle between the vertex and the point of intersection with another median is two-thirds of the whole median.

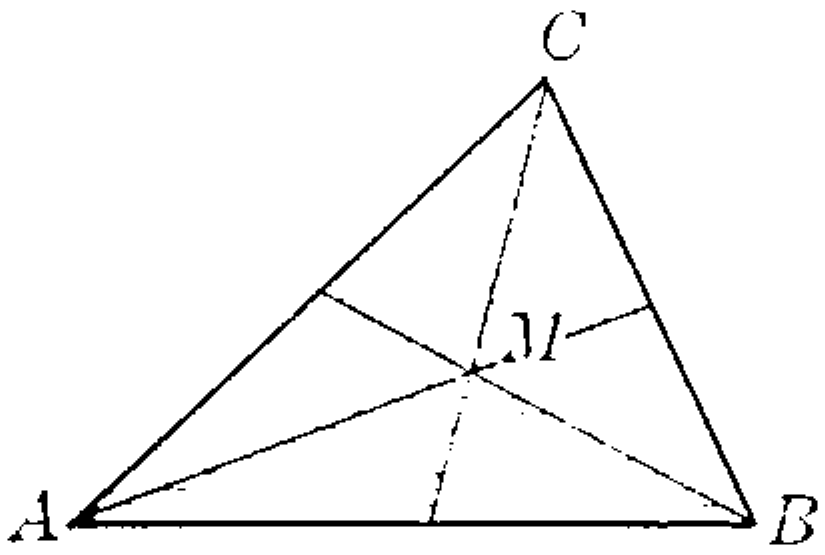


Given:
$CD = DA$
$CE = EB$
Prove:
$x = 2 \cdot y$
Cons.:
Draw DE and prove $\triangle MDE \sim \triangle MAB$.

8. Using the theorem in Ex. 7, prove that the medians of a triangle meet in a point

9. The point where the medians of a triangle meet is called the *centroid* or the center of gravity. In the figure, prove that:

$$\triangle MAB = \triangle MBC = \triangle MAC = \tfrac{1}{3}\triangle ABC$$



10. Prove: If two medians of a triangle are equal, the triangle is isosceles.

11. Prove: The perpendiculars from the ends of a diameter on a chord will meet the chord, prolonged if necessary, at equal distances from the ends of the chord. *Hint:* Draw a radius perpendicular to the chord.

12. (*a*) Write out the next three terms of this series: 4, 9, 14, 19, 24, · · · .
    (*b*) The series of numbers in (*a*) is called an arithmetic series or an arithmetic *progression;* 9 is called the arithmetic mean between 4 and 14.

    Draw a trapezoid whose bases are 21 and 39 units. Divide one of the legs into six equal parts and draw parallels to the bases through the points of division.
    (*c*) Prove that all the parallel segments between the legs (including the bases) are in an arithmetic progression.
    (*d*) Prove that the line segment between the midpoints of the legs of the original trapezoid is the arithmetic mean of all the parallel segments.

Help: The parallel line segments are 21, x, y, 30, z, w, and 39. Solve simultaneously:

$$x = \frac{21 + y}{2}$$

and

$$y = \frac{x + 30}{2}$$

General Definition of a Mean

S334. Thus far we have had two kinds of averages, the arithmetic mean (or *the average*) and the median. Consider the group of five numbers 1, 3, 4, 8, and 9. The arithmetic mean, the mean that is based on addition, is 5. If the only use of this group is to make a sum of 25, then every number of the group could be replaced by 5 and the group would be just as useful, because its sum would still be 25.

The median, or the middle term, of the same group is 4. Here the only use of the series is to determine the middle number 4. Therefore, if every member of the group were replaced by 4, then the series would be just as useful, because the middle term (like all the rest) would still be 4.

The following is a general definition of a mean or an average.

S335. Definition. If a group remains just as useful when every member is replaced by the same quantity, then that quantity is said to be the mean of the group.

If the usefulness of the group consists of forming a sum, then the mean is the arithmetic mean. If the usefulness is in locating the middle term, then the mean is the median.

A Mean Based on Multiplication

S336. Suppose that the usefulness of the same group 1, 3, 4, 8, and 9, consists in forming a product. Now the mean will be different. Let us call the new mean g.

Then $g \cdot g \cdot g \cdot g \cdot g = 1 \cdot 3 \cdot 4 \cdot 8 \cdot 9$, or $g^5 = 864$, and $g = \sqrt[5]{864}$

The most convenient way to approximate this answer is by logarithms. Prove that 3.87 is correct to three significant digits. This is called the *geometric mean.*

S1. (*a*) Find the geometric mean of 3, 4, 6, and 18.

(*b*) Find the geometric mean of 20 and 500.

The Harmonic Mean

S337. The usefulness of a group of numbers may be in the sum of their reciprocals. In that case the mean is called the *harmonic mean.* Let us call h the harmonic mean of the group that we have been using, 1, 3, 4, 8, and 9.

Then

$$\frac{1}{h} + \frac{1}{h} + \frac{1}{h} + \frac{1}{h} + \frac{1}{h} = \frac{1}{1} + \frac{1}{3} + \frac{1}{4} + \frac{1}{8} + \frac{1}{9}$$

$$\frac{5}{h} = \frac{72 + 24 + 18 + 9 + 8}{72} = \frac{131}{72},$$

$$131h = 360,$$

$$h = 2.75$$

S2. (*a*) Find the harmonic mean of 3, 4, and 6.

(*b*) Find the harmonic mean of m and n.

S3. Sound is caused by the vibration of the air. The pitch of a sound depends on the frequency of the vibrations. The frequencies of the notes in a musical scale are in the following ratios: 1.000, 1.125, 1.250, 1.333, 1.500, 1.667, 1.875, 2.000. Show that each of these numbers is approximately the harmonic mean of the number before it and the number after it.

S4. From the intersection of the diagonals of a trapezoid, a line segment x parallel to the bases b and b' ends in one of the legs. Prove that $2x$ is the harmonic mean between the bases b and b'; that is, $\frac{1}{2x} + \frac{1}{2x} = \frac{1}{b} + \frac{1}{b'}$.

From the similar triangles prove that $\frac{x}{b} + \frac{x}{b'} = 1$.

The Geometric Mean

You have seen that the arithmetic mean is based on addition. A mean that is based on multiplication is called a *geometric mean*.

338. Definition. The geometric mean of two numbers is a quantity whose square equals the product of the two numbers.

The geometric mean of 3 and 12 is 6, because $6^2 = 3 \cdot 12$. The geometric mean of 3 and 12 may be written as part of the proportion $\frac{3}{6} = \frac{6}{12}$, where 6 is the common middle term between the end terms 3 and 12 (§ 282). Do you see that 6 is the mean proportional between 3 and 12 (§ 283)?

339. Corollary. The geometric mean of two quantities is the mean proportional between them.

The Segments of Two Intersecting Chords

The geometric mean is used in problems where something is known about the product of two line segments. We therefore prove the following theorem.

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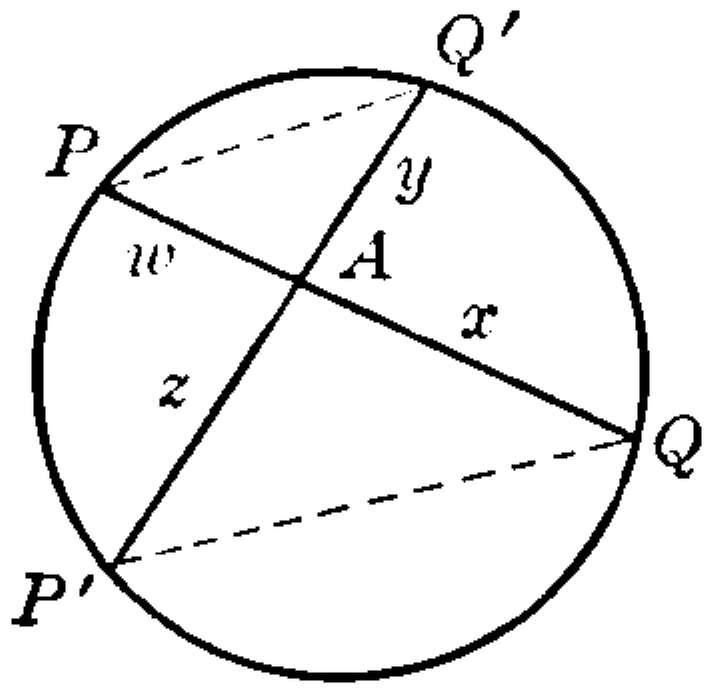
340. THEOREM. If two chords intersect within a circle, the product of the segments of one chord equals the product of the segments of the other chord.



Given: Chords PQ and $P'Q'$ intersect at A

Prove: $wx = yz$

Cons.: Draw PQ' and $P'Q$.

PROOF

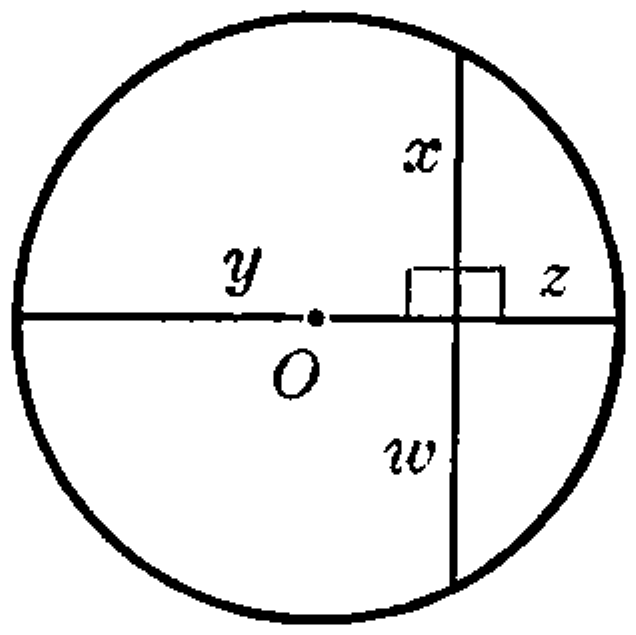
Statements	*Reasons*
1. $\angle P = \angle P'$	1. ∠s inscribed in same arc are = (§ 239).
2. $\angle PAQ' = \angle P'AQ$	2. Vert. ∠s are =.
3. $\triangle APQ' \sim \triangle AP'Q$	3. △s agreeing in 2 ∠s are ~.
4. $\frac{w}{z} = \frac{y}{x}$	4. Def. ~ polygons.
5. $w \cdot x = y \cdot z$	5. Product of means = product of extremes.

---

341. Corollary. If a chord is perpendicular to a diameter, then either segment of the chord is the mean proportional between the segments of the diameter.

From our theorem in § 340, $w \cdot x = y \cdot z$. But in this figure $x = w$ (§ 219). Therefore, $x \cdot x = y \cdot z$, or $x^2 = y \cdot z$.

This corollary gives us a simple method for constructing the mean proportional



between two given line segments. On the sum of the two given line segments as a diameter, we construct a circle (or a semicircle) and then we construct a perpendicular to the diameter.

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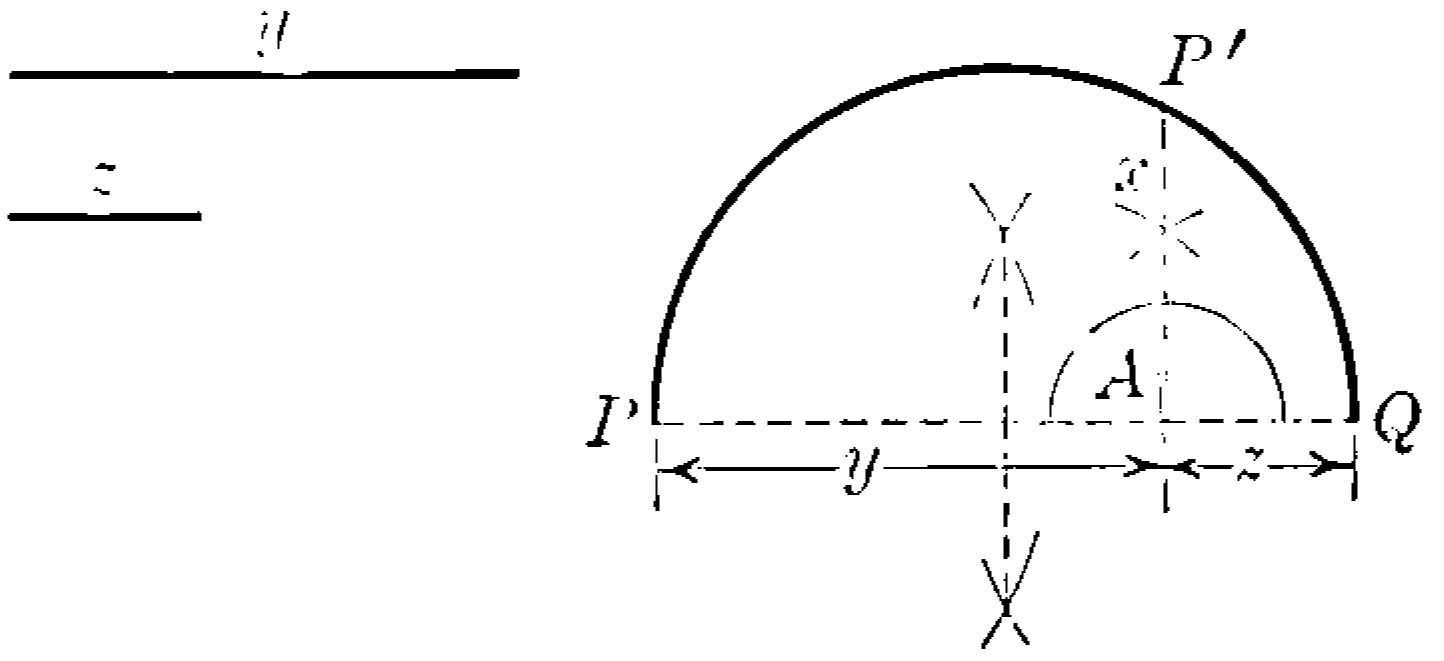
342. PROBLEM. Construct the mean proportional between two given line segments.



Given: Line segments y and z

Req.: To cons. the mean proportional between y and z

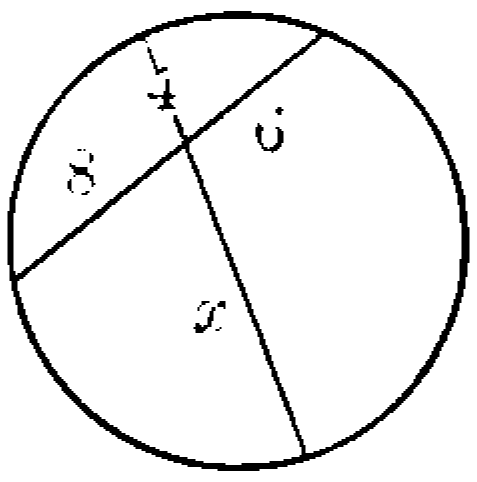
Construction

1. On the same line, lay off $PA = y$ and $AQ = z$.
2. Construct a semicircle on PQ as a diam.
3. At A, the end of segment y, cons. $AP' \perp PQ$.
4. Then $x^2 = yz$.

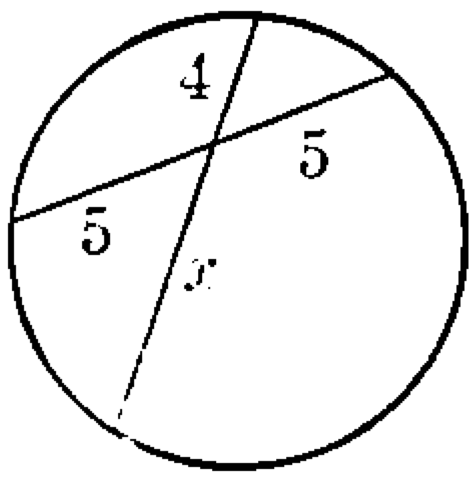
EXERCISES

1–3. In each figure find x.

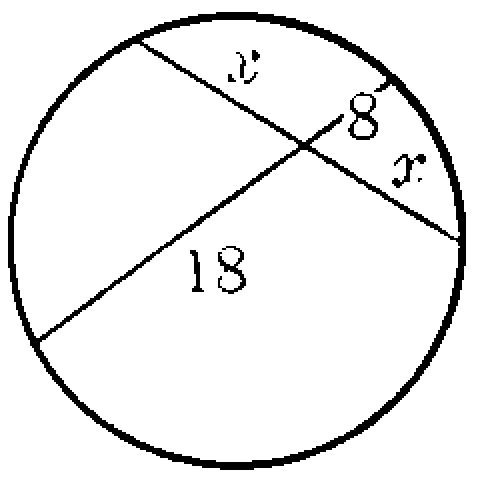
1.



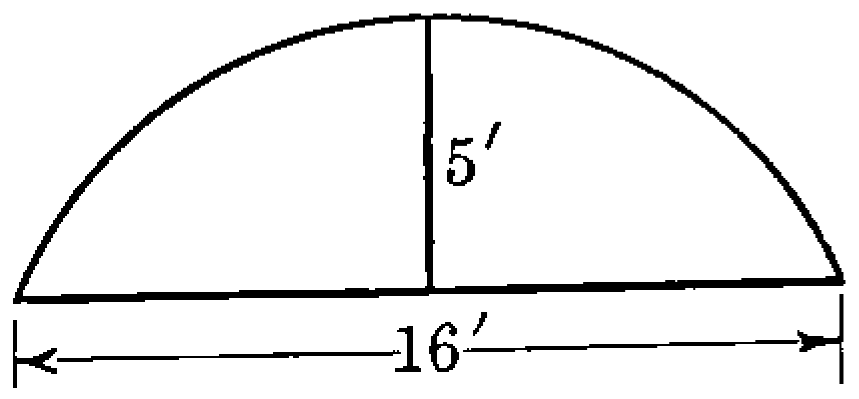
2.



3.



4. A circular arch has a span of 16 feet and is 5 feet high. Find the radius of the arch.



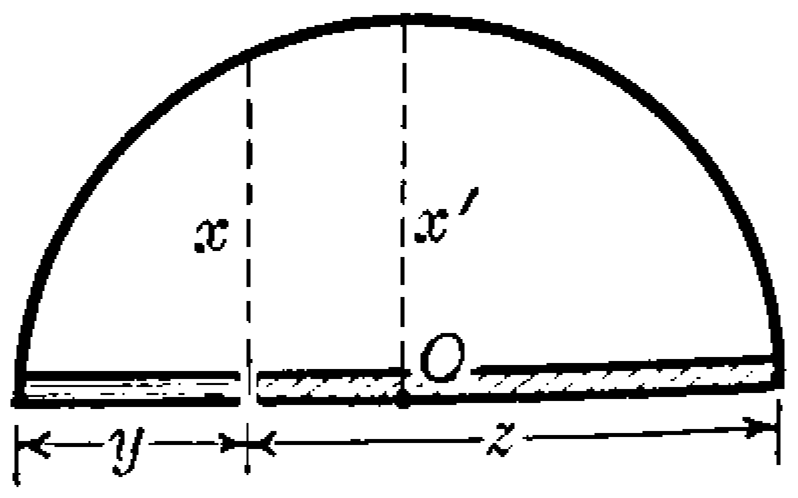
5. Find the span of a circular arch that has a radius of 12.5 feet and is 5 feet high.

6. (*a*) Given a line segment m, construct the mean proportional between m and $2m$.
   (*b*) Using the same line segment m of (*a*), construct a right triangle with the legs m and m. Compare the hypotenuse of this right triangle with the mean proportional as you have constructed it. Explain your result.

7. Construct a square equal to a given rectangle. (First construct the given rectangle.) *Help:* Calling the dimensions of the rectangle y and z, construct the side of the square x so that $x^2 = yz$.

8. Transform a given triangle into a square (§ 275).

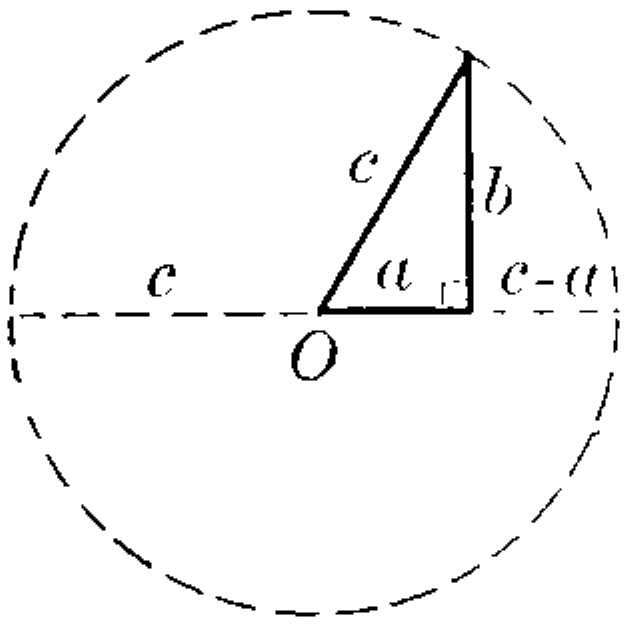
9. Complete the following statements by referring to the figure:
   (*a*) x is the —?— mean of y and z and x' is the —?— mean of y and z.
   (*b*) The geometric mean of two quantities is never —?— their arithmetic mean.
   (*c*) The geometric mean of two equal quantities is —?— their arithmetic mean.



(*d*) The more nearly two quantities are equal, the more nearly their geometric and arithmetic means are —?—.

10. Prove the Pythagorean Theorem by the method suggested in the diagram.

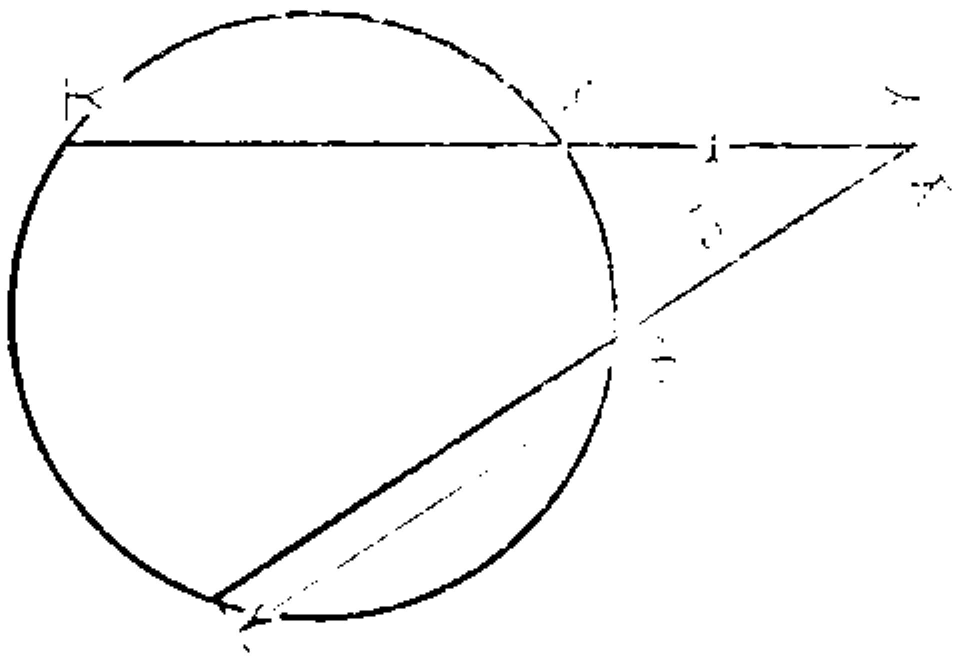
$$b^2 = (c + a)(c - a)$$



11. By first circumscribing a circle about a given right triangle, prove the following proposition:

Theorem. The altitude to the hypotenuse of a right triangle is the mean proportional between the segments of the hypotenuse (page 204, S3).

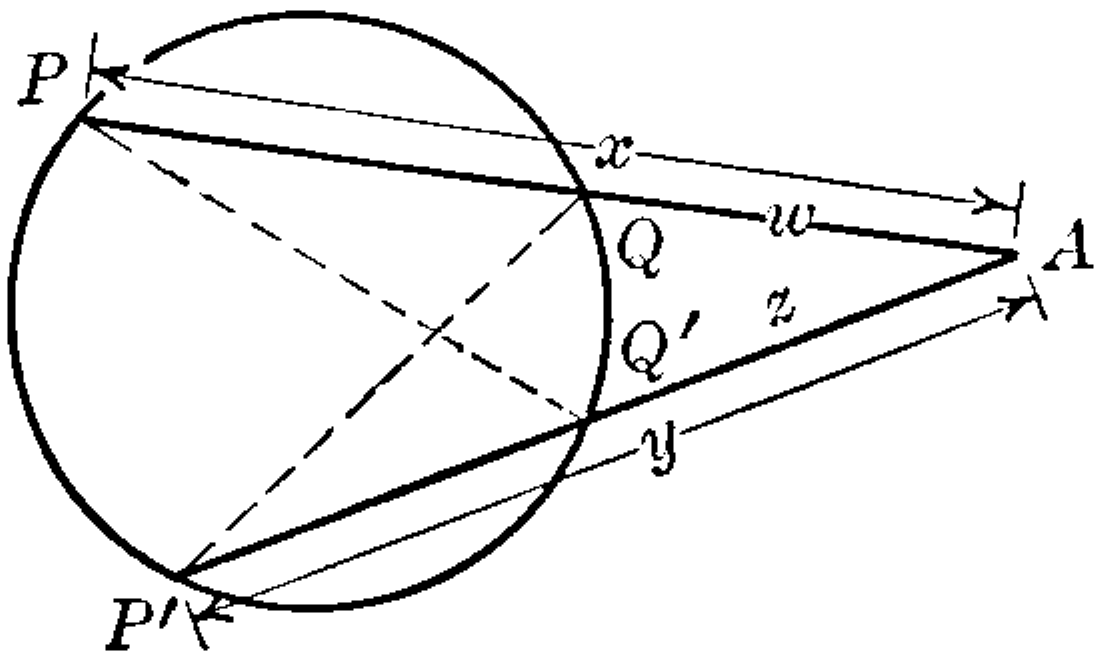
12. Copy the figure; then draw two lines that will form similar triangles. From these, find the value of x.



The Segments of Two Secants

In the figure for the Theorem in § 340, imagine that the two given chords turn about P and P'. Draw a position of this figure in which it is necessary to extend the chords to get their point of intersection A. The segment AP has become a secant and the segment AQ is called the external segment of the secant. The following theorem is therefore an extension of § 340.

343. THEOREM. If from an external point two secants are drawn to a circle, the product of one secant and its external segment equals the product of the other secant and its external segment.



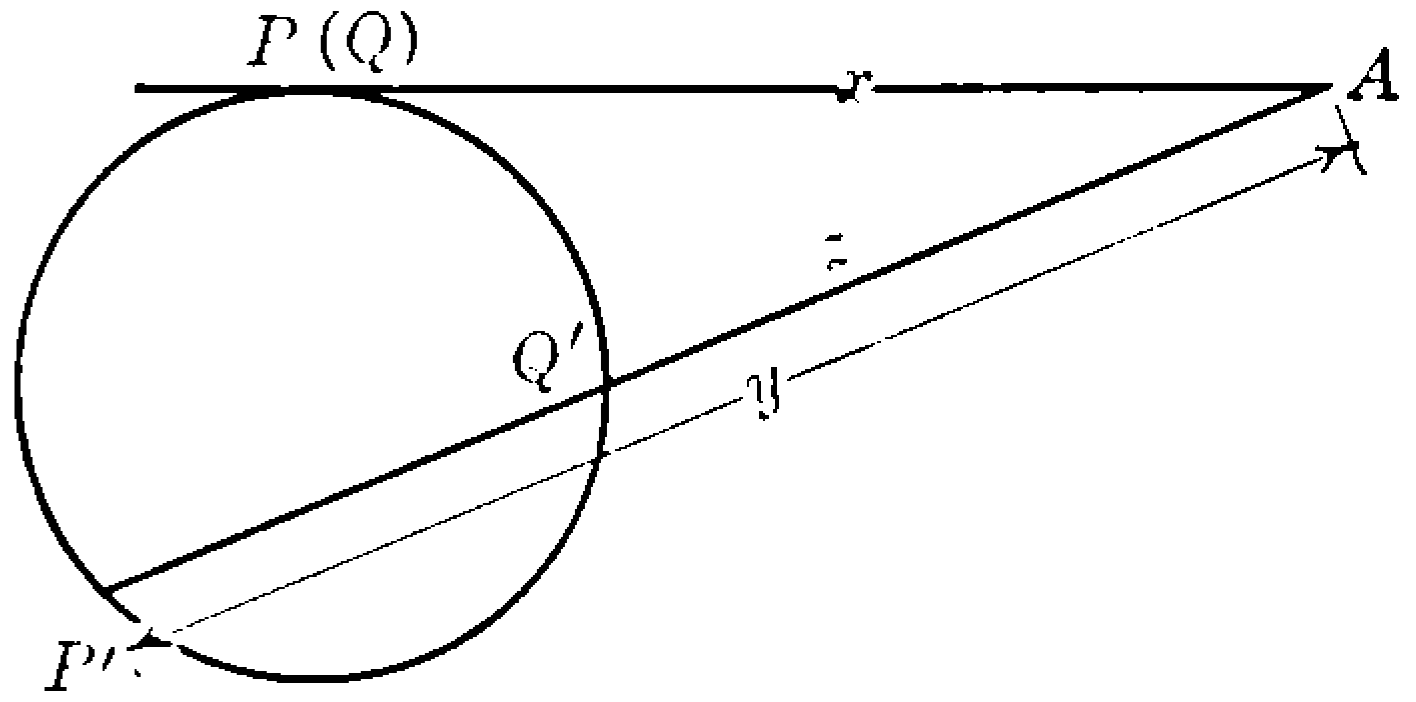
Given: Secants AP and AP'
Prove: $w \cdot x = y \cdot z$
Cons.: Draw PQ' and $P'Q$.

PROOF

Statements	*Reasons*
1. $\angle P = \angle P'$	1. —?—.
2. $\angle A = \angle A$	2. —?—.
3. $\triangle APQ' \sim \triangle AP'Q$	3. —?—.
4. $\frac{w}{z} = \frac{y}{x}$	4. —?—.
5. $wx = yz$	5. —?—.

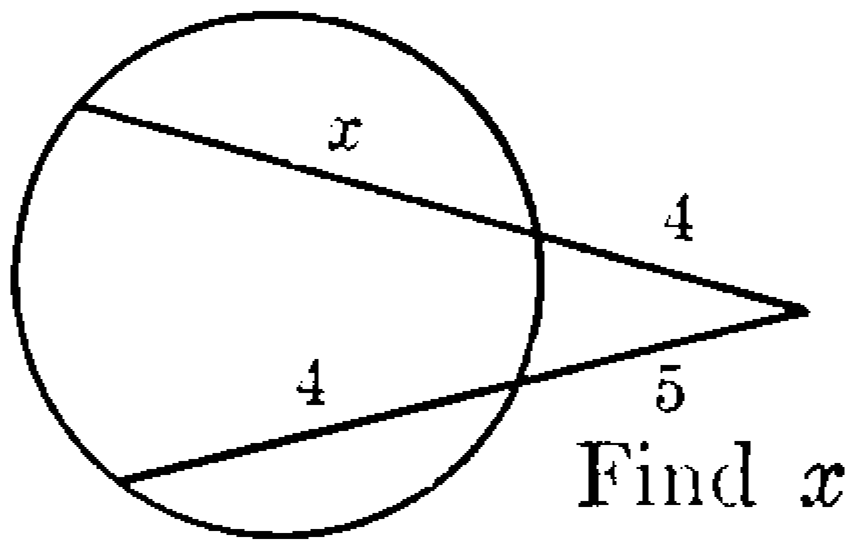
Complete the proof. Then compare it with § 340.

In our theorem, if AQ is a tangent we may think of it as the limiting position of a secant where $w = x$. We therefore substitute x for w in the equation $wx = yz$ and we get $x^2 = yz$.



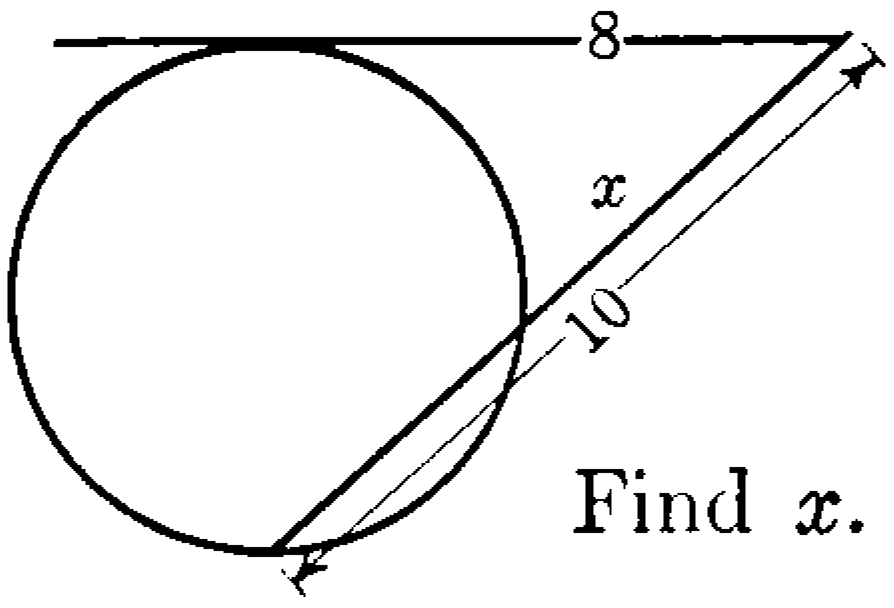
344. Corollary. If from an external point a tangent and a secant are drawn to a circle, the tangent is the mean proportional between the whole secant and its external segment.

13.



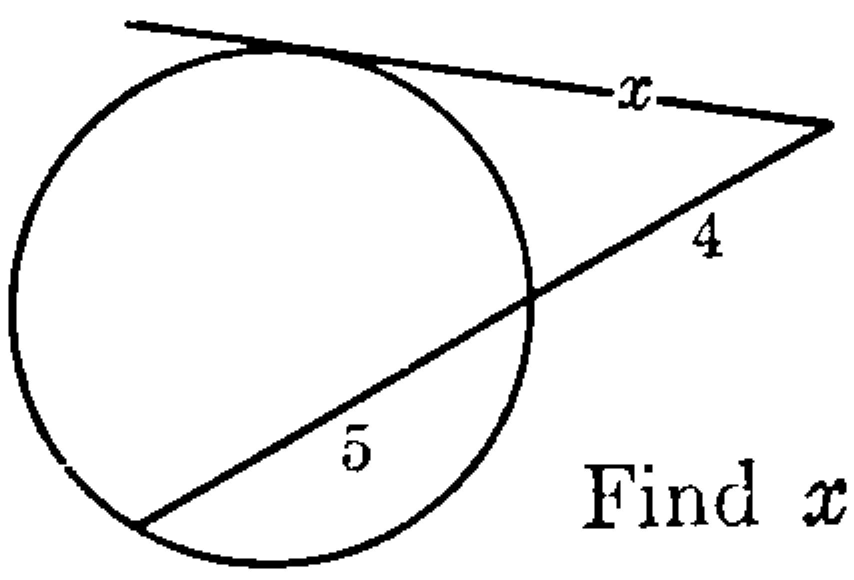
Find x.

14.



Find x.

15.

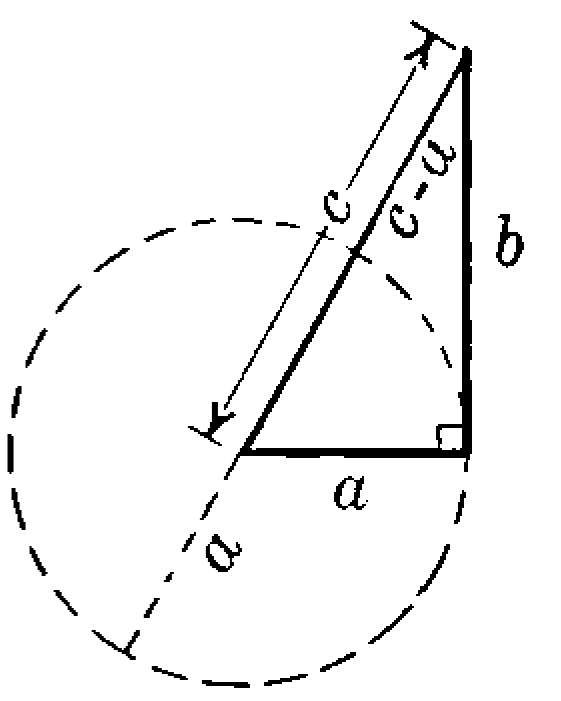


Find x.

16. Prove that the tangents drawn to two intersecting circles from a point in the prolongation of their common chord are equal.

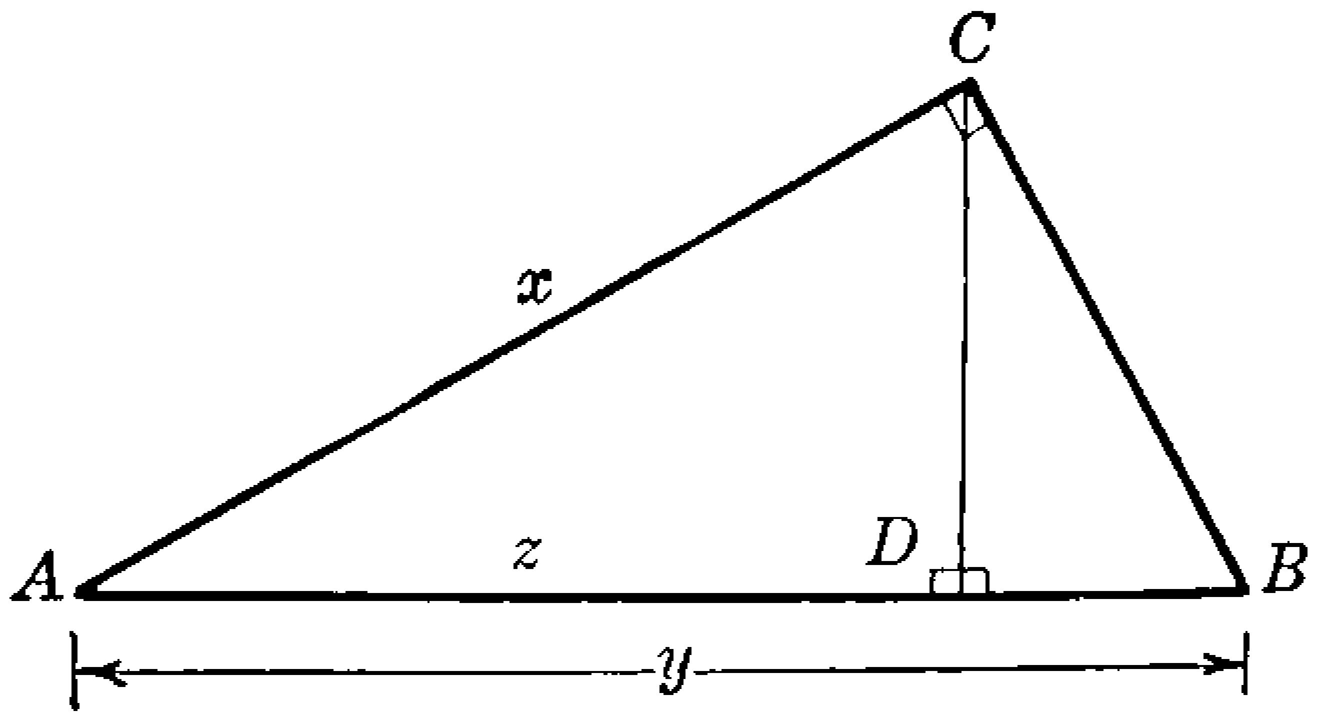
17. Prove the Pythagorean Theorem by the method suggested in the diagram.

$$b^2 = (c + a)(c - a)$$



18. Prove the following proposition:

Theorem. If the altitude is drawn upon the hypotenuse of a right triangle, either leg is the mean proportional between the whole hypotenuse and the adjoining segment of the hypotenuse. (See page 204, S3.)



Circumscribe a circle about $\triangle CBD$ and prove that $x^2 = y \cdot z$.

SUPERIOR WORK

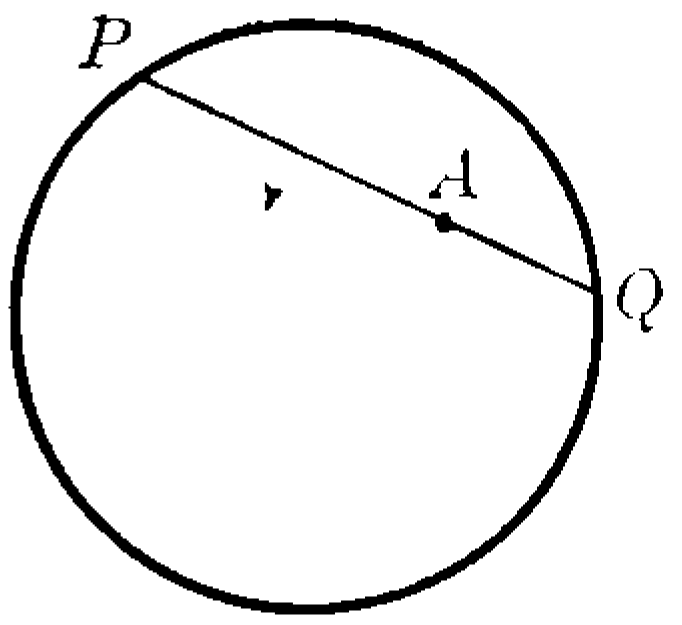
S1. A line MN is tangent to a circle O at C. Two parallel tangents to the circle O cut MN in A and B.

$$\text{Prove: } \frac{CA}{CB} = \left(\frac{OA}{OB}\right)^2$$

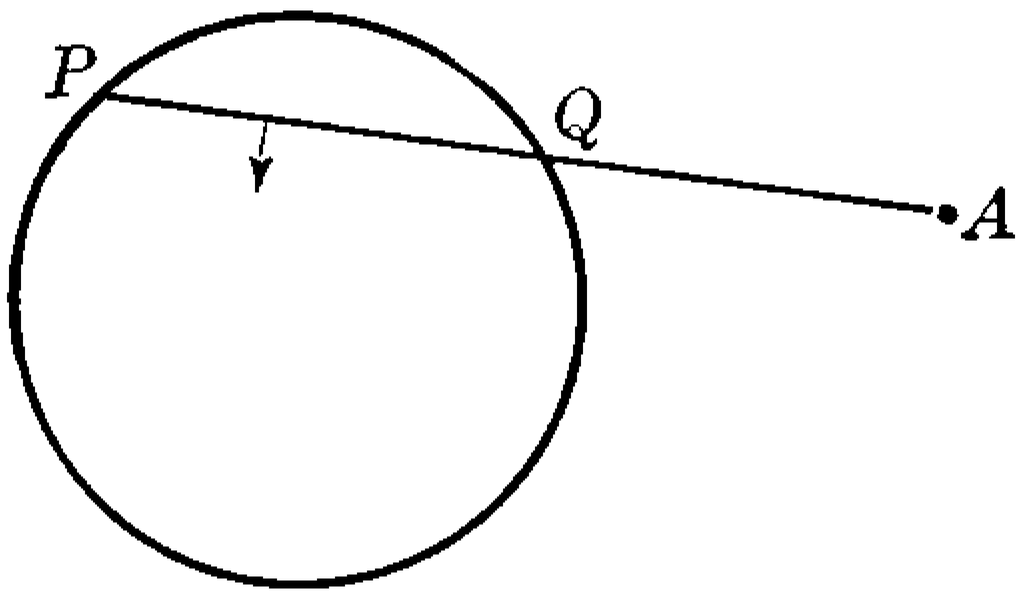
S2. Two circles intersect at M and N. A double chord passing through M cuts the circles again at A and B. From A a line is drawn tangent to circle MNB at C; from B a line is drawn tangent to circle MNA at D. Prove that AB, AC, and BD can be made the sides of a right triangle.

A Rotating Line

S345. In the figure, PQ rotates about the point A. We have proved that the product of the segments AP and AQ will be the same in any two positions: *i.e.*, if a chord rotates about a point, the product of the segments of the chord remains constant.



Similarly in the next figure, if secant AP rotates about the point A to a new position, we have proved that the product of $AP \cdot AQ$ will be the same in both positions. Therefore, if a secant rotates about an external point, then the product of the whole secant from the external point and the external segment remains constant.



In both of the above figures, we may call AP and AQ *collinear segments* from the point A to the circle. We would then have the general dynamic theorem that follows. (See pages 210–211, S1 and S3.)

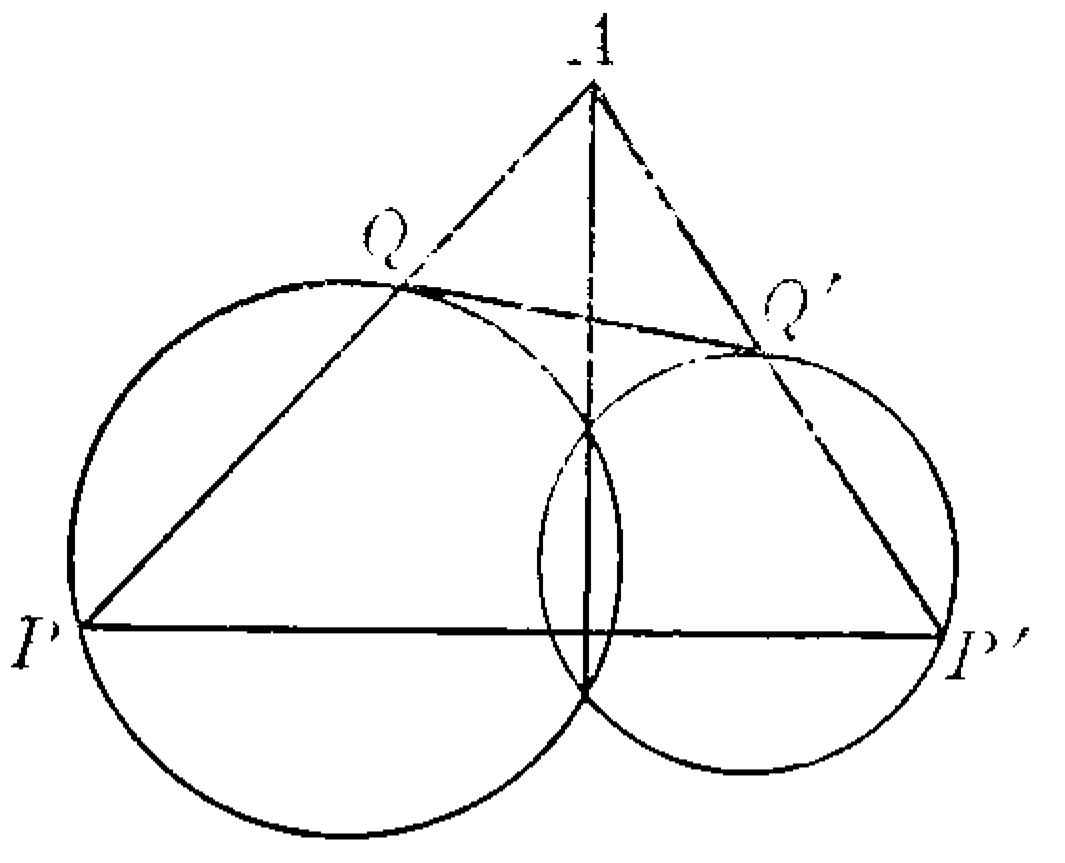
Dynamic Theorem. The product of two collinear segments from a given point to a given circle is constant.

S3. (*a*) Prove the following converse of the preceding Theorem (§ 345):

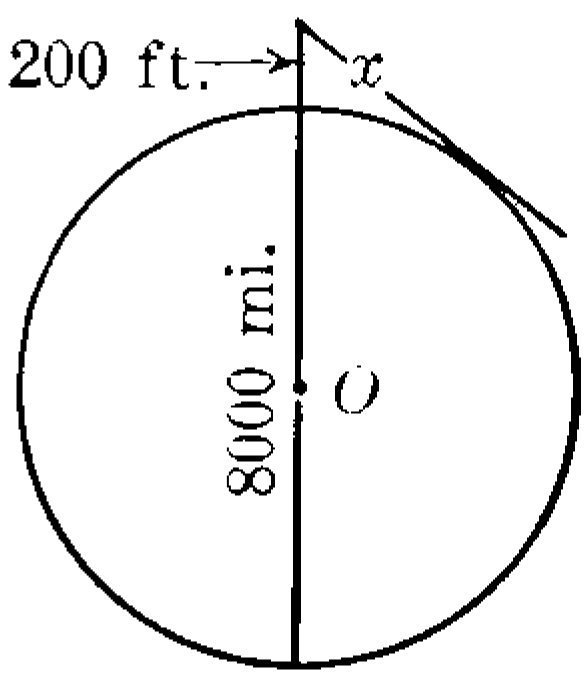
If the product of two collinear segments from a given point is constant, the ends of these segments are on a circle.

Help: Draw two positions of the collinear segments and join their ends to form a quadrilateral. Now use page 156, S4. Prove the two cases.

(*b*) In the figure, prove that quadrilateral $PQQ'P'$ can be inscribed in a circle.



S4. (*a*) Assuming that the diameter of the earth is 8000 miles, how far can you see from the top of a cliff 200 feet high?



(*b*) Prove that the number of miles m that you can see from a height of f feet is given by the formula: $m = 1.23\sqrt{f}$. (If the refraction of light is taken into account the distance that you can see is a little longer.)

S5. The photograph, reproduced as the frontispiece of this book, was taken by an automatic camera from a point 100 miles above the surface of the earth. Assuming that the diameter of the earth is 8000 miles, how long is the tangent to the earth from the point where the picture was taken?

S6. Using the formula in S4, solve the following:

(*a*) How far can you see from the top of the Empire State Building, which is 1250 feet high?

(*b*) How far can *you* see when standing at sea level?

S7. Unlike broadcast waves, which follow the curvature of the earth, television waves travel in straight lines. A television transmitter is located on the top of the Empire State Building. The television waves will strike the horizon before they reach Trenton, 60 miles distant. How high an antenna is needed to receive the signal in Trenton?

S8. Construct a circle that passes through two given points and is tangent to a given line.

Hint: Use § 344 and analyze.

S9. (*a*) Write the next three terms of the series 3, 6, 12, 24, etc. This is called a *geometric progression.*

(*b*) Prove that any term of the geometric progression is the geometric mean of the term before it and the term after it.

(*c*) Figure out how much $100 would amount to at 6% compounded annually at the end of 1, 2, 3, and 4 years. (Compound interest means that at the end of each interest period the interest is added to the principal to form a larger principal; that is, at the end of each successive interest period, interest is paid on the total sum—principal and interest—accumulated to date.)

(*d*) Using your answers to (*c*), prove that money invested at compound interest increases in a geometric progression.

(*e*) Prove that the amount of money that you would have at the end of two years is the geometric mean of the amounts that you would have at the end of one and three years.

(*f*) The population of a country multiplies by 106% during a certain year and 110% during the following year. Prove that a uniform annual growth that would produce the same increase in two years is the geometric mean of 106% and 110%.

S10. You have heard of the "middle man" in business. He is analogous to the mean proportional. Complete the following analogies in which middle terms are used:

(*a*) The —?— is to the *middle man* as the *middle man* is to the —?—.

(*b*) A *pentagon* is to a(n) —?— as a(n) —?— is to a *triangle*.

(*c*) —?— is to *father* as *father* is to —?—.

(*d*) *Dictatorship* is to —?— as —?— is to *anarchy*.

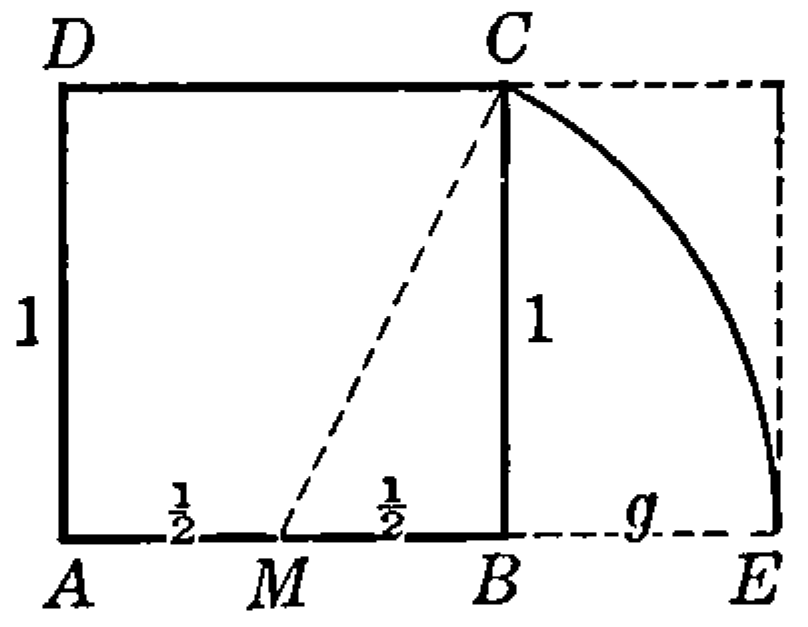
(*e*) An *adverb* is to a(n) —?— as a(n) —?— is to a *noun*.

(*f*) *Co-operation* is to *indifference* as *indifference* is to —?—.

HONOR WORK

The Golden Mean

H346. In the figure, $ABCD$ is a square whose side is 1. From M, the midpoint of AB, lay off $ME = MC$ and complete the rectangle DE. This rectangle is considered by many artists to have very "pleasing proportions."



BE, which we shall call g, is known as the *golden mean* of AB.

From the right $\triangle MBC$,

$$MC^2 = CB^2 + MB^2$$

Since $MC = ME = g + \frac{1}{2}$,

$CB = 1$, and $MB = \frac{1}{2}$,

We substitute these in the above

$$(g + \tfrac{1}{2})^2 = 1^2 + (\tfrac{1}{2})^2$$

$$g^2 + g + \tfrac{1}{4} = 1 + \tfrac{1}{4}$$

Hence, $$g^2 + g = 1 \qquad (1)$$

This equation defines g.

Definition. The *golden mean* (g) is a number that satisfies the equation $g^2 + g = 1$.

H1. By finding the length of MC (in the figure above) to four significant digits, prove that $g = .618$.

H2. Using the relationship $g^2 + g = 1$, prove that $\square DE \sim \square CE$.

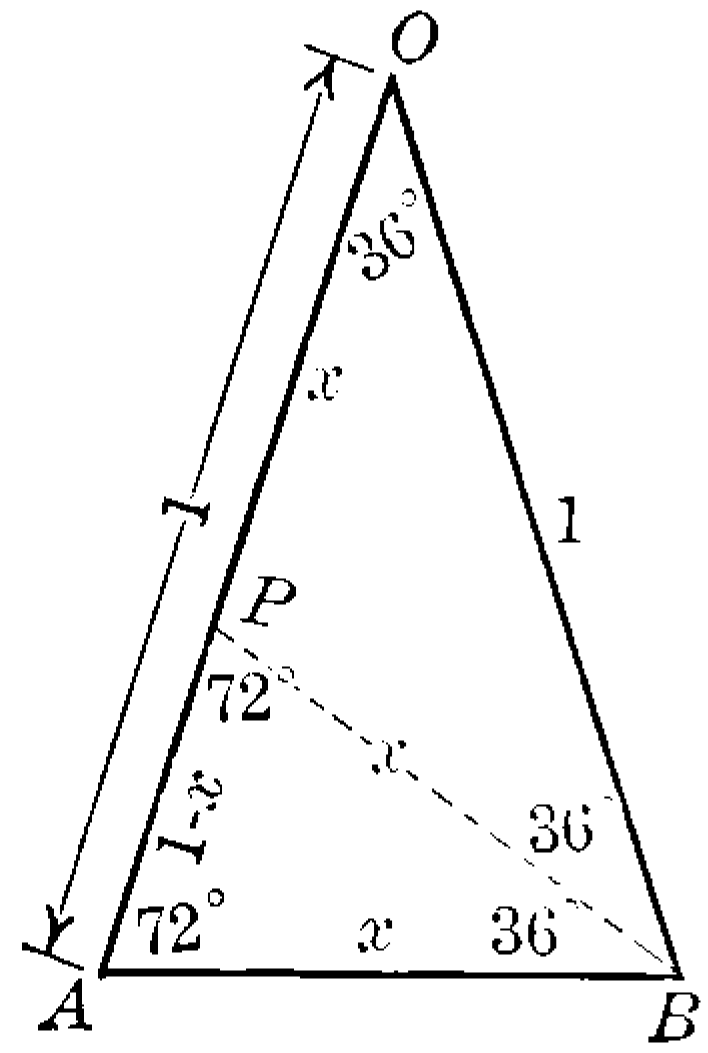
H3. Prove that if a line segment is divided into two parts, of which one is the golden mean, then the golden mean is the mean proportional between the whole line segment and the other part.

The line segment is also said to be divided in *extreme and mean ratio.* If the length of the line segment is m, then the golden mean is mg.

H4. Construct the golden mean of a given line segment AB, using as few construction lines as you can.

H5. (*a*) *Theorem.* If the vertex angle of an isosceles triangle is 36°, the base is the golden mean of either leg.

We shall let each leg of the triangle be one unit.



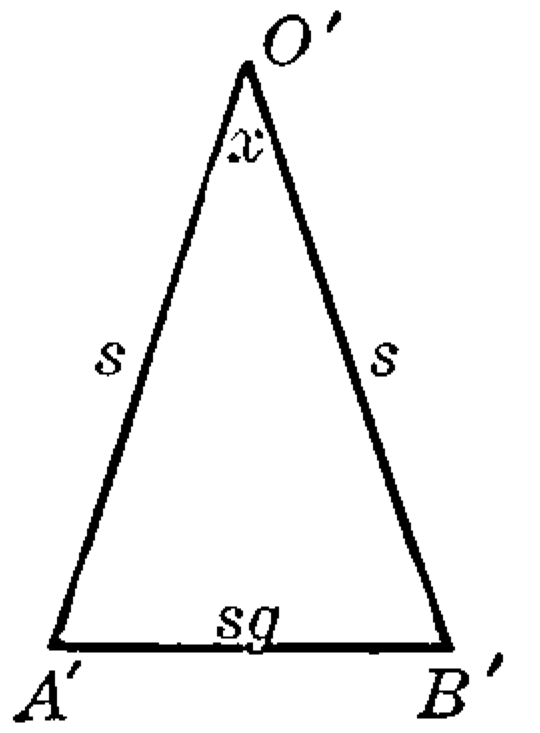
Given:
$\angle O = 36°$
$OA = OB = 1$
Prove:
$AB = g$
Cons.:
Bisect $\angle B$

PROOF

Statements	*Reasons*
1. $\angle A + \angle B = 144°$	1. —?—.
2. $\angle A = \angle B$	2. —?—.
3. $\angle A = \angle B = 72°$	3. —?—.
4. $\angle ABP = \angle PBO = 36°$	4. —?—.
5. $\angle APB = 72°$	5. —?—.
6. Then $AB = PB = PO = x$	6. —?—.
7. $\triangle ABO \sim \triangle ABP$	7. —?—.
8. $\frac{AO}{AB} = \frac{AB}{AP}$, or $\frac{1}{x} = \frac{x}{1-x}$	8. —?—.
9. $x^2 = 1 - x$, or $x^2 + x = 1$	9. —?—.
10. $x = g$	10. —?—.

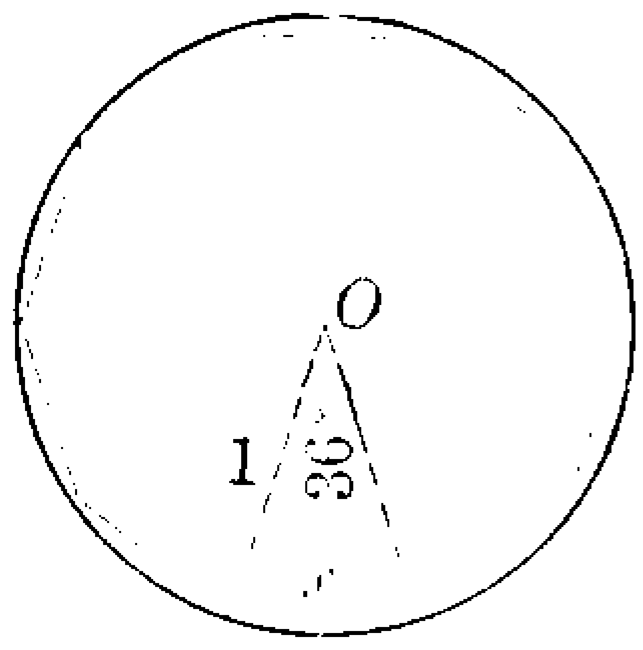
Supply the reasons in the proof.

(*b*) Prove the following converse theorem: If the base of an isosceles triangle is the golden mean of either leg, then the vertex angle is 36°.



Help: Prove that $\triangle O'A'B'$ is similar to $\triangle OAB$ above.

(*c*) Prove that the side of a regular decagon inscribed in a circle is the golden mean of the radius (§ 347).



(*d*) *Construction Problem:* Inscribe a regular decagon in a given circle.

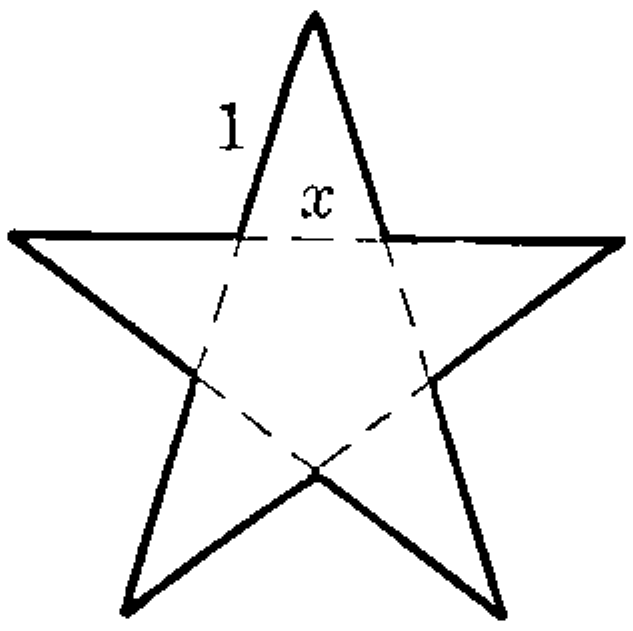
(*e*) *Construction Problem:* Inscribe a regular pentagon in a given circle.

(*f*) *Construction Problem:* Inscribe a regular 15-sided polygon in a given circle.

Help: Each central angle $= \frac{360°}{15} = 24° = 60° - 36°$.

H6. The golden mean is found in many works of art and in nature.

(*a*) In the figure, the five-pointed star is formed by prolonging the sides of a regular pentagon. Prove that the side of the original pentagon is the golden mean of the side of the star. *Hint:* Assume that the side of the star is 1 and prove that $x = g$.



(*b*) The series 1, 2, 3, 5, 8, 13, 21, 34, etc., is known as the *summation series* because each term after the first two is the sum of the two terms before it. Show that the higher up you go in this series, the more nearly is the ratio of two successive terms equal to the golden ratio g (.618).

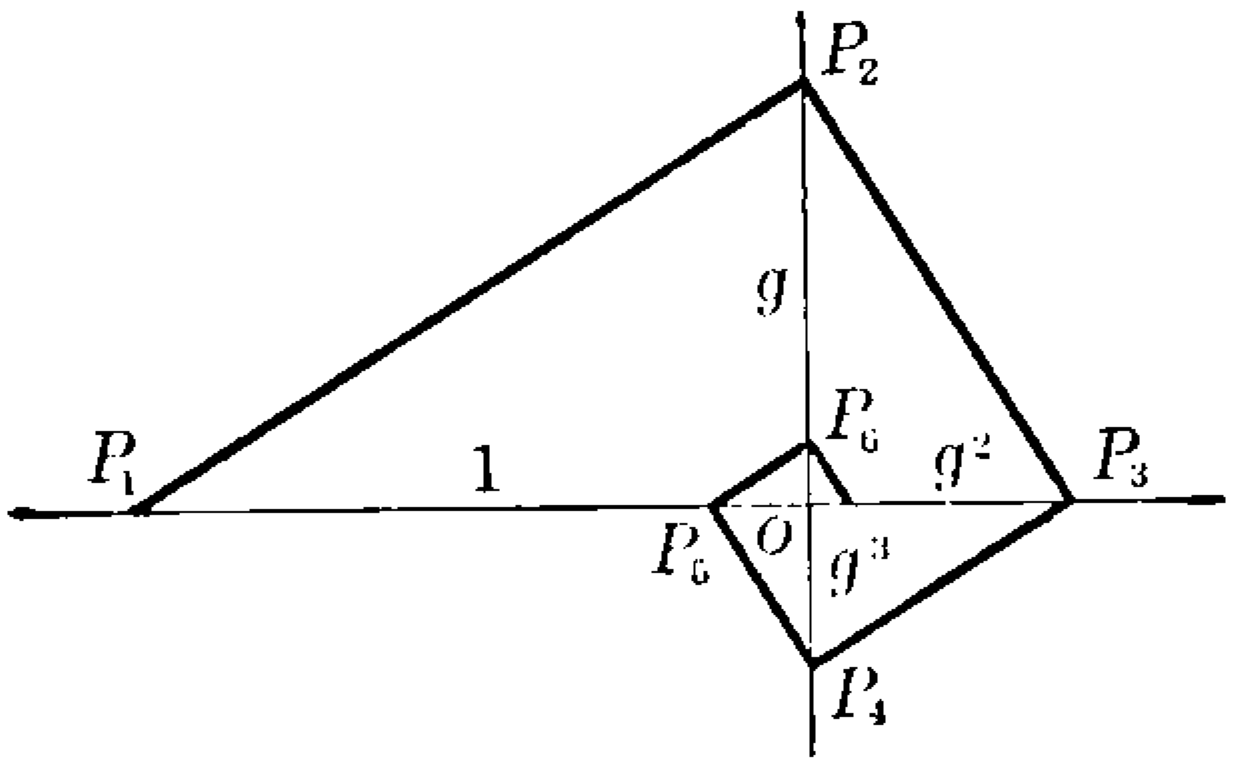
Students of nature have found that growth often follows this series. The number of seeds in the concentric circles of a sunflower, for example, approximates the summation series.

(*c*) Measure the length and width of your desk. How close to g is their ratio?

(*d*) Look up the distances of the earth and the planet Mars from the sun. How does their ratio compare with g?

H7. In the figure, P_1P_3 and P_2P_4 are perpendicular axes.

$OP_1 = 1$, $OP_2 = g$, $OP_3 = g^2$, $OP_4 = g^3$, etc.



(*a*) Prove: $P_1P_2 = \sqrt{1 + g^2}$

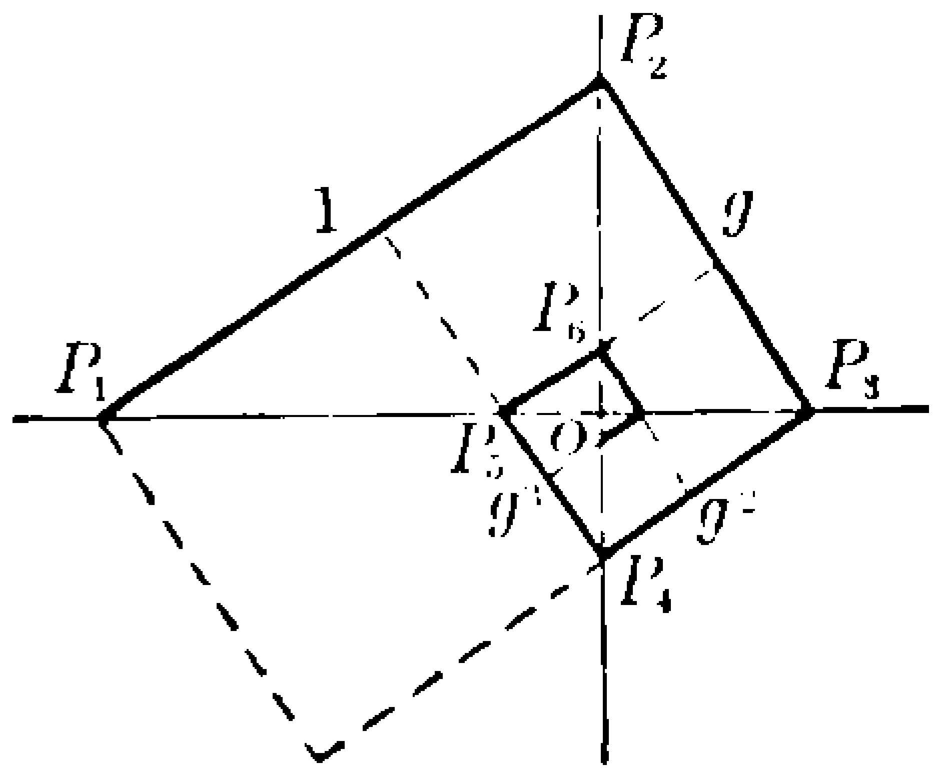
$$P_2P_3 = g\sqrt{1 + g^2}$$

(*b*) Prove: $\dfrac{P_2P_3}{P_1P_2} = g$; $\dfrac{P_3P_4}{P_2P_3} = g$; etc.

(*c*) Prove that $\angle P_1P_2P_3 = 90°$ by showing that the sides of $\triangle P_1P_2P_3$ obey the Pythagorean relationship.

Also prove that $\angle P_2P_3P_4 = 90°$, etc. The figure is called a *rectangular spiral*.

H8. The figure below is a rectangular spiral with the sides prolonged to form rectangles. The diagonals of these rectangles are along the axes of the spiral.



Let $P_1P_2 = 1$
then from H7 (*b*), $P_2P_3 = g$
$P_3P_4 = g^2$, etc.

(*a*) Prove $\square P_1P_3 \sim \square P_2P_4$,
$\square P_2P_4 \sim \square P_3P_5$, etc.

(*b*) Prove that P_1P_4 is a square; also that P_2P_5 is a square, etc.

(*c*) The above figure is sometimes called the "whirling squares." Find the ratio of the areas of two successive whirling squares.

LARGE AND INFINITE GROUPS

Regular Polygons

Up to the present we have worked mainly with groups of only two things. We shall now study groups having a larger number of members. One such group consists of the sides of a polygon.

347. Definition. A regular polygon is a polygon that is equilateral and equiangular.

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348. THEOREM. If a circle is divided into n equal arcs, the chords of these arcs will form a regular polygon of n sides.

Draw a figure and write out the proof.

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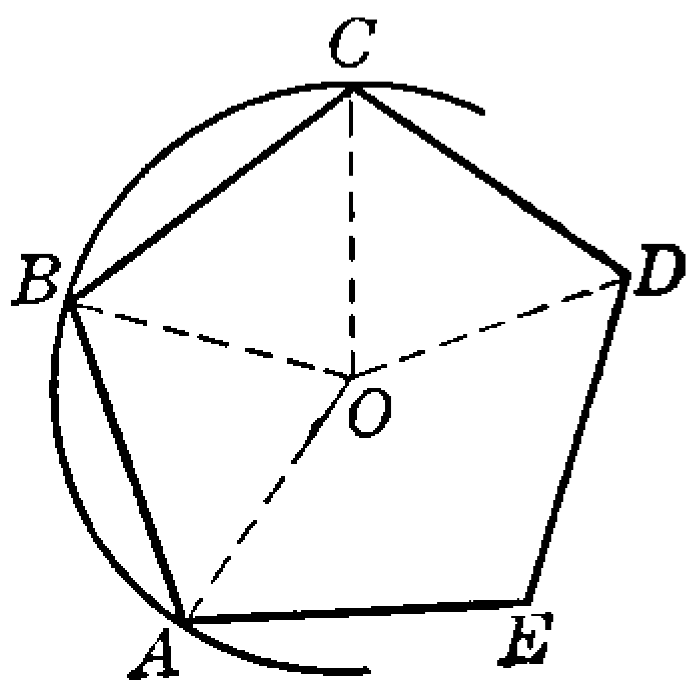
349. CONVERSE THEOREM. A circle can be circumscribed about any regular polygon.



Given: Regular polygon $ABCD \cdots$

Prove: A circle can be circumscribed about polygon $ABCD \cdots$

Let O be the center of the circle passing through A, B, and C. (A circle can be drawn through three points not in a straight line.) Draw OA, OB, OC, and OD.

By proving $\triangle OAB \cong \triangle OCD$, show that the circle passes through D.

---

350. Definitions.

The center of a regular polygon is the center of the circle circumscribed about the polygon.

A radius of a regular polygon is a line segment from the center to a vertex of the polygon.

A central angle of a regular polygon is the angle formed by two radii of the polygon.

An apothem of a regular polygon is a perpendicular from the center of the polygon to one of the sides.

351. Corollary. All the radii of a regular polygon are equal.

352. Corollary. All the apothems of a regular polygon are equal (§221).

EXERCISES

1. Prove that the lines connecting the ends of two perpendicular diameters form a regular quadrilateral.
2. Inscribe a square in a given circle.
3. Inscribe a regular octagon in a given circle.
4. Prove that the side of a regular hexagon equals the radius of the hexagon.
5. Inscribe a regular hexagon in a given circle.
6. Inscribe a regular dodecagon (12-sided polygon) in a given circle.
7. Inscribe an equilateral triangle in a given circle.
8. Prove: If two regular polygons have the same number of sides, they are similar.
9. The perimeters of two regular polygons of the same number of sides are 24 and 36 units. If the radius of the smaller polygon is 3 units, find the radius of the larger polygon.
10. The apothems of two regular polygons of the same number of sides are 5 and 8 units. If the perimeter of the smaller polygon is 40 units, find the perimeter of the larger polygon.

INFINITE GROUPS

353. Thus far the groups we have studied had a *finite* number of terms. Let us now consider a group like

$$.3,\ .33,\ .333,\ .3333,\ \cdots$$

The number of terms in this group is infinite, so we cannot write them all. But we know exactly which numbers belong in this group and which numbers do not belong in it. We know this because the terms follow a definite scheme. Such a group is called a *series*.

How can we find a measure of this group? Since the number of terms is infinite, we cannot add them all. Yet the fraction $\frac{1}{3}$ describes the group better than any other number. Thus, $\frac{1}{3}$ is closer to all the terms than any other number. In fact, there are members of this group as close to $\frac{1}{3}$ as you please. If you wish to find any number of these decimals that are within one-billionth of $\frac{1}{3}$, for example, you need only write down decimals that have enough 3's.

We may therefore call $\frac{1}{3}$ a measure of this group. To define this new measure, we think of the members of the group dynamically as a variable that approaches the measure as nearly as you wish without actually reaching it. This measure is called the *limit* of the series.

354. **Definition.** **A variable is said to approach a constant as a limit if it can be made to differ from the constant by a quantity as small as you please except zero.**

The Circumference of a Circle

From our definition of a variable approaching a limit, we have the following corollary.

355. **Corollary.** **If the number of sides of an inscribed regular polygon is repeatedly doubled, the limit of the polygon is the circle in which it is inscribed and the limit of the apothem is the radius.**

In our study of the circle up to this point, we have defined the angular measure of an arc but not its length. The length of an arc means the number of *linear* units that the arc contains. Now, a linear unit is part of a straight line but an arc is part of a circle. A circle and a line cannot be compared directly.

To define the length of a circle we must proceed indirectly. We inscribe a regular polygon in the circle and continually double the number of sides. We now define the length of the circle, called its circumference, as the limit of the perimeter of the inscribed polygon.

356. **Definition.** **The length of a circle, called its circumference, is the limit of the perimeter of a regular inscribed polygon as the number of sides is continually doubled.**

By definition, the length of a circle is the limit of the perimeter of a regular polygon inscribed in the circle, as the number of sides is repeatedly doubled. The border around this paragraph is a regular polygon of 40 sides. It differs little from a circle drawn around it. The difference can be made still smaller by doubling the number of sides of the polygon.

The Regular Polygon and the Circle

We have seen that when taking up a new topic we must usually make certain assumptions. If a statement about a regular polygon remains true no matter how large the number of sides becomes, then it seems reasonable to assume that it will remain true when the number of sides becomes infinite and the polygon approaches the circumscribed circle.

357. **Postulate.** **Any proposition about a regular polygon that does not depend on the number of sides is equally valid for the circumscribed circle.**

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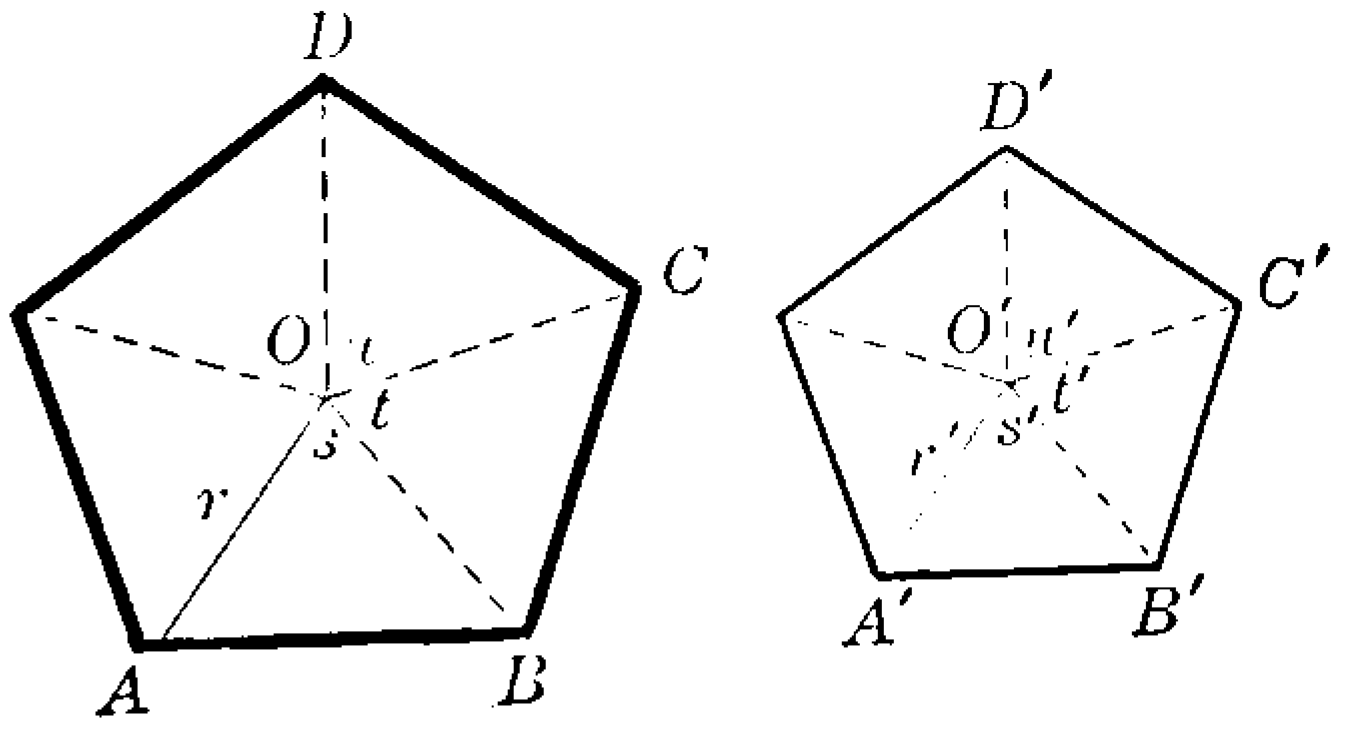
358. **THEOREM.** **The perimeters of two regular polygons of the same number of sides are proportional to the radii of the polygons.**



Given: Regular polygons $ABCD \cdots$ and $A'B'C'D' \cdots$ having n sides
p and p' the perimeters
r and r' the radii

Prove: $\frac{p}{p'} = \frac{r}{r'}$

Cons.: Draw all the radii.

PROOF

Statements	*Reasons*
1. $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong \cdots$	1. s.s.s. (Radii of a regular polygon are =.)
2. $\angle s = \angle t = \angle u = \cdots$	2. Def. ~ ⚠.
3. $\angle s + \angle t + \angle u + \cdots = 360°$	3. Sum of all ∠ about pt. = 360° (§ 136).
4. $\angle s = \dfrac{360°}{n}$ Similarly, $\angle s' = \dfrac{360°}{n}$	4. Div. Ax.
5. $\angle s = \angle s'$	5. Subs. Ax.
6. $OA = OB$ $O'A' = O'B'$	6. Radii of a regular polygon are =.
7. $\dfrac{OA}{O'A'} = \dfrac{OB}{O'B'}$	7. Div. Ax.
8. $\triangle OAB \sim \triangle O'A'B'$	8. If 2 ⚠ agree in one angle and have including sides proportional, they are ~.
9. $\dfrac{AB}{A'B'} = \dfrac{r}{r'}$ or $\dfrac{n \cdot AB}{n \cdot A'B'} = \dfrac{r}{r'}$	9. Def. ~ polygons.
10. $\dfrac{p}{p'} = \dfrac{r}{r'}$	10. Subs. Ax.

Using the postulate that any proposition about a regular polygon not dependent on the number of sides can be applied to a circle (§ 357), we have:

359. Corollary 1. The circumferences of any two circles are proportional to their radii.

This corollary means that $\frac{C}{C'} = \frac{r}{r'}$. By alternation $\frac{C}{r} = \frac{C'}{r'}$. Let us now think of a growing circle. For any two positions of this changing circle, the ratios of the circumferences to the radii are equal. In other words, as the circle changes, the circumference divided by the radius, $\frac{C}{r}$, is constant. The circumference divided by the diameter, $\frac{C}{d}$, is also constant. The constant $\frac{C}{d}$ is of special importance in many problems about the circle. It is denoted by the Greek letter π (pronounced like *pie*). This symbol is the initial letter of the Greek word for *perimeter*. The value of π cannot be expressed in numbers exactly. By higher mathematics it can be approximated as accurately as one wishes. Its value has been calculated correct to 707 places. Correct to five significant digits, $\pi = 3.1416$.

360. Corollary 2. The circumference of a circle equals π times the diameter. $C = \pi d$.

EXERCISES

Illustrative Example. A boy observes that a merry-go-round makes a complete revolution in 5 seconds, and he estimates its diameter as 50 feet. How fast in miles per hour does a rider near the outer edge of the merry-go-round travel?

From the nature of the measurements given, it appears that we shall not need more than three digit places of the approximation of π. We therefore round off the approximation of π and set $\pi = 3.14$. π should generally be carried to one more digit place than the least accurate of the other measurements (§ 277).

Circumference of merry-go-round $= \pi d = 3.14 \times 50$ ft. $= 157$ ft.

Speed of outer rim

$$= 157 \text{ ft.} \div 5 \text{ sec.}$$

$$= \frac{157}{5280} \text{ mi.} \div \frac{5}{60 \times 60} \text{ hours}$$

$$= \frac{157}{5280} \times \frac{60 \times 60}{5} \text{ mi/hr}$$

$$= \frac{157}{\underset{\underset{22}{\cancel{88}}}{\cancel{5280}}} \cdot \frac{\overset{1}{\cancel{60}} \times \overset{\overset{3}{\cancel{12}}}{\cancel{60}}}{\underset{1}{\cancel{5}}} = \frac{471}{22} = 21 \text{ mi/hr}$$

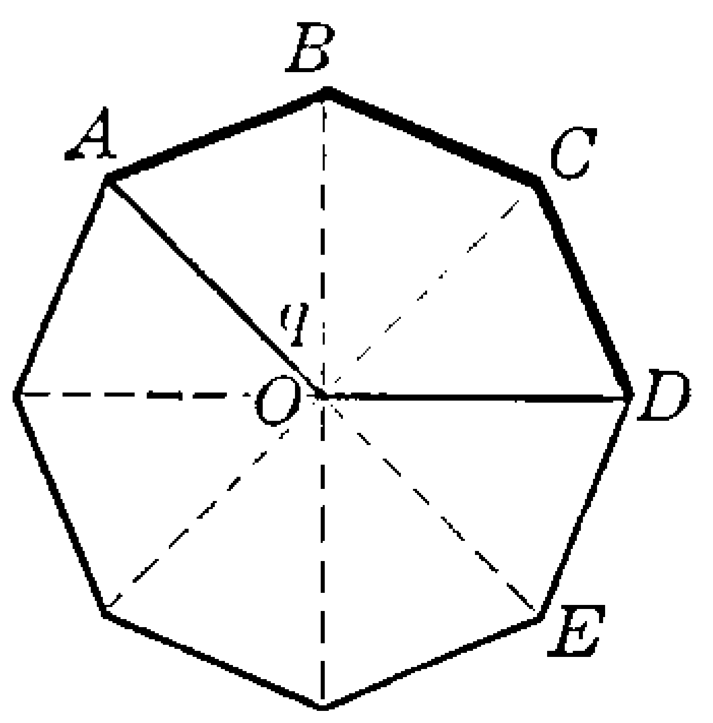
Expressed in decimals, $3\frac{1}{7}$ agrees with π to three significant digits. In problems that require only three digit places of accuracy, we may set $\pi = 3\frac{1}{7}$ or $\frac{22}{7}$.

1. The radius of a circle is 12.3 feet. Find the circumference.
2. The circumference of a circle is 18.42 ft. Find the radius.
3. (*a*) The circumference of a circle is 15π. Find the radius.
   (*b*) A square whose side is 15 units has a circle inscribed in it. Find the circumference of the circle in terms of π.
4. A circular depression around a fountain has a concrete rim 50 feet long. Adjoining the outside of the rim is a border of flowers 3 feet wide. How long is the fence enclosing the flowers?
5. The distance around the inside of a circular track is 440 yards and the distance around the outside is 500 yards. Find the width of the track in terms of π.
6. A plane flies a circle of 1400 feet radius in one minute. What is the speed of the plane in miles per hour?
7. By taking the diameter of the earth as 8000 miles, how fast does a point on the equator travel because of the earth's rotation about its axis?
8. A boy whirls a stone tied to a 5-foot string, making three turns a second. If the string breaks, at how many miles per hour will the stone fly off?
9. The legs of a right triangle are 5 units and 12 units. Find the circumference of the circle circumscribed about the triangle in terms of π.
10. One side of an equilateral triangle is $2\sqrt{3}$. Find the circumference of the circle circumscribed about the triangle in terms of π.

The Length of an Arc

To learn how to find the length of an arc, we again go back to a regular polygon. We shall first see how to compare the part of the perimeter between two radii of a regular polygon with the whole perimeter. It is the same part of the whole perimeter as the central angle formed by the radii is of 360°. Bear in mind that the radius of a regular polygon has been defined as a line segment from the center to a vertex of the polygon (§ 350).

361. THEOREM. The part of the perimeter of a regular polygon between two radii is to the whole perimeter as the central angle between these radii is to 360°.



Given: Regular polygon $ABCDE \cdots$ having n sides, and perimeter p. $ABCD$ is part of the perimeter made up of m sides.

Prove: $\frac{ABCD}{p} = \frac{\angle AOD}{360°}$

Cons.: Draw all the radii of the polygon.

PROOF

Statements	*Reasons*
1. $\triangle OAB \cong \triangle OBC \cong \triangle OCD \cong \cdots$	1. s.s.s.
2. $\angle AOB = \angle BOC = \angle COD = \cdots = \angle q$	2. Def. $\cong$ $\triangle$s.
3. $\angle AOD = m \cdot q$	3. Add. Ax.
4. $360° = n \cdot q$	4. Sum of all $\angle$s about a pt. = 360°.
5. $\frac{\angle AOD}{360°} = \frac{m}{n}$	5. Div. Ax.
6. $ABCD = m \cdot AB$ $p = n \cdot AB$	6. Given.
7. $\frac{ABCD}{p} = \frac{m}{n}$	7. Div. Ax.
8. $\frac{ABCD}{p} = \frac{\angle AOD}{360°}$	8. Subs. Ax.

Since the preceding theorem does not depend on the number of sides in the polygon, we apply it to a circle (§ 357) and we have the following corollary.

362. Corollary 1. The length of an arc of a circle is to the circumference of the circle as the angular measure of the arc is to 360°.

Where c is the length of the arc, θ (theta) the angular measure of the arc, and C the circumference of the circle, $\frac{c}{C} = \frac{\theta}{360°}$.

363. Corollary 2. In the same circle, or in equal circles, the lengths of two arcs are proportional to their angular measures.

Let c_1 and c_2 be the lengths of two arcs whose angular measures are θ_1 and θ_2, and let C be the circumference of the circle to which these arcs belong.

From Corollary 1, $\frac{c_1}{C} = \frac{\theta_1}{360°}$, and

$$\frac{c_2}{C} = \frac{\theta_2}{360°}.$$

Therefore $\frac{c_1}{c_2} = \frac{\theta_1}{\theta_2}$ (Div. Ax.)

364. Corollary 3. In the same circle, or in equal circles, if two arcs have equal angular measures they have equal lengths.

That is, if $\frac{\theta_1}{\theta_2} = 1$, then $\frac{c_1}{c_2} = 1$, and $c_1 = c_2$.

EXERCISES

1. The radius of a circle is 16.2 inches. Find the length of a 40° arc.
2. In a circle whose radius is 16.0 inches, an arc is 12.0 inches long. Find the angular measure of the arc.
3. A 35° angle is inscribed in a circle of 6-inch radius. How long is the arc that it cuts?
4. A circle is circumscribed about a right triangle whose legs are 5 and $5\sqrt{3}$ units. Find the lengths of the three arcs between the vertices of the triangle.
5. The sides of a 40° angle are tangent to a circle of 10-inch radius. Find the lengths of the arcs between the points of contact in terms of π.
6. The length of a 50° arc is 15 feet. Find the radius of the circle.
7. A chord 18 inches long is 9 inches from the center of a circle. Find the length of its major arc in terms of π and the simplest radical.
8. The length of one minute ($\frac{1}{60}$ of a degree) on the equator is called a nautical mile. If the diameter of the equator is taken as 7930 land miles, by what percent is a nautical mile greater than a land mile?

The Area of a Circle

To arrive at the area of a circle, we again start with a regular polygon.

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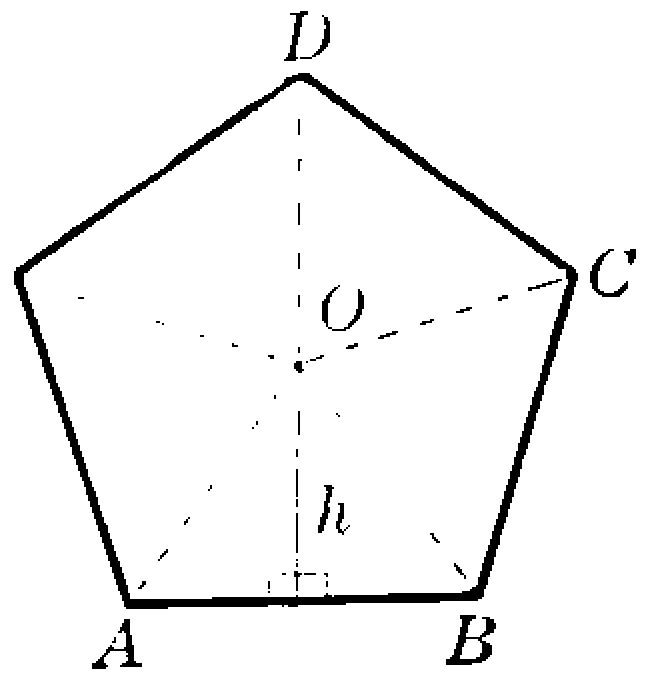
365. THEOREM. The area of a regular polygon equals half the perimeter times the apothem.



Given: S = Area of regular polygon $ABCD \cdots$ of n sides
h the apothem
p the perimeter

Prove: $S = \frac{1}{2}ph$

Cons.: Draw the radii of the polygon, dividing the figure into triangles.

PROOF

Statements	*Reasons*
1. $\angle OAB \simeq \triangle OBC \cong \triangle OCD \cong \cdots$	1. s.s.s.
2. $\triangle OAB = \frac{1}{2} \cdot AB \cdot h$	2. Area of $\triangle = \frac{1}{2}$ base × alt.
3. $n \cdot \triangle OAB = \frac{1}{2}(n \cdot AB) \cdot h$	3. Mult. Ax.
4. But $n \cdot \triangle OAB = S$ and $n \cdot AB = p$	4. Given. (The polygon has n sides.)
5. $S = \frac{1}{2}ph$	5. Subs. Ax.

Applying this theorem to a circle, and recalling that the apothem approaches the radius as a limit, we arrive at the following corollary.

366. Corollary 1. Tne area of a circle equals half the circumference times the radius.

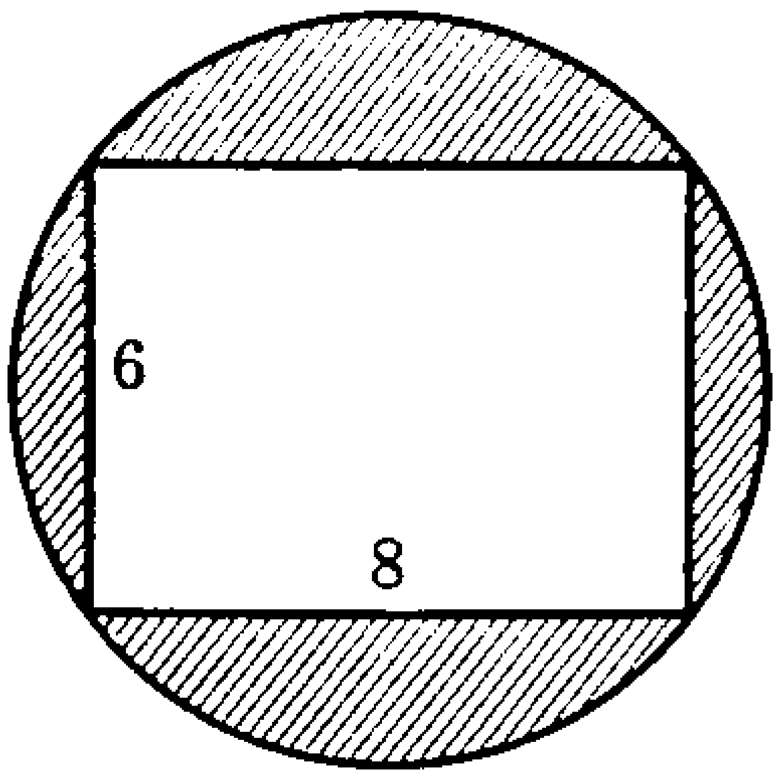
When the area is represented by S, the circumference by C, and the radius by r, this corollary may be abbreviated to $S = \frac{1}{2} \cdot C \cdot r$.

Since $C = 2\pi r$, we have by substitution $S = \frac{1}{2} \cdot 2\pi r \cdot r = \pi r^2$.

367. Corollary 2. The area of a circle equals π times the square of the radius.

EXERCISES

1. Prove that if d is the diameter of a circle, the area of the circle $S = \frac{1}{4}\pi d^2$.
2. Prove that the areas of two circles are proportional to the squares of their radii or to the squares of their diameters.
3. Find the area of a circle whose radius is 14.25 inches.
4. The diameter of a circle is 12 yards. Find the area of the circle.
5. The side of a square is 1. Find the area of the circle constructed with the diagonal of the square as a diameter. Give your answer in terms of π.
6. The side of an equilateral triangle is $\sqrt{3}$. In terms of π, find the area of the circle circumscribed about the triangle.
7. Find the area of the circle inscribed in the equilateral triangle of Ex. 6. Leave your answer in terms of π.
8. A rectangle whose dimensions are 6 inches by 8 inches is inscribed in a circle. Find the area of the shaded section.

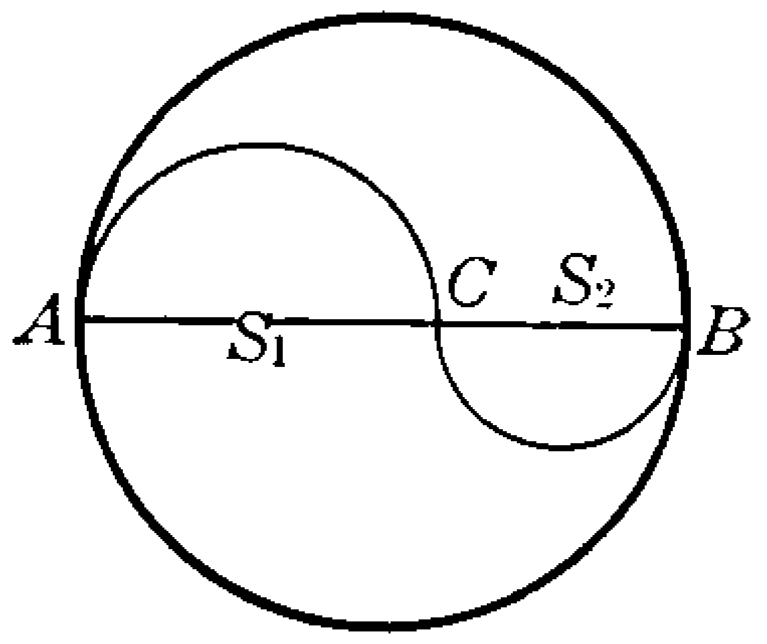


9. A rectangular sheet of tin is 8 feet by 5 feet. Forty circular plates, each one foot in diameter, are cut from it. What percent of the metal was wasted?
10. If a circular 6-inch bunch of asparagus costs 60 cents, what should be the cost of a 3-inch bunch?
11. The area of a circle is 16.4 square feet. Find the radius.
12. The area of a circle is 10π. Find the area of an equilateral triangle inscribed in the circle in terms of the simplest radical.
13. An ancient rule for finding the area of a circle was to square $\frac{8}{9}$ of the diameter. This rule amounts to using the value of π to how many significant digits?
14. The area of a circle is 25 square inches. Find the circumference of the circle.
15. The circumference of a circle is 10 inches. Find the area of the circle.

SUPERIOR WORK

S1. Prove that the area of a regular hexagon inscribed in a circle is the geometric mean of the inscribed and the circumscribed equilateral triangles.

S2. In the figure, AB is the diameter of the whole circle; AC and CB are the diameters of the semicircles. Prove that the semicircles divide the whole circle into two parts S_1 and S_2 such that:



$$\frac{S_1}{S_2} = \frac{AC}{CB}$$

Help: Let $AC = m$ and $CB = n$

$$S_1 = \frac{\pi}{8}(m + n)^2 + \frac{\pi}{8}m^2 - \frac{\pi}{8}n^2$$

S3. (*a*) The diameters of two circles are 3 and 4 units. Find the diameter of a circle that equals their sum.

(*b*) The diameters of two circles are a and b. Find the diameter of a circle that equals their sum.

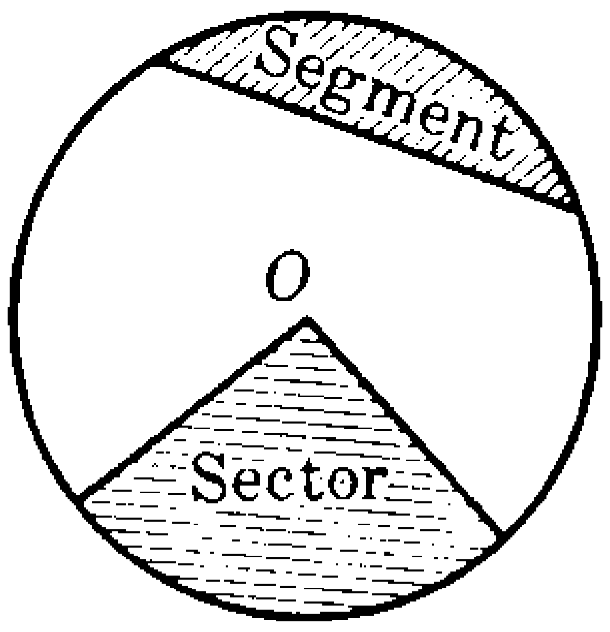
S4. A tank is now being filled by a 2-inch pipe and a 3-inch pipe. It is proposed to replace these with a single pipe that will fill the tank in the same time. What should be the diameter of the new pipe? (Disregard friction.)

85. The radius of a given circle is 30 inches. Find the radii of two circles concentric with it that will divide the original area into three equal parts.

The Area of a Sector

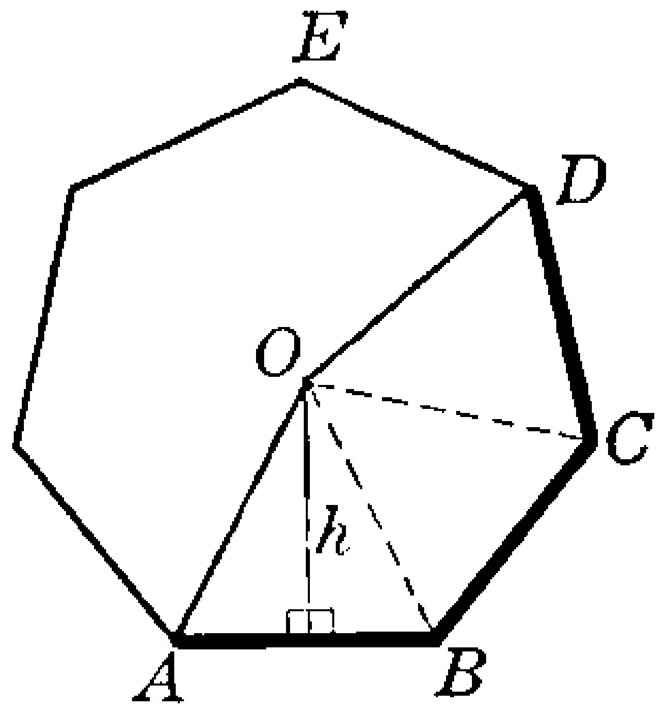
368. Definition. A sector of a circle is a figure formed by two radii of the circle and the arc they intercept.

369. Definition. A segment of a circle is a figure formed by a chord of the circle and its arc.



A sector or a segment of a circle is said to be minor or major according as its arc is minor or major.

370. THEOREM. The area of the part of a regular polygon between two radii equals half the part of the perimeter between the radii times the apothem.



Given: Regular polygon $ABCDE \cdots$ with apothem h
s = Area $OABCD$
Prove: $s = \frac{1}{2}(AB + BC + CD) \cdot h$
Cons.: Draw the radii OB and OC.

Write out the proof. It is similar to § 365.

By applying this theorem to the circle, we arrive at the corollary that follows.

371. Corollary 1. The area of a sector of a circle equals half the arc times the radius.

372. Corollary 2. The area of a sector of a circle is to the whole circle as the angular measure of the sector is to 360°.

From Corollary 1, $s = \frac{1}{2}cr$, where c is the arc of the sector, s is the area, and r is the radius.

Since the area of the whole circle $S = \frac{1}{2}Cr$, we have by division

$$\frac{s}{S} = \frac{\frac{1}{2}cr}{\frac{1}{2}Cr} = \frac{c}{C}$$

From § 362, $\frac{c}{C} = \frac{\theta}{360°}$, where θ is the angular measure of arc c.

By substitution, $\frac{s}{S} = \frac{\theta}{360°}$.

EXERCISES

1. The radius of a circle is 2.2 feet. Find the area of a sector whose arc is 16 inches.
2. The radius of a circle is 10 inches. Find the area of a sector whose angle is 25°.
3. The legs of a right triangle are 3 units and 4 units. On each side of the triangle as a diameter, a semicircle is constructed. Prove that the sum of the areas of the

two smaller semicircles equals the area of the largest semicircle.

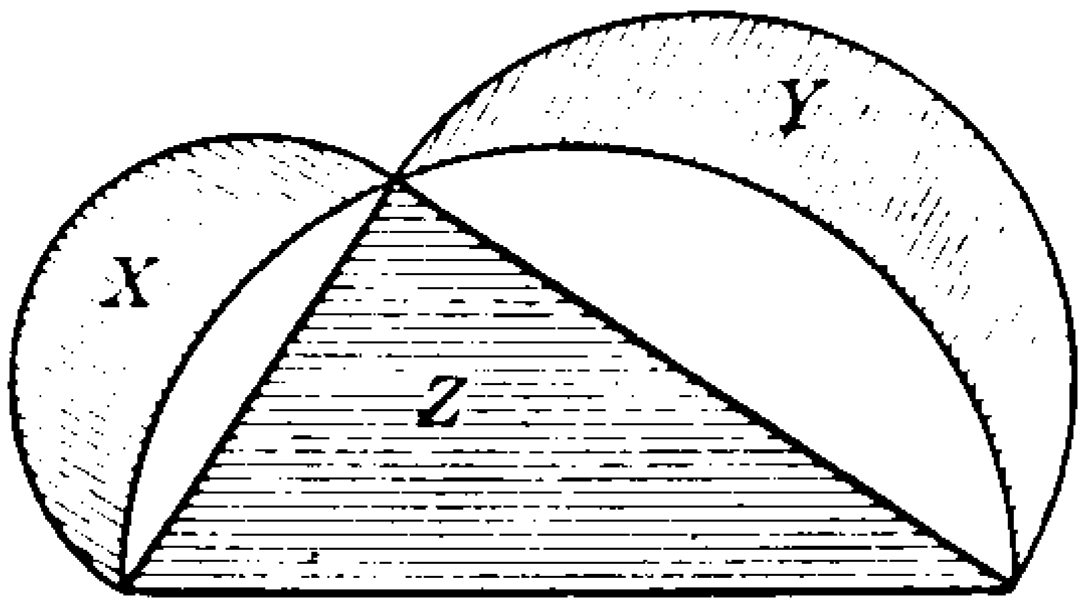
4. A horse is tied to the corner of a square barn with a 15-foot rope. Find the area over which the horse can graze.
5. The radius of a circle is 1 foot. Find the angular measure of a sector whose area is 100 square inches.
6. A circle graph is to picture the fact that a family spends \$660 for rent out of a total income of \$3000. If the whole circle represents the entire income, what is the angular measure of the sector that will show the amount spent for rent?
7. The area of a 35° sector is 12.43 square feet. Find the radius of the circle.
8. The arc of a 40° sector is 20 feet. Find the area of the sector.

SUPERIOR WORK

S1. The radius of a circle is 14.5 inches. Find the area of a 60° segment.

S2. In two circles of radii r_1 and r_2, s_1 and s_2 are the areas of two sectors having the angular measures θ_1 and θ_2. Prove that $\frac{s_1}{s_2} = \frac{\theta_1 r_1^2}{\theta_2 r_2^2}$.

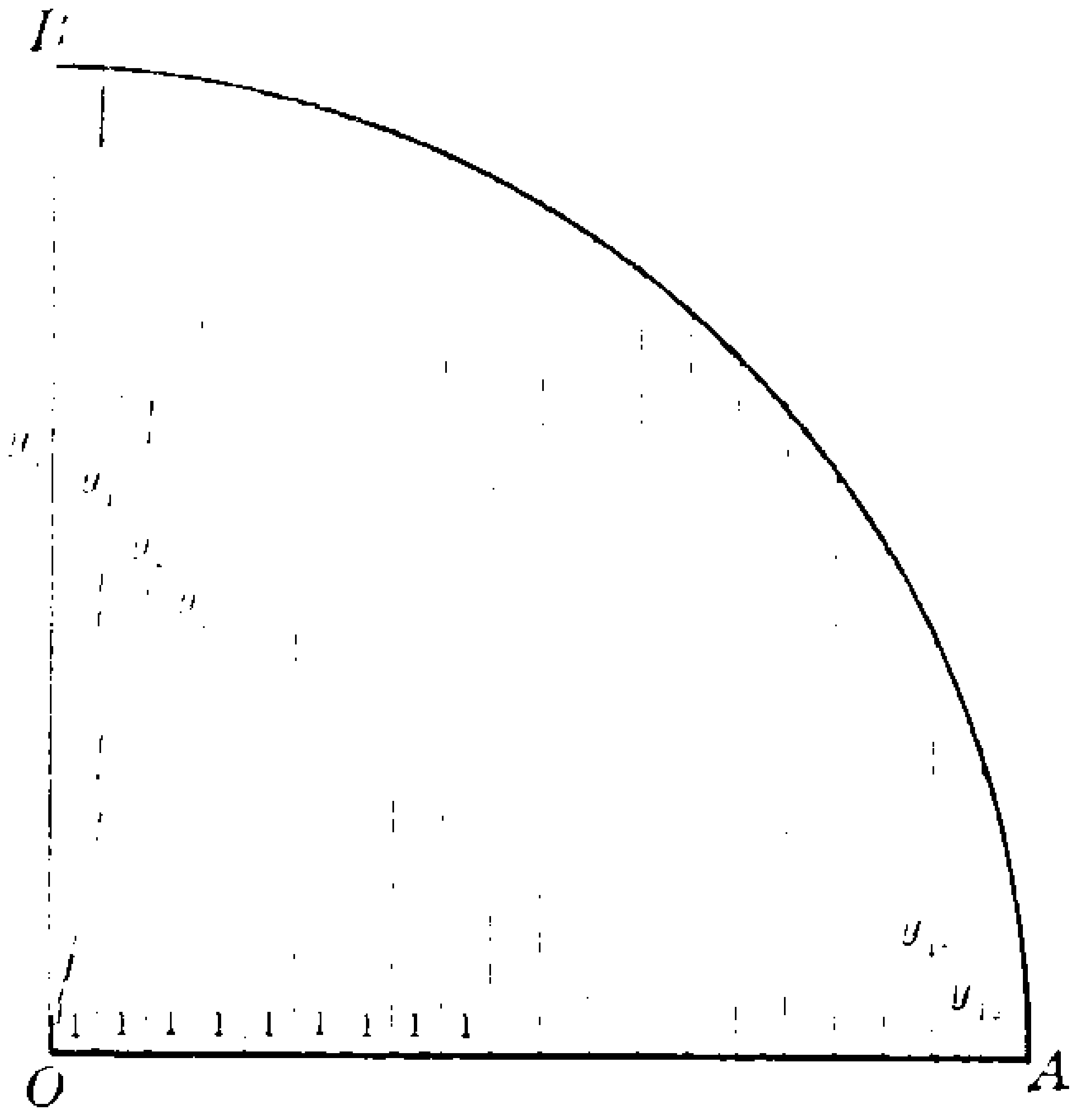
S3. The figure shows three semicircles whose diameters are the sides of a right triangle. Prove that the sum of the two shaded crescents X and Y equals the triangle Z.



Prove: $X + Y = Z$

HONOR WORK

H1. *Approximating π.* The figure shows a 90° sector OAB, where $OA = 20$ units. Perpendiculars to OA, called *offsets*, were constructed 1 unit apart. By connecting the ends of these offsets with chords, 20 trapezoids (the last one a triangle) were formed.



The sum of these trapezoids will approximate the area of the sector OAB.

On page 196, S2, you have derived a convenient formula for finding the area of the trapezoids. It may be stated as follows:

Trapezoidal Rule. To the average of the first and last offsets, add the intervening offsets and multiply by the common distance between the offsets.

Applied to our trapezoids,

Sector OAB =

$$1\left(\frac{y_0 + y_{20}}{2} + y_1 + y_2 + y_3 + \cdots + y_{19}\right)$$

We can calculate any offset by drawing the radius to its end:

$$y_0 = 20$$

$$y_1 = \sqrt{20^2 - 1^2}$$

$$y_2 = \sqrt{20^2 - 2^2}$$

$$y_3 = \sqrt{20^2 - 3^2}$$

$$\cdots$$

$$y_{20} = 0$$

Find the area of the trapezoids correct to three digit places. From the fact that the exact area of the sector equals $\frac{1}{4} \cdot 20^2\pi$ or 100π, find the approximate value of π.

H2. *Three Famous Problems.* Three construction problems have occupied students of geometry for over 2000 years. It has now been proved that none of the three problems that follow can be solved by straightedge and compass alone—the only tools permitted in demonstrative geometry (§ 149).

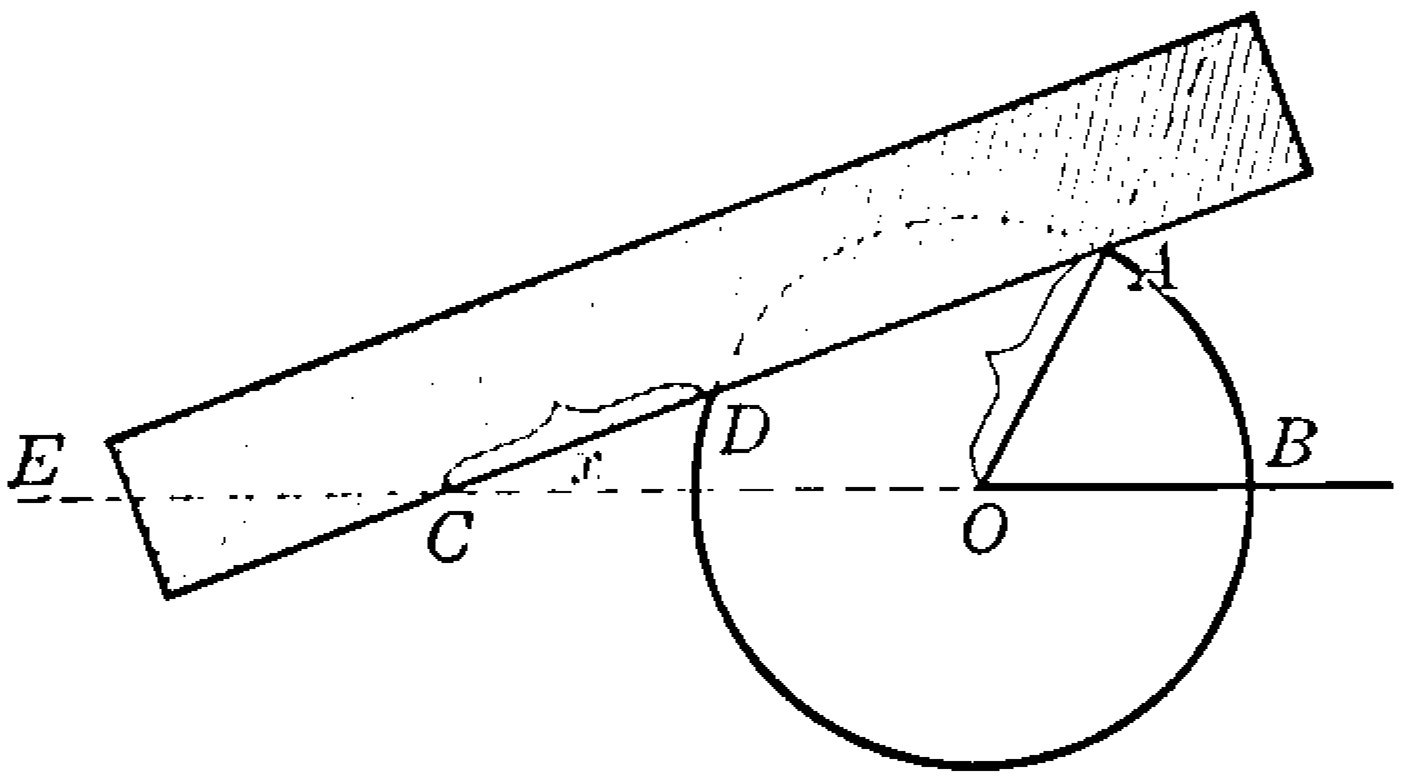
(1) *Squaring the Circle.* This problem calls for the construction of a square that has the same area as a given circle. If this problem could be done, it would mean that π could be found exactly in numbers or as the square root of a number.

(2) *Duplication of a Cube.* In this problem we have given the side of a cube, and we are asked to construct the side of another cube having twice the volume of the first cube.

(3) *Trisection of an Angle.* The problem of trisecting a given angle has interested young students because it looks easy. Following are two methods for solving this problem. Both methods are "illegal" in demonstrative geometry because they use more than a straightedge and compass. Constructions like these sometimes appear in newspapers where it is claimed that the age-old problem has finally been solved.

(*a*) Given $\angle AOB$. Draw circle O and prolong BO through O.

On a straightedge or a card, mark off $CD = OA$.



Place the straightedge through A so that C falls on OE and D falls on the circle.

Then $\angle x = \frac{1}{3} \angle AOB$.

Draw OD and prove that $\angle AOB = 3\angle x$.

Is it permissible in our geometry to use a straightedge with CD marked on it (§ 149)? In placing the marked straightedge as indicated above, was its position determined exactly or only approximately?

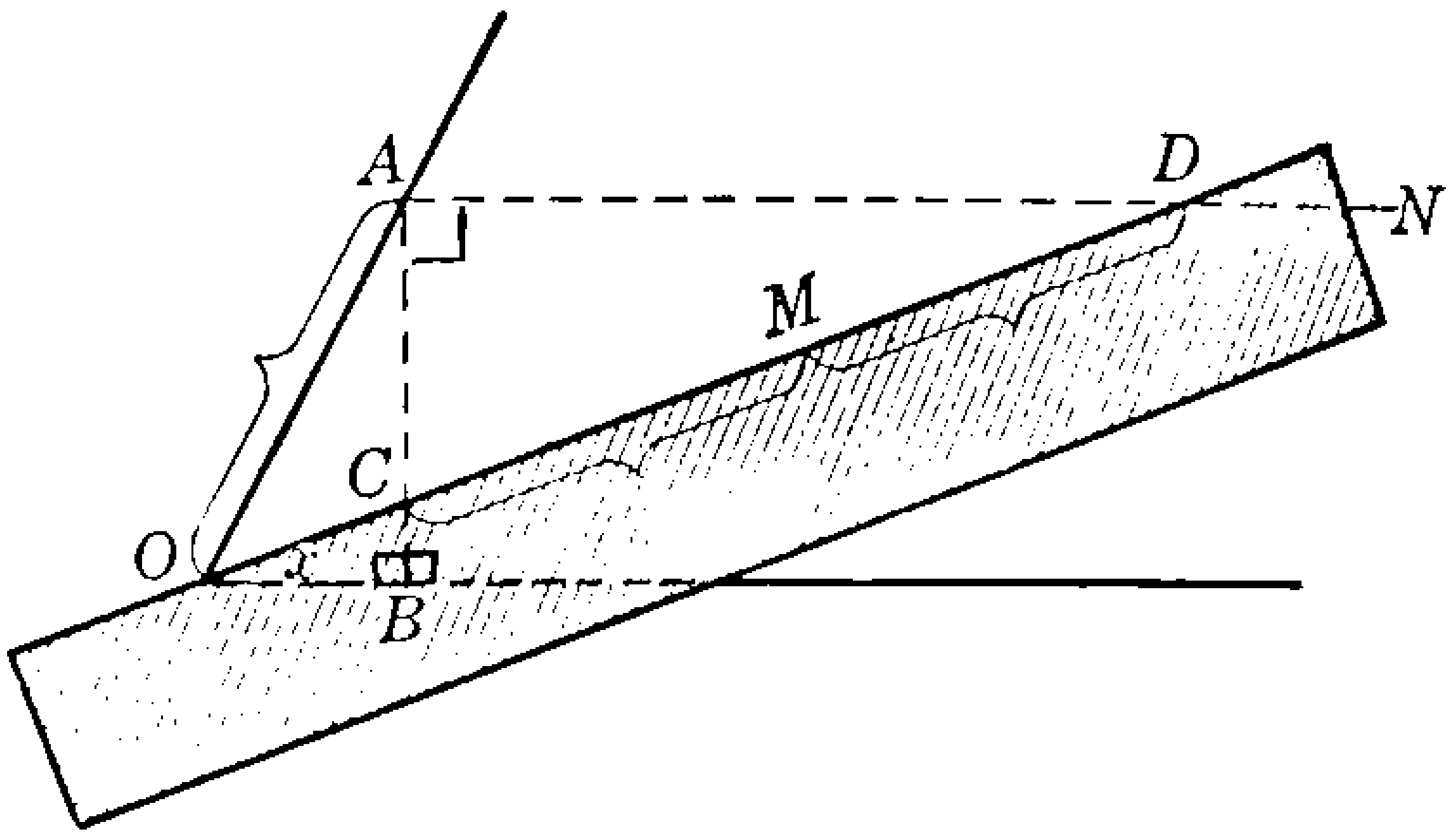
(*b*) Given $\angle AOB$.

Construct $AB \perp OB$ and $AN \perp AB$.

On a straightedge or a card, mark off $CM = MD = OA$. Place the straightedge through O so that C falls on AB and D falls on AN.

Then $\angle x = \frac{1}{3} \angle AOB$.

Draw AM and prove that $\angle AOC = 2 \angle x$.



Exactly what has been done that it is not permissible to do?

SUMMARY

373. For a group that has a finite number of members, the most common measure is obtained by dividing the *sum* of all the members by the number of them. This is called the *arithmetic mean* or simply the average. It was proved that the line segment connecting the midpoints of the legs of a trapezoid is parallel to the bases and is the arithmetic mean of the bases.

A measure that has many applications in geometry is the *geometric mean.* It is used mostly when there are but two terms in the group and it is based on multiplication. The product of the two terms equals the square of their geometric mean. Since the geometric mean can replace each of the two middle terms of a proportion in which two other numbers are the extremes, it is the *mean proportional* between the two extreme terms.

If two chords intersect within a circle, the product of the segments of one chord equals the product of the segments of the other chord. In the special case where one chord bisects another chord, either segment of the bisected chord is the geometric mean of the segments of the other chord.

If two secants are drawn to a circle from an external point, the product of one whole secant and its external segment equals the product of the other secant and its external segment. In the special case where one secant has become a tangent, the tangent is the geometric mean of the whole secant and its external segment.

It often happens that the members of an infinite group are so built up that they come closer and ever closer to a certain quantity without actually reaching it. This quantity can then serve as a measure of the group. It is called the *limit* of the variable members of the group.

If the number of sides of a regular polygon inscribed in a circle is doubled again and again indefinitely, then the perimeter of the polygon becomes ever greater. Yet, no matter how many sides the polygon may have, the perimeter of the polygon will always be less than the perimeter of the circle—but by a quantity as small as we choose. In other words, the perimeter of the polygon approaches the length of the circle as a limit. The circumference of a circle is therefore defined as this limit.

By assuming that propositions about regular polygons that do not depend on the number of sides can be applied to circles, we arrive at the following useful propositions:

(1) The circumference of a circle equals π times the diameter. π is an irrational constant whose approximate value is $3.14159 \cdots$.

(2) The area of a circle equals π times the square of the radius.

(3) An arc of a circle is the same part of the circumference as the angular measure of the arc is of 360°.

(4) A sector of a circle is the same part of the area of the whole circle as the angular measure of the sector is of 360°.

RECOGNIZING NEW TERMS

Can you describe each word or term below? If not, go back to the section indicated.

measure of a group § 330
arithmetic mean § 330
median of a group § 330
geometric mean § 338
regular polygon § 347
  center § 350
  radius § 350
  central angle § 350
  apothem § 350
limit of a variable § 354
circumference of a circle § 356
constant π § 359
sector of a circle § 368
segment of a circle § 369

REVIEW EXERCISES

1. (*a*) The distance between the midpoints of the legs of a trapezoid is 17 units and one base is 15 units. Find the other base.

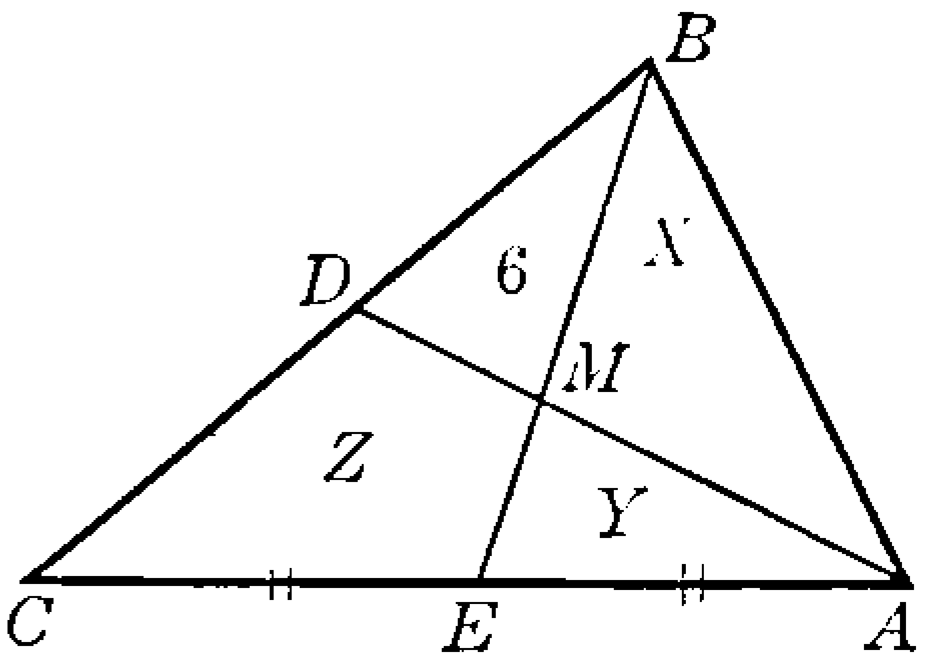
(*b*) The bases of a trapezoid are 9 units and 15 units. A point *P* is midway between the midpoints of the bases. How far is *P* from the midpoints of the legs of the trapezoid?

(*c*) A line segment is drawn between the midpoints of the bases of a trapezoid. After one base of the trapezoid has shrunk to a point, what name is given to this line segment?

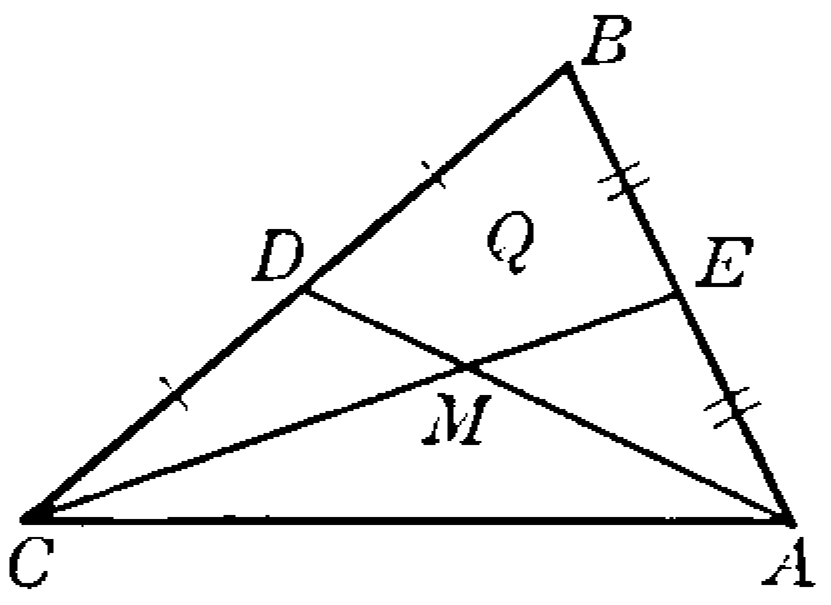
2. Prove that the triangle whose vertices are the midpoints of the sides of another triangle is similar to it.

3. Prove: If the midpoints of the sides of a rectangle are joined in succession then a rhombus is formed. (See page 247, Ex. 3.)

4. In the figure, *AD* and *BE* are medians of the triangle *ABC*. The area of the triangle *BDM* is 6. Find the areas of the triangles *X* and *Y*, and of the quadrilateral *Z*.

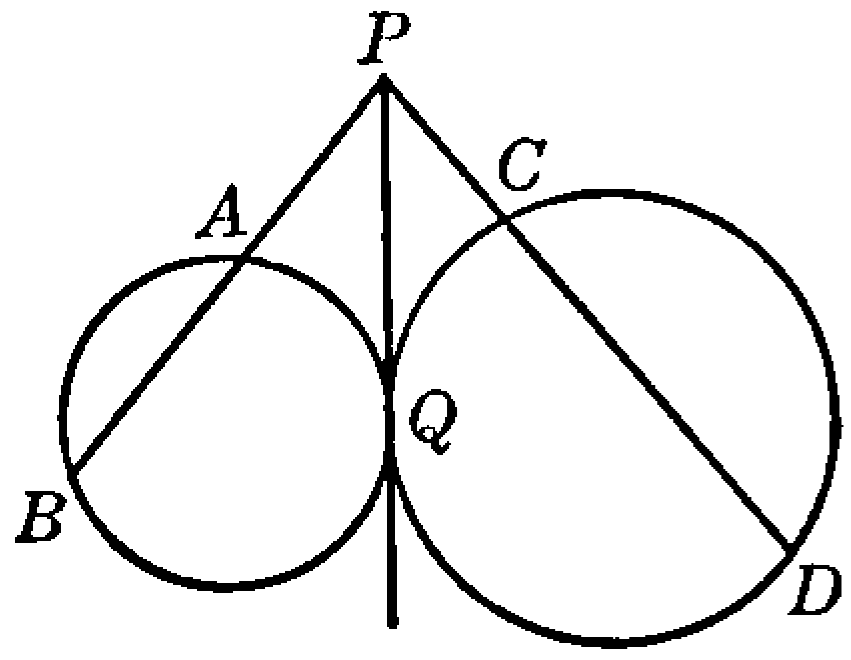


5. In the figure, *AD* and *CE* are medians of the triangle *ABC*. Find the areas of the triangles *MEA*, *MCD*, *MAC*, and *ABC* in terms of *Q*, the area of the quadrilateral *BEMD*.



6. Construct a square that has twice the area of a given triangle.

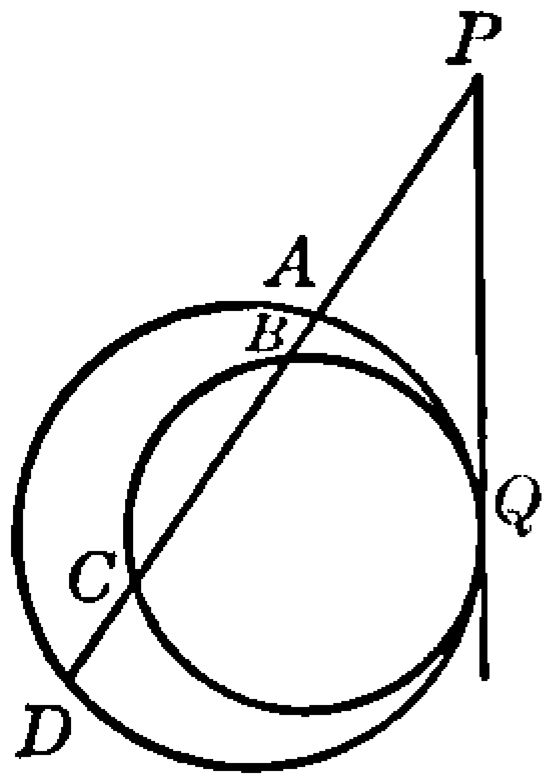
7.



Given:
Two circles
tan. to PQ at Q

Prove:
$$\frac{PA}{PC} = \frac{PD}{PB}$$

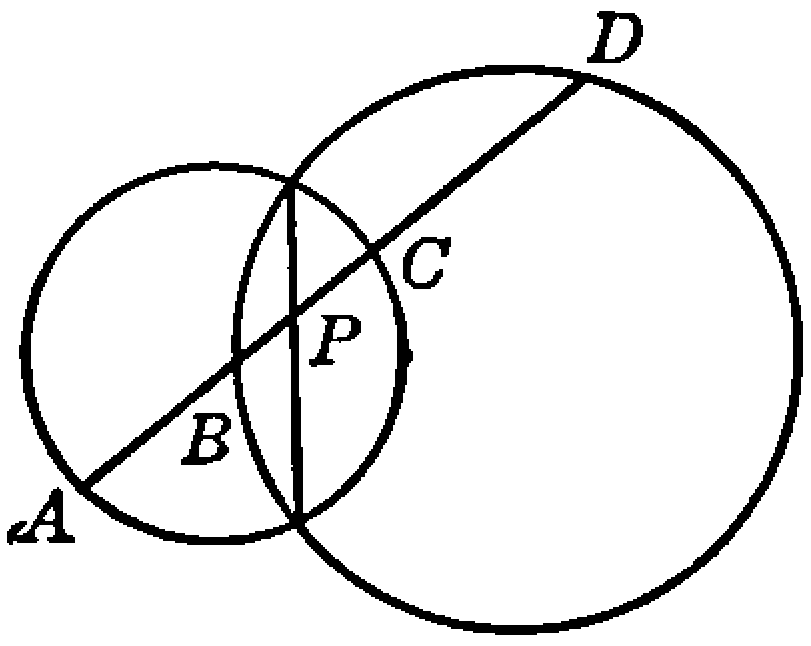
8.



Given:
Two circles
tan. to PQ at Q

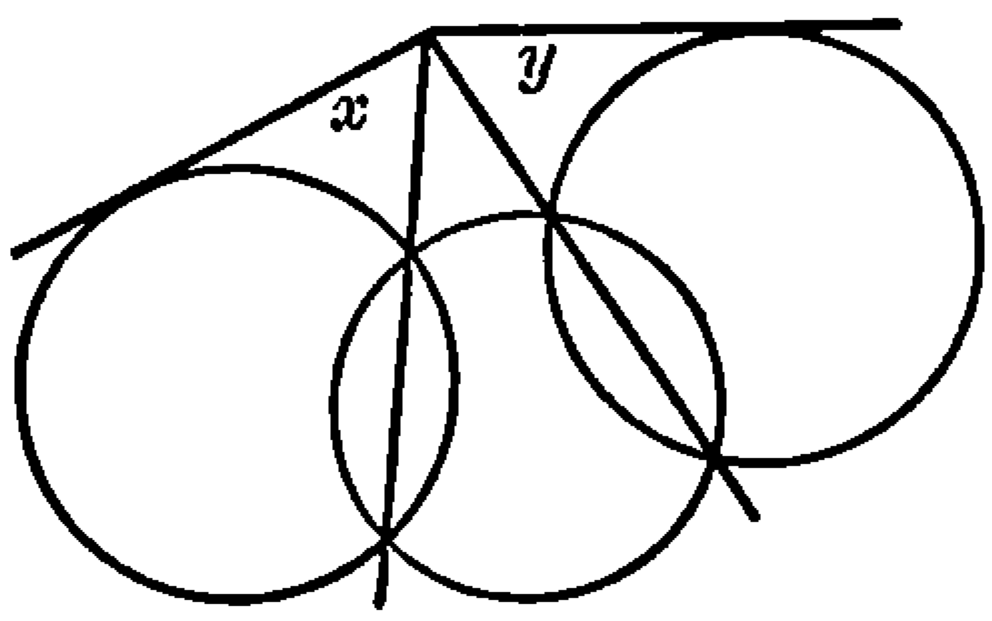
Prove:
$$\frac{PA}{PB} = \frac{PC}{PD}$$

9.



Prove:
$$\frac{PA}{PB} = \frac{PD}{PC}$$

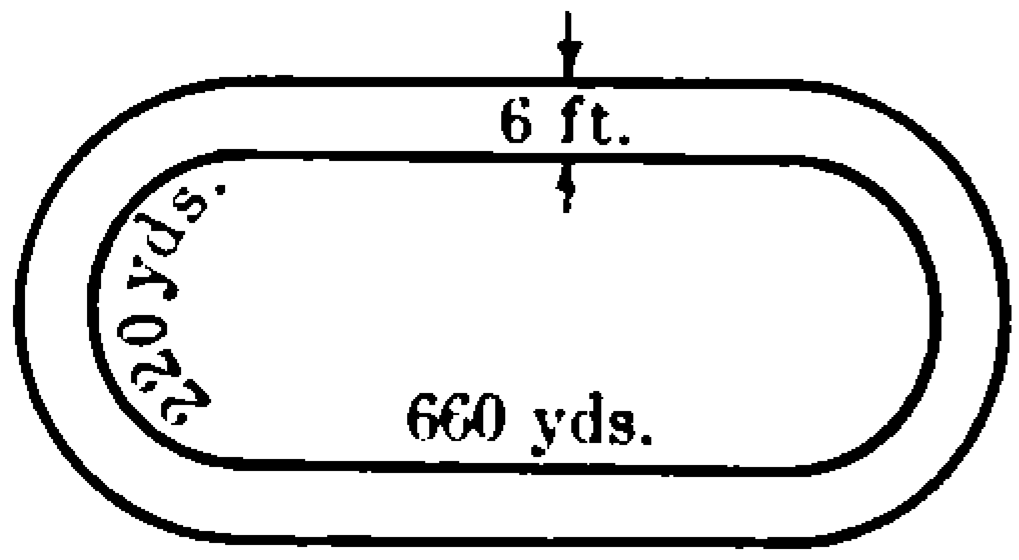
10.



Prove:
Tan. x = tan. y

11. The earth is 93,000,000 miles from the sun. Assuming that the earth revolves around the sun in a circle, what is the speed of the earth in its orbit in miles per hour?

12. The inner lane of a mile track is laid out as shown in the figure. The straight sides are each 660 yards, and the semicircular ends are each 220 yards. One boy runs in the inner lane and another boy runs in the outer lane which is 6 feet from the inner lane. How many feet handicap should the second boy be given to equalize the distances?



13. The bases of a trapezoid are 11 units and 17 units. Find the area of a circle constructed on the medial line of the trapezoid as a diameter. Use $\frac{22}{7}$ for π. (The medial line connects the midpoints of the legs.)

14. What is the largest area that can be enclosed with 100 feet of fencing? *Hint:* Try out figures like the square, the equilateral triangle, and the circle.

15. Prove that the areas of two circles are proportional to the squares of their circumferences.

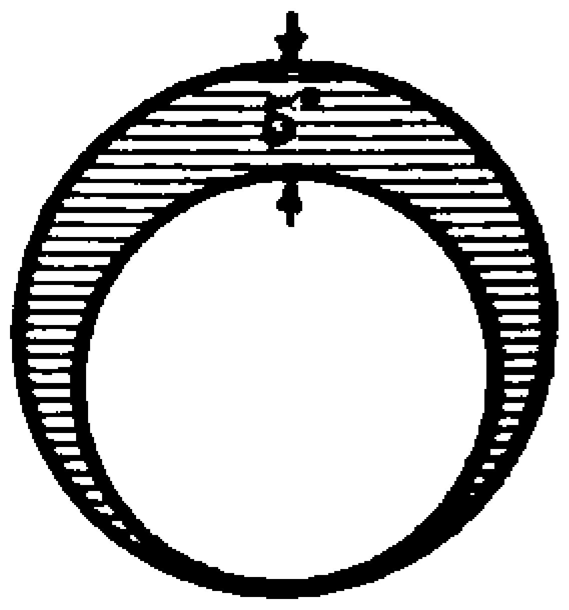
16. A circular plot is 12 feet in diameter. How many square feet are there in a 2-foot border of flowers planted along the inner edge of the plot?

17. If a circle of 1-inch diameter represents \$120,000, the annual amount that a community spent on its schools 20 years ago, what should be the diameter of the circle that will represent the annual cost of \$300,000 today?

18. A beacon light that can be seen 15 miles swings back and forth through an angle of 160°. Over how many square miles can it be seen?

19. The area of a 40-degree sector is 100 square inches. Find the length of the arc.

20. The shaded design in the diagram has an area of 150 square inches. It is made by two tangent circles, and the width at the



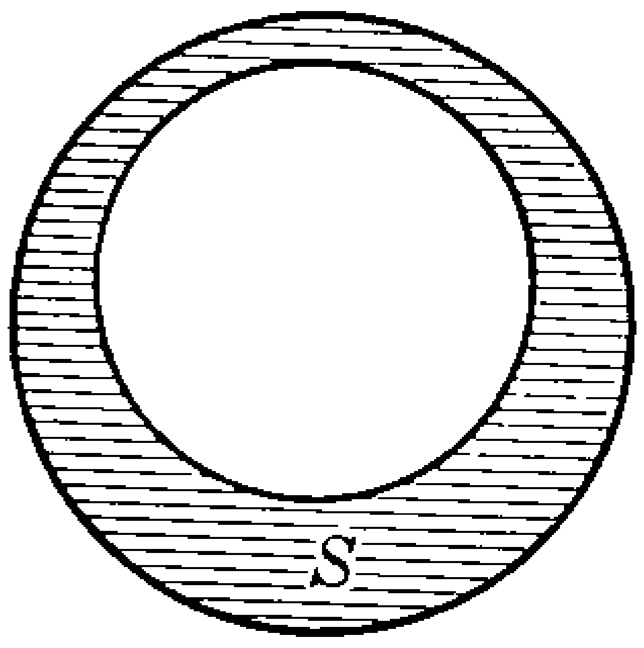
top is 5 inches. Find the diameter of the inner circle.

21. One side of an equilateral triangle is 10 units. Find the area of the ring between the inscribed and circumscribed circles in terms of π.

22. (*a*) The radii of two concentric circles are 15 and 12 units. Find the area of the ring in terms of π.

(*b*) Prove that the area of this ring equals the area of a circle whose radius is the geometric mean of the sum and the difference of the two given radii; that is, the geometric mean of $15 + 12$ and $15 - 12$.

(*c*) Prove that the area S in the figure, bounded by two circles, one inside the other, equals the area of a circle whose radius is the geometric mean of the sum and the difference of the radii of the circles forming the area S.

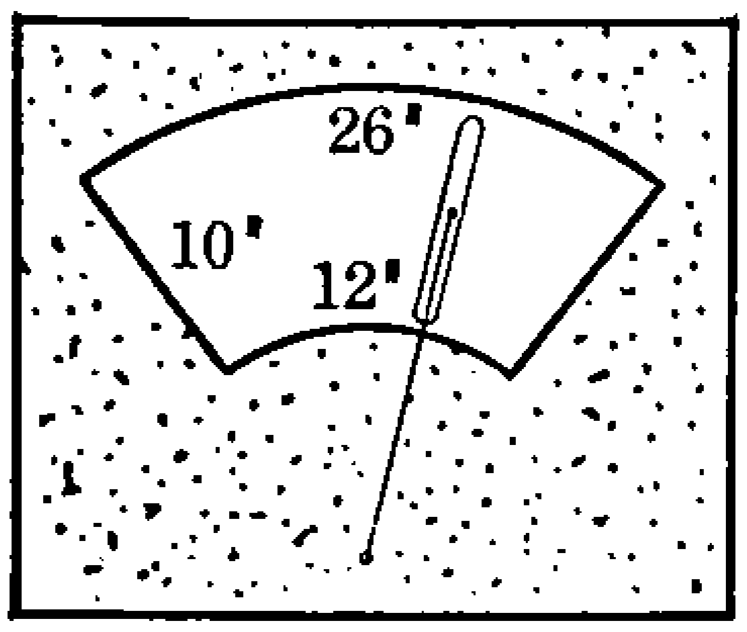


Help: Let a and b be the radii of the two given circles. Then $S = \pi a^2 - \pi b^2$. Factor this completely.

SUPERIOR WORK

S1. Prove that the line segment connecting the midpoints of the diagonals of a trapezoid equals one-half the difference of the bases.

S2. (*a*) A windshield wiper cleans an area as shown in the figure. The arcs are 26 inches and 12 inches long. The wiper blade is 10 inches long. Find the area that is kept clean.



(*b*) Show that you would get the same answer if you considered the figure analogous to a trapezoid.

S3. The idea of measuring a group by using a representative of the group is common in everyday thinking. Our whole system of democratic government is based on it. You should therefore give some thought to the following questions:

(*a*) Do you consider a certain student in your school average because he is the arithmetic mean or because he is the median of the group?

(*b*) Do you think that the average student in your school is completely honest and trustworthy?

(*c*) Do you want an average student or a superior student to represent you on the student council? on the varsity football team? in interclass basketball games?

(*d*) How would you find out what the average student in your class thinks about student government?

(*e*) How do popular opinion polls determine what the average person thinks? Can sampling be compared to working with averages?

(*f*) In addressing your school assembly, do you direct your speech to the whole group or do you try to talk to one average individual? What is the advantage of each method?

(*g*) Is yours an average school? If so, in what things?

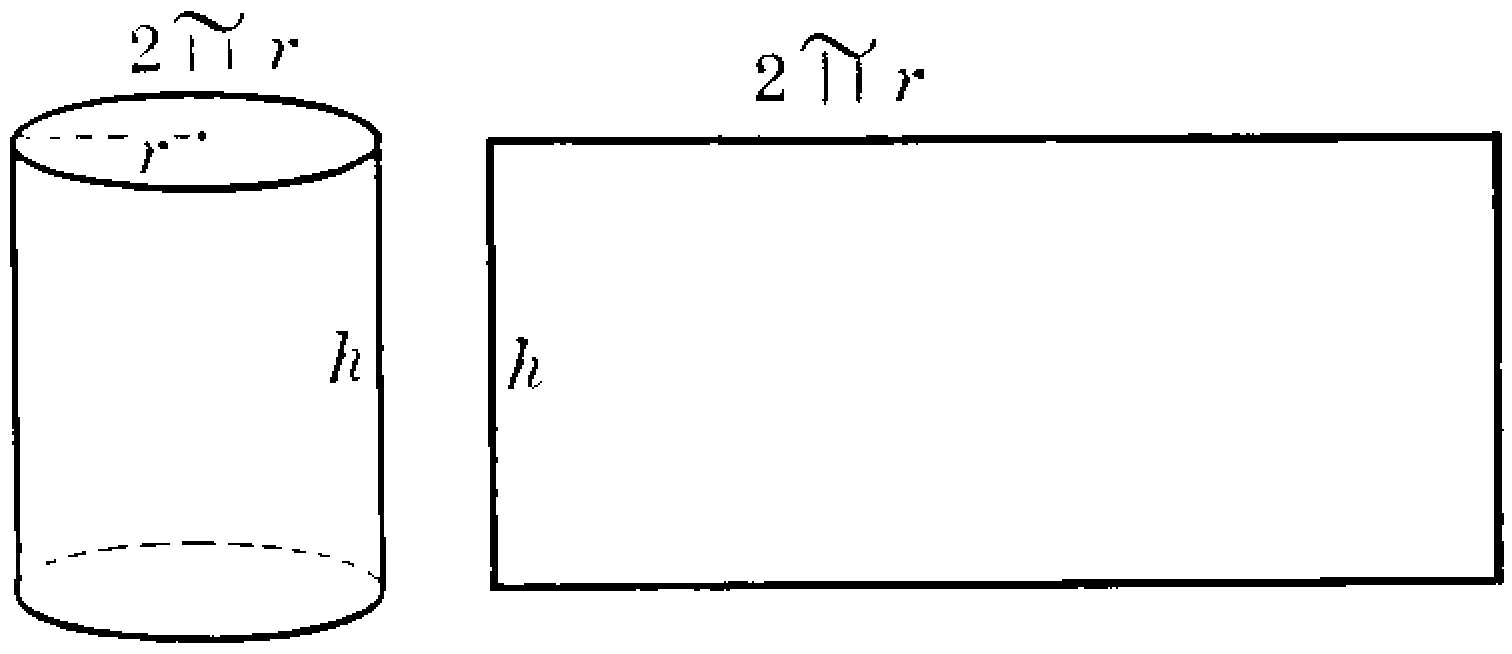
(*h*) In what respects is your community average and in what respects is it not average?

(*i*) In 1925, and again in 1935, a certain midwestern town was studied by a group of students. Since this town was believed to be a cross section of American communities, the reports were of national importance. They are contained in two books, *Middletown* and *Middletown in Transition*, by Lynd and Lynd. Look at one of these books in your library and make a list of the main topics that were studied.

General Space

S4. *Lateral Area.* The curved surface of a cylinder or a cone is called its *lateral surface*.

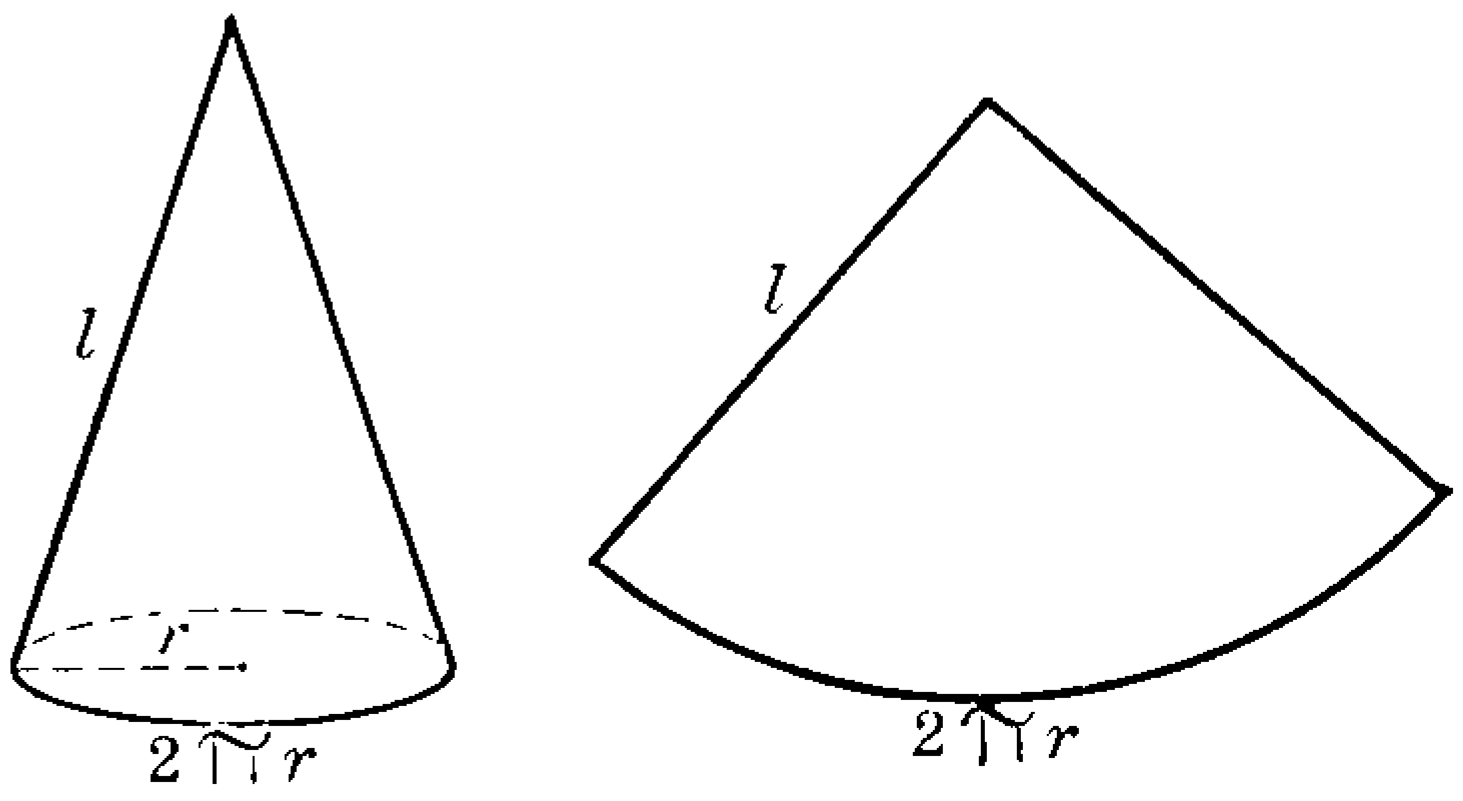
In a right circular cylinder, the bases are circles in parallel planes and the straight line segments in the lateral surface, called the *elements*, are perpendicular to the bases. (See *right prism*, p. 197, S6.) The figure shows the lateral surface of a right circular cylinder rolled out into a rectangle.



(*a*) Write the formula for the lateral area of a right circular cylinder in terms of the radius of the base *r* and the altitude *h*.

(*b*) The altitude of a right circular cylinder is 15 units and the radius of the base is 10 units. Find the lateral area in terms of π.

S5. In a right circular cone, the base is a circle and all line segments from the vertex to the circle, called the *elements*, are equal. The figure shows how the lateral surface of such a cone can be rolled out into a sector of a circle whose radius *l* is an element of the cone.

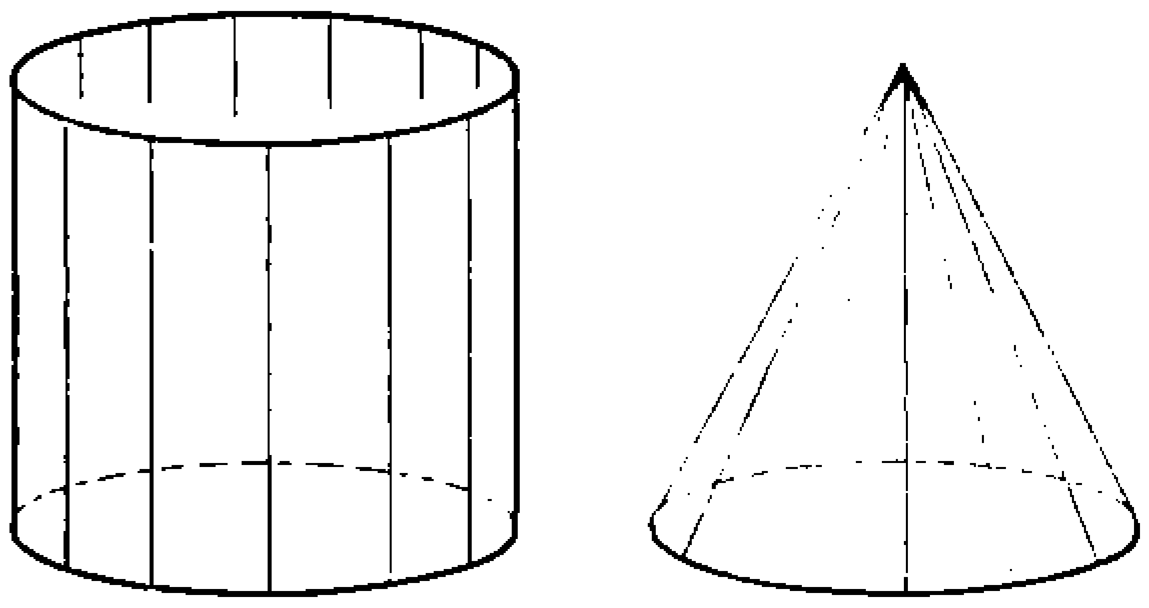


(*a*) Write a formula for the lateral area of a right circular cone in terms of the radius of the base and an element.

(*b*) Write this formula in terms of the radius of the base and the altitude of the cone.

(*c*) The radius of the base of a right circular cone is 5 units and the altitude is 12 units. Find the lateral area in terms of π.

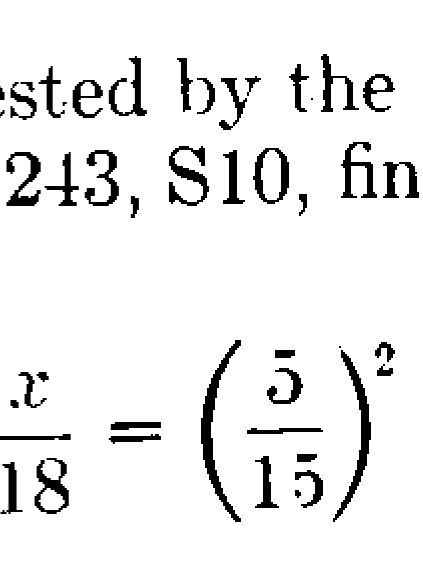
S6. Just as we arrived at the measurement of the circle from a regular polygon of an increasing number of sides, so we can learn how to measure a circular cylinder or a circular cone by working with a prism or a pyramid whose base is a regular polygon inscribed in the base of the cylinder or the cone. Since we already learned how to find the volume of a prism and a pyramid (pages 217-218, S15, S18), we see that the volume of a cylinder is the area of the base times the altitude, and the volume of a cone is one-third the base times the altitude.



(*a*) Find the volume of a cylinder whose base is a circle of 5-inch radius and whose altitude is 12 inches.

(*b*) Find the volume of a cone having the same base and altitude as the cylinder in (*a*).

S7. The area of the base of a circular cone is 18 square inches and the altitude is 15 inches. It is cut by a plane parallel to the base and 10 inches from the base.



(*a*) As suggested by the theorem about a pyramid on page 243, S10, find the area of the cross section x:

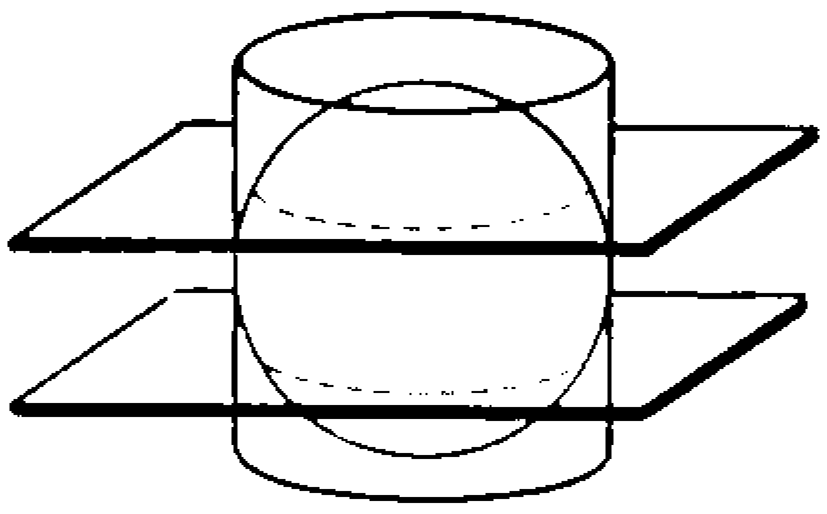
$$\frac{x}{18} = \left(\frac{5}{15}\right)^2$$

(*b*) Find the volumes of the two cones and, by subtracting them, find the volume of the part of the cone between the parallel planes.

S8. We shall assume the following striking proposition that can be proved in solid geometry:

Situation. A hollow rubber ball fits snugly into a tin can. Two planes parallel to the bases of the can, cut the figure.

Conclusion: The area of the rubber between the parallel planes equals the area of the tin between these planes.



The surface of a sphere between two parallel planes is called a *zone*.

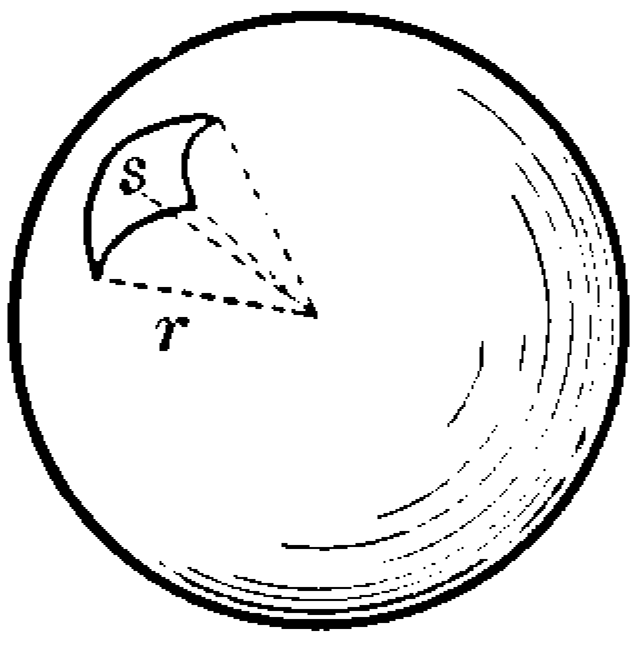
(*a*) Prove that the area of a zone equals the circumference of a great circle of the sphere (page 169, S12) times the distance between the parallel planes.

(*b*) Prove that the area of a sphere is four times the area of a great circle, $4\pi R^2$ or πD^2 (diameter = distance between planes).

(*c*) Find the area of the surface of the earth (diameter 8000 miles).

(*d*) Show that the surface of the earth between the 30°-latitude circles is half the whole surface of the earth.

S9. *Volume of a Sphere.* The volume of a sphere is the sum of all the sections like that shown in the figure. This section is the limit of a series of pyramids. Its volume is therefore $\frac{1}{3} \cdot s \cdot r$. When these sections are added to give the volume of the sphere, the sum of their volumes will be $\frac{1}{3}Sr$ where S is the surface of the sphere.



(*a*) Complete the following proposition: The volume of a sphere equals one-third the —?— of the sphere times the —?—.

(*b*) Write the formula for the volume of a sphere in terms of the radius r and π.

(*c*) Find the volume of the earth.

(*d*) Prove that the volumes of two spheres are proportional to the cubes of their diameters.

(*e*) The diameter of the planet Jupiter is about 10 times the diameter of the earth. How many times the volume of the earth is the volume of Jupiter?

(*f*) The diameter of the sun is 10 times the diameter of Jupiter. Compare the volume of the sun with that of the earth.

TEST

1. Prove: If the midpoints of the sides of a quadrilateral are joined in succession, the area of the new quadrilateral so formed is one-half the area of the original quadrilateral.
2. Prove: If the line segment that connects the midpoints of the legs of a right triangle equals one of the legs, then the larger triangle has a 60° angle.
3. Construct a square equal to half of a given square.
4. It is desired to plant two dozen tulips 18 inches apart measured along the circumference of a circle. Find the radius of the circle.
5. A pendulum 30 inches long swings through an arc of 20°. How long is the arc?
6. Beginning at the center of a circular grass plot, a boy mows 18 rings each 12 inches wide. How many square feet of grass did he cut?
7. On the outside of a grass plot 15 feet in diameter, a concrete walk 2 feet 6 inches wide is to be constructed. At $1.20 a square foot, what will be the cost of the walk?
8. From a circular plate 10 inches in diameter, two equal circles are cut. If the circles are the largest possible, what percent of the material is wasted?
9. One side of a square is 10 units. Find the area of the ring between the inscribed and the circumscribed circles in terms of π.
10. A searchlight that can turn through an angle of 50° covers an area of 40 square miles. Find the range of the light.

Appendix

Inequalities

374. Definition. A statement that a is greater than b (written $a > b$), or a is less than b ($a < b$), is called an *inequality*.

Following are some basic propositions about inequalities.

375. Axiom. The whole of any magnitude is greater than any of its parts. Thus, if $a = b + c$, then $a > b$.

376. Corollary. An exterior angle of a triangle is greater than either remote interior angle. (§ 64)

377. Postulate. A straight line is the shortest distance between two points.

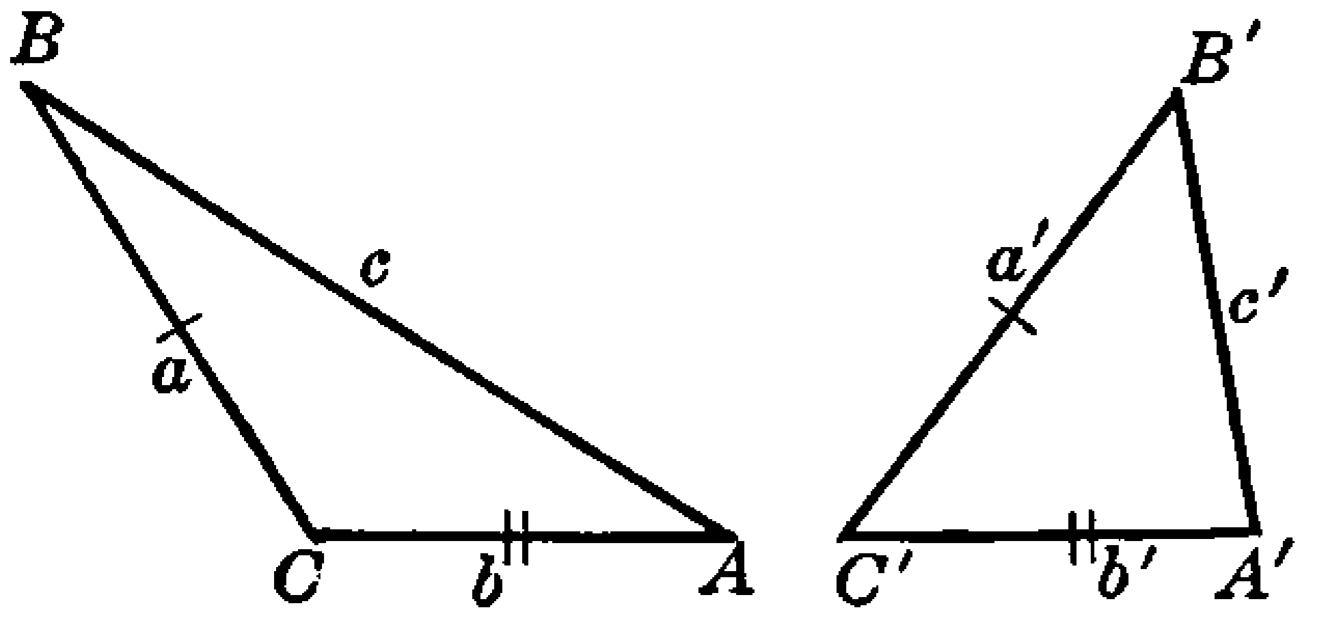
378. Axiom. If unequals are added to equals, or to unequals, the sums are unequal in the same order. Thus,

if	$5 = 5$	if	$10 = 10$	if	$4 > 3$	if	$2 < 3$
and	$\underline{8 > 6}$	and	$\underline{4 < 7}$	and	$\underline{7 > 5}$	and	$\underline{4 < 5}$
then	$13 > 11$	then	$14 < 17$	then	$11 > 8$	then	$6 < 8$

379. Axiom. If equals are subtracted from unequals, the remainders are unequal in the same order.

380. Axiom. If unequals are subtracted from equals, the remainders are unequal in the opposite order.

381. Theorem If two sides of one triangle are equal to two sides of another triangle, but the included angle of the first is greater than the included angle of the second, then the third side of the first is greater than the third side of the second.



Given: In $\triangle ABC$ and $A'B'C'$,

$a = a'$

$b = b'$

$\angle C > \angle C'$

Prove: $c > c'$

Suggested proof:

$$c^2 = a^2 + b^2 - 2\,ab \cos C \qquad (\S\ 321)$$

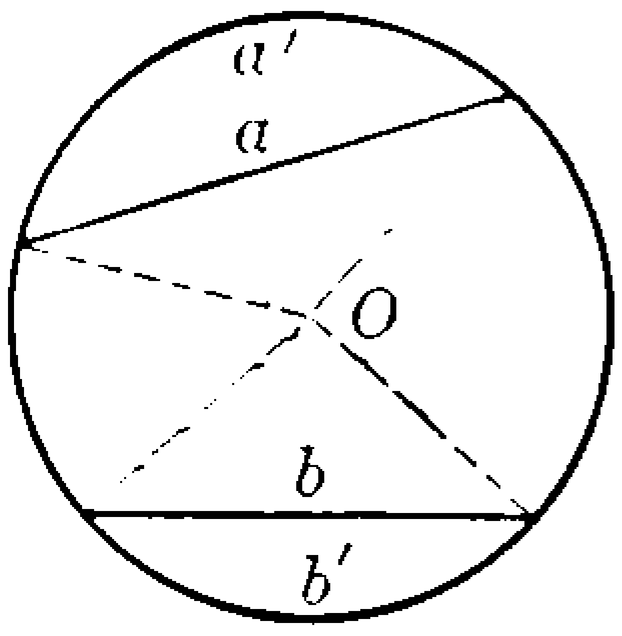
$$c'^2 = (a')^2 + (b')^2 - 2a'b' \cos C'$$

Since $C > C'$, $\cos C < \cos C'$ (§ 314). Hence the right-hand member of the first equation is greater than the right-hand member of the second equation (§ 380).

382. Converse Theorem. If two sides of one triangle are equal to two sides of another triangle, but the third side of the first is greater than the third side of the second, then the angle opposite the third side of the first is greater than the angle opposite the third side of the second.

Use the same method of proof as in § 381.

383. Theorem. If two chords of a circle are unequal, their minor arcs are unequal in the same order.

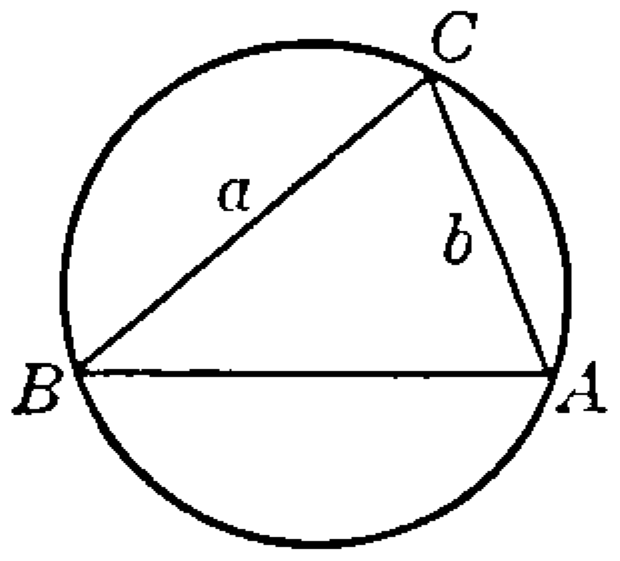


Given: Chord a > chord b
Prove: $\widehat{a'} > \widehat{b'}$
Hint: Draw the radii and use § 382.

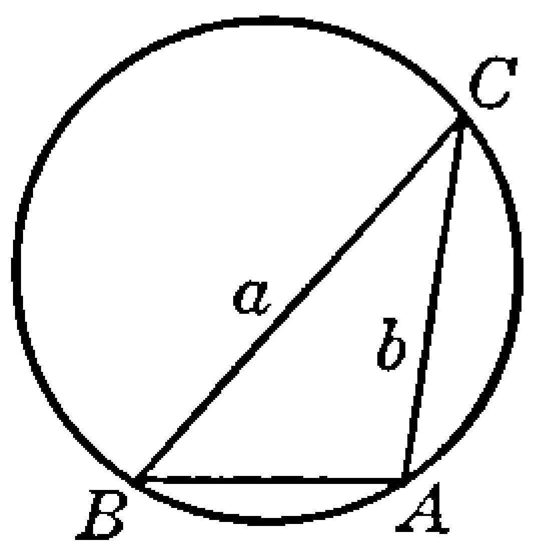
384. Converse Theorem. If two minor arcs of a circle are unequal their chords are unequal in the same order.

Write out the proof.

385. Theorem. If two angles of a triangle are unequal, the sides opposite are unequal in the same order.



Case 1. $\angle A$ acute



Case 2. $\angle A$ obtuse

Given: In $\triangle ABC$
$\angle A > \angle B$
Prove: $a > b$
Hint: Circumscribe a circle about the triangle and use § 384.

386. Converse Theorem. If two sides of a triangle are unequal the angles opposite are unequal in the same order.

Use the same method of proof as in § 385.

387. Theorem. The perpendicular is the shortest distance from a given point to a given line.

Use § 385 to prove this.

EXERCISES

1. Prove that a line segment from the vertex of an isosceles triangle to any point in the base is less than either leg of the triangle.
2. Prove: The difference between two sides of a triangle is less than the third side.
3. Prove: The sum of the distances from a point within a triangle to the ends of one side is less than the sum of the other two sides.
4. Prove: The angle formed by two line segments joining a point within a triangle to two vertices is greater than the angle at the third vertex.
5. Prove: If two sides of a triangle are unequal, the median through their intersection makes the greater angle with the smaller side. (See § 200, Table VIII, No. 5.)

Summary of Fundamental Theorems

ANGLES AND LINES

Supplements of the same or equal angles are equal. §§ 44, 43

Complements of the same or equal angles are equal. §§ 50, 49

Vertical angles are equal. § 46

The perpendicular bisector of a line segment is the locus of a point equidistant from the ends of the line segment. § 154

The bisector of an angle is the locus of a point equidistant from the sides of the angle. § 164

PARALLEL LINES

If two lines are cut by a transversal and the corresponding angles are equal, the lines are parallel. § 54

If two lines are cut by a transversal and the alternate interior angles are equal, the lines are parallel. § 57

If two lines are cut by a transversal and the interior angles on the same side of the transversal are supplementary, the lines are parallel. § 59

If two parallels are cut by a transversal, then the

(1) corresponding angles are equal. § 54
(2) alternate interior angles are equal. § 58
(3) interior angles on the same side are supplementary. § 60

If two lines are perpendicular to the same line, they are parallel to each other. § 55

If two lines are parallel to the same line, they are parallel to each other. § 61

If a line is perpendicular to one of two parallels, it is perpendicular to the other. § 56

Three or more parallels cut proportional segments on two transversals. § 305

If three or more parallels cut equal parts on one transversal, they cut equal parts on any other transversal. § 306

If a line is parallel to one side of a triangle, it divides the other two sides proportionally. § 302

If a line divides two sides of a triangle proportionally, it is parallel to the third side. § 326

TRIANGLES

The sum of the angles of a triangle is 180°. § 62

An exterior angle of a triangle equals the sum of the two remote interior angles. § 64

If two angles of one triangle are equal respectively to two angles of another triangle, the remaining angles are equal. § 65

Two triangles are congruent if they agree in

(1) one side and two corresponding angles. (a.s.a., s.a.a.) §§ 114, 116
(2) two sides and the included angle. (s.a.s.) § 113
(3) two sides and the right angle opposite one of them. (h.l.) § 122
(4) three sides. (s.s.s.) § 120

If two sides of a triangle are equal, the angles opposite are equal. § 118

If two angles of a triangle are equal, the sides opposite are equal. § 119

The area of a triangle is one-half the base times the altitude. § 268

If an altitude of one triangle equals an altitude of another triangle, the triangles are proportional to the bases. § 291

If an angle of one triangle equals an angle of another triangle, the triangles are proportional to the products of the sides forming the equal angles. § 292

Two triangles are similar if they

(1) agree in two angles. § 298
(2) agree in one angle and have the including sides proportional. § 325
(3) have all their sides proportional. § 324

The areas of two similar triangles are proportional to the squares of the corresponding sides. § 301

SECONDARY PARTS OF A TRIANGLE

The altitudes of a triangle meet in a point. Page 206, H1

The medians of a triangle meet in a point that is two-thirds from each vertex to the middle of the opposite side. Page 247, Ex. 7, Ex. 8

The angle bisectors of a triangle meet in a point that is equidistant from the sides of the triangle. Page 114, Ex. 1, Ex. 2

The perpendicular bisectors of the sides of a triangle meet in a point that is equidistant from the vertices of the triangle. Page 109, Ex. 8

The line segment between the midpoints of two sides of a triangle is parallel to the third side and equal to half of it. § 332

RIGHT TRIANGLES

The median to the hypotenuse of a right triangle equals one-half the hypotenuse. § 191

The square of the hypotenuse of a right triangle equals the sum of the squares of the legs. § 318

If the altitude is drawn to the hypotenuse of a right triangle, it is the mean proportional between the segments of the hypotenuse; and either leg is the mean proportional between the whole hypotenuse and the adjoining segment. Page 251, Ex. 11; page 252, Ex. 18

In a right triangle, if one angle is 60°, then the hypotenuse is twice the adjacent leg. § 192

In a right triangle, if the hypotenuse is twice one leg, then the included angle is 60°. § 193

POLYGONS

The sum of the interior angles of a polygon of n sides is $(n - 2)$ straight angles. § 169

The sum of the exterior angles of a polygon of n sides is 2 straight angles. § 168

The area of a polygon circumscribed about a circle is half the perimeter of the polygon times the radius of the circle. Page 180, Ex. 10

The perimeters of two similar polygons are proportional to the corresponding sides. Page 208, Ex. 13

The areas of two similar polygons are proportional to the squares of the corresponding sides. Page 238, 82

QUADRILATERALS

A quadrilateral is a parallelogram if (1) the opposite sides are equal. § 175
(2) two sides are equal and parallel. § 178
(3) the opposite angles are equal. § 176
(4) the diagonals bisect each other. § 177

In a parallelogram (1) the opposite sides are equal. § 171
(2) the opposite angles are equal. § 172
(3) the diagonals bisect each other. § 173

The area of a trapezoid equals the average of its bases times the altitude. § 265

The line segment connecting the midpoints of the legs of a trapezoid is parallel to the bases and equals half their sum. § 333

If the base angles of a trapezoid are equal, the trapezoid is isosceles. § 195

If the diagonals of a trapezoid are equal, the trapezoid is isosceles. Page 124, Ex. 2

In an isosceles trapezoid the base angles are equal. § 194

In an isosceles trapezoid the diagonals are equal. Page 124, Ex. 2

The diagonals of a rectangle are equal. § 189

If the diagonals of a parallelogram are equal, the figure is a rectangle. § 190

The area of a parallelogram equals the base times the height (altitude). § 267

CIRCLES

In the same circle, or in equal circles, equal chords have equal arcs. § 217

In the same circle, or in equal circles, equal chords are equally distant from the center. § 221

In the same circle, or in equal circles, two chords that have equal arcs are equal. § 216

In the same circle, or in equal circles, two chords that are equally distant from the center are equal. § 222

A diameter perpendicular to a chord bisects the chord and its arcs. § 219

Parallel lines intercept equal arcs on a circle. § 230

A line perpendicular to a diameter at one end is tangent to the circle. § 224

A diameter is perpendicular to a tangent at the point of contact. § 225

If the sides of an angle cut or touch a circle, the angle equals half the algebraic sum of the arc degrees between the sides and between the extensions of the sides of the angle, one arc being negative if the vertex is outside the circle. § 237

An inscribed angle, or an angle between a tangent and a chord to the point of contact, equals half the intercepted arc in degrees. § 238

Angles inscribed in the same arc are equal. § 239

An angle inscribed in a semicircle is a right angle. § 240

Tangents to a circle from an external point are equal and make equal angles with the line joining that point to the center. § 247

The common external tangents to two circles are equal. § 252

The common internal tangents to two circles are equal. § 251

A circle can be circumscribed about or inscribed in any triangle. Page 109, Ex. 8 (b); § 248

A circle can be circumscribed about or inscribed in any regular polygon. §§ 349, 352

In an inscribed quadrilateral the opposite angles are supplementary. § 241

In a circumscribed quadrilateral the sum of two opposite sides equals the sum of the other two sides. Page 159, Ex. 6

If in a quadrilateral the opposite angles are supplementary, then a circle can be circumscribed about it. Page 156, S4

If in a quadrilateral the sum of two opposite sides equals the sum of the other two sides, then a circle can be inscribed in it. Page 159, S1

If two chords intersect within a circle, the product of the segments of one chord equals the product of the segments of the other chord. § 340

If a chord is perpendicular to a diameter, either segment of the chord is the mean proportional between the segments of the diameter. § 341

If two secants are drawn to a circle from an external point, the product of one secant and its external segment equals the product of the other secant and its external segment. § 343

If a tangent and a secant are drawn to a circle from an external point, the tangent is the mean proportional between the whole secant and its external segment. § 344

The circumference of a circle equals the diameter times π. § 360

The area of a circle equals π times the square of the radius. § 367

In the same circle, or in equal circles, the length of an arc is to the circumference of the circle as the angular measure of the arc is to 360°. § 362

In the same circle, or in equal circles, the area of a sector is to the area of the circle as the angular measure of the sector is to 360°. § 372

TABLE OF SINES, COSINES, AND TANGENTS

Angle	Sine	Cosine	Tangent	Angle	Sine	Cosine	Tangent
1°	.0175	.9998	.0175	46°	.7193	.6947	1.0355
2°	.0349	.9994	.0349	47°	.7314	.6820	1.0724
3°	.0523	.9986	.0524	48°	.7431	.6691	1.1106
4°	.0698	.9976	.0699	49°	.7547	.6561	1.1504
5°	.0872	.9962	.0875	50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900
45°	.7071	.7071	1.0000	90°	1.0000	.0000	

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
1.0	1.000	1.005	1.010	1.015	1.020	1.025	1.030	1.034	1.039	1.044
1.1	1.049	1.054	1.058	1.063	1.068	1.072	1.077	1.082	1.086	1.091
1.2	1.095	1.100	1.105	1.109	1.114	1.118	1.122	1.127	1.131	1.136
1.3	1.140	1.145	1.149	1.153	1.158	1.162	1.166	1.170	1.175	1.179
1.4	1.183	1.187	1.192	1.196	1.200	1.204	1.208	1.212	1.217	1.221
1.5	1.225	1.229	1.233	1.237	1.241	1.245	1.249	1.253	1.257	1.261
1.6	1.265	1.269	1.273	1.277	1.281	1.285	1.288	1.292	1.296	1.300
1.7	1.304	1.308	1.311	1.315	1.319	1.323	1.327	1.330	1.334	1.338
1.8	1.342	1.345	1.349	1.353	1.356	1.360	1.364	1.367	1.371	1.375
1.9	1.378	1.382	1.386	1.389	1.393	1.396	1.400	1.404	1.407	1.411
2.0	1.414	1.418	1.421	1.425	1.428	1.432	1.435	1.439	1.442	1.446
2.1	1.449	1.453	1.456	1.459	1.463	1.466	1.470	1.473	1.476	1.480
2.2	1.483	1.487	1.490	1.493	1.497	1.500	1.503	1.507	1.510	1.513
2.3	1.517	1.520	1.523	1.526	1.530	1.533	1.536	1.539	1.543	1.546
2.4	1.549	1.552	1.556	1.559	1.562	1.565	1.568	1.572	1.575	1.578
2.5	1.581	1.584	1.587	1.591	1.594	1.597	1.600	1.603	1.606	1.609
2.6	1.612	1.616	1.619	1.622	1.625	1.628	1.631	1.634	1.637	1.640
2.7	1.643	1.646	1.649	1.652	1.655	1.658	1.661	1.664	1.667	1.670
2.8	1.673	1.676	1.679	1.682	1.685	1.688	1.691	1.694	1.697	1.700
2.9	1.703	1.706	1.709	1.712	1.715	1.718	1.720	1.723	1.726	1.729
3.0	1.732	1.735	1.738	1.741	1.744	1.746	1.749	1.752	1.755	1.758
3.1	1.761	1.764	1.766	1.769	1.772	1.775	1.778	1.780	1.783	1.786
3.2	1.789	1.792	1.794	1.797	1.800	1.803	1.806	1.808	1.811	1.814
3.3	1.817	1.819	1.822	1.825	1.828	1.830	1.833	1.836	1.838	1.841
3.4	1.844	1.847	1.849	1.852	1.855	1.857	1.860	1.863	1.865	1.868
3.5	1.871	1.873	1.876	1.879	1.881	1.884	1.887	1.889	1.892	1.895
3.6	1.897	1.900	1.903	1.905	1.908	1.910	1.913	1.916	1.918	1.921
3.7	1.924	1.926	1.929	1.931	1.934	1.936	1.939	1.942	1.944	1.947
3.8	1.949	1.952	1.954	1.957	1.960	1.962	1.965	1.967	1.970	1.972
3.9	1.975	1.977	1.980	1.982	1.985	1.987	1.990	1.992	1.995	1.997
4.0	2.000	2.002	2.005	2.007	2.010	2.012	2.015	2.017	2.020	2.022
4.1	2.025	2.027	2.030	2.032	2.035	2.037	2.040	2.042	2.045	2.047
4.2	2.049	2.052	2.054	2.057	2.059	2.062	2.064	2.066	2.069	2.071
4.3	2.074	2.076	2.078	2.081	2.083	2.086	2.088	2.090	2.093	2.095
4.4	2.098	2.100	2.102	2.105	2.107	2.110	2.112	2.114	2.117	2.119
4.5	2.121	2.124	2.126	2.128	2.131	2.133	2.135	2.138	2.140	2.142
4.6	2.145	2.147	2.149	2.152	2.154	2.156	2.159	2.161	2.163	2.166
4.7	2.168	2.170	2.173	2.175	2.177	2.179	2.182	2.184	2.186	2.189
4.8	2.191	2.193	2.195	2.198	2.200	2.202	2.205	2.207	2.209	2.211
4.9	2.214	2.216	2.218	2.220	2.223	2.225	2.227	2.229	2.232	2.234
5.0	2.236	2.238	2.241	2.243	2.245	2.247	2.249	2.252	2.254	2.256
5.1	2.258	2.261	2.263	2.265	2.267	2.269	2.272	2.274	2.276	2.278
5.2	2.280	2.283	2.285	2.287	2.289	2.291	2.293	2.296	2.298	2.300
5.3	2.302	2.304	2.307	2.309	2.311	2.313	2.315	2.317	2.319	2.322
5.4	2.324	2.326	2.328	2.330	2.332	2.335	2.337	2.339	2.341	2.343

Note: To find the square root of a number not between 1 and 100, shift the decimal point. Thus, $\sqrt{175} = \sqrt{1.75} \times 10$; $\sqrt{.175} = \sqrt{17.5} \div 10$ (see § 277).

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
5.5	2.345	2.347	2.349	2.352	2.354	2.356	2.358	2.360	2.362	2.364
5.6	2.366	2.369	2.371	2.373	2.375	2.377	2.379	2.381	2.383	2.385
5.7	2.387	2.390	2.392	2.394	2.396	2.398	2.400	2.402	2.404	2.406
5.8	2.408	2.410	2.412	2.415	2.417	2.419	2.421	2.423	2.425	2.427
5.9	2.429	2.431	2.433	2.435	2.437	2.439	2.441	2.443	2.445	2.447
6.0	2.449	2.452	2.454	2.456	2.458	2.460	2.462	2.464	2.466	2.468
6.1	2.470	2.472	2.474	2.476	2.478	2.480	2.482	2.484	2.486	2.488
6.2	2.490	2.492	2.494	2.496	2.498	2.500	2.502	2.504	2.506	2.508
6.3	2.510	2.512	2.514	2.516	2.518	2.520	2.522	2.524	2.526	2.528
6.4	2.530	2.532	2.534	2.536	2.538	2.540	2.542	2.544	2.546	2.548
6.5	2.550	2.551	2.553	2.555	2.557	2.559	2.561	2.563	2.565	2.567
6.6	2.569	2.571	2.573	2.575	2.577	2.579	2.581	2.583	2.585	2.587
6.7	2.588	2.590	2.592	2.594	2.596	2.598	2.600	2.602	2.604	2.606
6.8	2.608	2.610	2.612	2.613	2.615	2.617	2.619	2.621	2.623	2.625
6.9	2.627	2.629	2.631	2.632	2.634	2.636	2.638	2.640	2.642	2.644
7.0	2.646	2.648	2.650	2.651	2.653	2.655	2.657	2.659	2.661	2.663
7.1	2.665	2.666	2.668	2.670	2.672	2.674	2.676	2.678	2.680	2.681
7.2	2.683	2.685	2.687	2.689	2.691	2.693	2.694	2.696	2.698	2.700
7.3	2.702	2.704	2.706	2.707	2.709	2.711	2.713	2.715	2.717	2.718
7.4	2.720	2.722	2.724	2.726	2.728	2.729	2.731	2.733	2.735	2.737
7.5	2.739	2.740	2.742	2.744	2.746	2.748	2.750	2.751	2.753	2.755
7.6	2.757	2.759	2.760	2.762	2.764	2.766	2.768	2.769	2.771	2.773
7.7	2.775	2.777	2.778	2.780	2.782	2.784	2.786	2.787	2.789	2.791
7.8	2.793	2.795	2.796	2.798	2.800	2.802	2.804	2.805	2.807	2.809
7.9	2.811	2.812	2.814	2.816	2.818	2.820	2.821	2.823	2.825	2.827
8.0	2.828	2.830	2.832	2.834	2.835	2.837	2.839	2.841	2.843	2.844
8.1	2.846	2.848	2.850	2.851	2.853	2.855	2.857	2.858	2.860	2.862
8.2	2.864	2.865	2.867	2.869	2.871	2.872	2.874	2.876	2.877	2.879
8.3	2.881	2.883	2.884	2.886	2.888	2.890	2.891	2.893	2.895	2.897
8.4	2.898	2.900	2.902	2.903	2.905	2.907	2.909	2.910	2.912	2.914
8.5	2.915	2.917	2.919	2.921	2.922	2.924	2.926	2.927	2.929	2.931
8.6	2.933	2.934	2.936	2.938	2.939	2.941	2.943	2.944	2.946	2.948
8.7	2.950	2.951	2.953	2.955	2.956	2.958	2.960	2.961	2.963	2.965
8.8	2.966	2.968	2.970	2.972	2.973	2.975	2.977	2.978	2.980	2.982
8.9	2.983	2.985	2.987	2.988	2.990	2.992	2.993	2.995	2.997	2.998
9.0	3.000	3.002	3.003	3.005	3.007	3.008	3.010	3.012	3.013	3.015
9.1	3.017	3.018	3.020	3.022	3.023	3.025	3.027	3.028	3.030	3.032
9.2	3.033	3.035	3.036	3.038	3.040	3.041	3.043	3.045	3.046	3.048
9.3	3.050	3.051	3.053	3.055	3.056	3.058	3.059	3.061	3.063	3.064
9.4	3.066	3.068	3.069	3.071	3.072	3.074	3.076	3.077	3.079	3.081
9.5	3.082	3.084	3.085	3.087	3.089	3.090	3.092	3.094	3.095	3.097
9.6	3.098	3.100	3.102	3.103	3.105	3.106	3.108	3.110	3.111	3.113
9.7	3.114	3.116	3.118	3.119	3.121	3.122	3.124	3.126	3.127	3.129
9.8	3.130	3.132	3.134	3.135	3.137	3.138	3.140	3.142	3.143	3.145
9.9	3.146	3.148	3.150	3.151	3.153	3.154	3.156	3.158	3.159	3.161

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
55.	7.416	7.423	7.430	7.436	7.443	7.450	7.457	7.463	7.470	7.477
56.	7.483	7.490	7.497	7.503	7.510	7.517	7.523	7.530	7.537	7.543
57.	7.550	7.556	7.563	7.570	7.576	7.583	7.589	7.596	7.603	7.609
58.	7.616	7.622	7.629	7.635	7.642	7.649	7.655	7.662	7.668	7.675
59.	7.681	7.688	7.694	7.701	7.707	7.714	7.720	7.727	7.733	7.740
60.	7.746	7.752	7.759	7.765	7.772	7.778	7.785	7.791	7.797	7.804
61.	7.810	7.817	7.823	7.829	7.836	7.842	7.849	7.855	7.861	7.868
62.	7.874	7.880	7.887	7.893	7.899	7.906	7.912	7.918	7.925	7.931
63.	7.937	7.944	7.950	7.956	7.962	7.969	7.975	7.981	7.987	7.994
64.	8.000	8.006	8.012	8.019	8.025	8.031	8.037	8.044	8.050	8.056
65.	8.062	8.068	8.075	8.081	8.087	8.093	8.099	8.106	8.112	8.118
66.	8.124	8.130	8.136	8.142	8.149	8.155	8.161	8.167	8.173	8.179
67.	8.185	8.191	8.198	8.204	8.210	8.216	8.222	8.228	8.234	8.240
68.	8.246	8.252	8.258	8.264	8.270	8.276	8.283	8.289	8.295	8.301
69.	8.307	8.313	8.319	8.325	8.331	8.337	8.343	8.349	8.355	8.361
70.	8.367	8.373	8.379	8.385	8.390	8.396	8.402	8.408	8.414	8.420
71.	8.426	8.432	8.438	8.444	8.450	8.456	8.462	8.468	8.473	8.479
72.	8.485	8.491	8.497	8.503	8.509	8.515	8.521	8.526	8.532	8.538
73.	8.544	8.550	8.556	8.562	8.567	8.573	8.579	8.585	8.591	8.597
74.	8.602	8.608	8.614	8.620	8.626	8.631	8.637	8.643	8.649	8.654
75.	8.660	8.666	8.672	8.678	8.683	8.689	8.695	8.701	8.706	8.712
76.	8.718	8.724	8.729	8.735	8.741	8.746	8.752	8.758	8.764	8.769
77.	8.775	8.781	8.786	8.792	8.798	8.803	8.809	8.815	8.820	8.826
78.	8.832	8.837	8.843	8.849	8.854	8.860	8.866	8.871	8.877	8.883
79.	8.888	8.894	8.899	8.905	8.911	8.916	8.922	8.927	8.933	8.939
80.	8.944	8.950	8.955	8.961	8.967	8.972	8.978	8.983	8.989	8.994
81.	9.000	9.006	9.011	9.017	9.022	9.028	9.033	9.039	9.044	9.050
82.	9.055	9.061	9.066	9.072	9.077	9.083	9.088	9.094	9.099	9.105
83.	9.110	9.116	9.121	9.127	9.132	9.138	9.143	9.149	9.154	9.160
84.	9.165	9.171	9.176	9.182	9.187	9.192	9.198	9.203	9.209	9.214
85.	9.220	9.225	9.230	9.236	9.241	9.247	9.252	9.257	9.263	9.268
86.	9.274	9.279	9.284	9.290	9.295	9.301	9.306	9.311	9.317	9.322
87.	9.327	9.333	9.338	9.343	9.349	9.354	9.359	9.365	9.370	9.375
88.	9.381	9.386	9.391	9.397	9.402	9.407	9.413	9.418	9.423	9.429
89.	9.434	9.439	9.445	9.450	9.455	9.460	9.466	9.471	9.476	9.482
90.	9.487	9.492	9.497	9.503	9.508	9.513	9.518	9.524	9.529	9.534
91.	9.539	9.545	9.550	9.555	9.560	9.566	9.571	9.576	9.581	9.586
92.	9.592	9.597	9.602	9.607	9.612	9.618	9.623	9.628	9.633	9.638
93.	9.644	9.649	9.654	9.659	9.664	9.670	9.675	9.680	9.685	9.690
94.	9.695	9.701	9.706	9.711	9.716	9.721	9.726	9.731	9.737	9.742
95.	9.747	9.752	9.757	9.762	9.767	9.772	9.778	9.783	9.788	9.793
96.	9.798	9.803	9.808	9.813	9.818	9.823	9.829	9.834	9.839	9.844
97.	9.849	9.854	9.859	9.864	9.869	9.874	9.879	9.884	9.889	9.894
98.	9.899	9.905	9.910	9.915	9.920	9.925	9.930	9.935	9.940	9.945
99.	9.950	9.955	9.960	9.965	9.970	9.975	9.980	9.985	9.990	9.995

SQUARE ROOTS

N	0	1	2	3	4	5	6	7	8	9
10.	3.162	3.178	3.194	3.209	3.225	3.240	3.256	3.271	3.286	3.302
11.	3.317	3.332	3.347	3.362	3.376	3.391	3.406	3.421	3.435	3.450
12.	3.464	3.479	3.493	3.507	3.521	3.536	3.550	3.564	3.578	3.592
13.	3.606	3.619	3.633	3.647	3.661	3.674	3.688	3.701	3.715	3.728
14.	3.742	3.755	3.768	3.782	3.795	3.808	3.821	3.834	3.847	3.860
15.	3.873	3.886	3.899	3.912	3.924	3.937	3.950	3.962	3.975	3.987
16.	4.000	4.012	4.025	4.037	4.050	4.062	4.074	4.087	4.099	4.111
17.	4.123	4.135	4.147	4.159	4.171	4.183	4.195	4.207	4.219	4.231
18.	4.243	4.254	4.266	4.278	4.290	4.301	4.313	4.324	4.336	4.347
19.	4.359	4.370	4.382	4.393	4.405	4.416	4.427	4.438	4.450	4.461
20.	4.472	4.483	4.494	4.506	4.517	4.528	4.539	4.550	4.561	4.572
21.	4.583	4.593	4.604	4.615	4.626	4.637	4.648	4.658	4.669	4.680
22.	4.690	4.701	4.712	4.722	4.733	4.743	4.754	4.764	4.775	4.785
23.	4.796	4.806	4.817	4.827	4.837	4.848	4.858	4.868	4.879	4.889
24.	4.899	4.909	4.919	4.930	4.940	4.950	4.960	4.970	4.980	4.990
25.	5.000	5.010	5.020	5.030	5.040	5.050	5.060	5.070	5.079	5.089
26.	5.099	5.109	5.119	5.128	5.138	5.148	5.158	5.167	5.177	5.187
27.	5.196	5.206	5.215	5.225	5.235	5.244	5.254	5.263	5.273	5.282
28.	5.292	5.301	5.310	5.320	5.329	5.339	5.348	5.357	5.367	5.376
29.	5.385	5.394	5.404	5.413	5.422	5.431	5.441	5.450	5.459	5.468
30.	5.477	5.486	5.495	5.505	5.514	5.523	5.532	5.541	5.550	5.559
31.	5.568	5.577	5.586	5.595	5.604	5.612	5.621	5.630	5.639	5.648
32.	5.657	5.666	5.675	5.683	5.692	5.701	5.710	5.718	5.727	5.736
33.	5.745	5.753	5.762	5.771	5.779	5.788	5.797	5.805	5.814	5.822
34.	5.831	5.840	5.848	5.857	5.865	5.874	5.882	5.891	5.899	5.908
35.	5.916	5.925	5.933	5.941	5.950	5.958	5.967	5.975	5.983	5.992
36.	6.000	6.008	6.017	6.025	6.033	6.042	6.050	6.058	6.066	6.075
37.	6.083	6.091	6.099	6.107	6.116	6.124	6.132	6.140	6.148	6.156
38.	6.164	6.173	6.181	6.189	6.197	6.205	6.213	6.221	6.229	6.237
39.	6.245	6.253	6.261	6.269	6.277	6.285	6.293	6.301	6.309	6.317
40.	6.325	6.332	6.340	6.348	6.356	6.364	6.372	6.380	6.387	6.395
41.	6.403	6.411	6.419	6.427	6.434	6.442	6.450	6.458	6.465	6.473
42.	6.481	6.488	6.496	6.504	6.512	6.519	6.527	6.535	6.542	6.550
43.	6.557	6.565	6.573	6.580	6.588	6.595	6.603	6.611	6.618	6.626
44.	6.633	6.641	6.648	6.656	6.663	6.671	6.678	6.686	6.693	6.701
45.	6.708	6.716	6.723	6.731	6.738	6.745	6.753	6.760	6.768	6.775
46.	6.782	6.790	6.797	6.804	6.812	6.819	6.826	6.834	6.841	6.848
47.	6.856	6.863	6.870	6.877	6.885	6.892	6.899	6.907	6.914	6.921
48.	6.928	6.935	6.943	6.950	6.957	6.964	6.971	6.979	6.986	6.993
49.	7.000	7.007	7.014	7.021	7.029	7.036	7.043	7.050	7.057	7.064
50.	7.071	7.078	7.085	7.092	7.099	7.106	7.113	7.120	7.127	7.134
51.	7.141	7.148	7.155	7.162	7.169	7.176	7.183	7.190	7.197	7.204
52.	7.211	7.218	7.225	7.232	7.239	7.246	7.253	7.259	7.266	7.273
53.	7.280	7.287	7.294	7.301	7.308	7.314	7.321	7.328	7.335	7.342
54.	7.348	7.355	7.362	7.369	7.376	7.382	7.389	7.396	7.403	7.409

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